JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

Christian AXLER

On the sum of the first n prime numbers

Tome 31, nº 2 (2019), p. 293-311.

<http://jtnb.cedram.org/item?id=JTNB_2019__31_2_293_0>

Journal de Théorie des Nombres de Bordeaux **31** (2019), 293–311

On the sum of the first n prime numbers

par Christian AXLER

Résumé. Dans cet article, nous établissons une formule asymptotique pour la somme des n premiers nombres premiers, plus précise que celle donnée par Massias et Robin en 1996. En outre, nous prouvons un certain nombre de résultats concernant l'inégalité de Mandl pour la somme des n premiers nombres premiers. Nous utilisons ces résultats pour établir de nouvelles estimations explicites de la somme des n premiers nombres premiers, qui améliorent les meilleures estimations actuellement connues.

Abstract. In this paper we establish an asymptotic formula for the sum of the first n prime numbers, more precise than the one given by Massias and Robin in 1996. Further we prove a series of results concerning Mandl's inequality on the sum of the first n prime numbers. We use these results to find new explicit estimates for the sum of the first n prime numbers, which improve the currently best known estimates.

1. Introduction

Let $\pi(x)$ denote the number of primes not exceeding x. Hadamard [13] and de la Vallée-Poussin [25] independently proved a result concerning the asymptotic behavior for $\pi(x)$, namely $\pi(x) \sim \mathrm{li}(x)$ as $x \to \infty$, which is known as the *Prime Number Theorem*. Here, the *logarithmic integral* $\mathrm{li}(x)$ is defined for every real $x \geq 0$ as

$$\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t} = \lim_{\varepsilon \to 0+} \left\{ \int_0^{1-\varepsilon} \frac{\mathrm{d}t}{\log t} + \int_{1+\varepsilon}^x \frac{\mathrm{d}t}{\log t} \right\}. \tag{1.1}$$

In a later paper [26], where the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\mathrm{Re}(s) = 1$ was proved, de la Vallée-Poussin also estimated the error term in the Prime Number Theorem by showing that

$$\pi(x) = \mathrm{li}(x) + O(xe^{-a\sqrt{\log x}}), \tag{1.2}$$

where a is a positive absolute constant. Denoting the sum of the first prime numbers not exceeding x by $S(x)$, Szalay [24, Lemma 1] used (1.2) to find

$$S(x) = \mathrm{li}(x^2) + O(x^2 e^{-a\sqrt{\log x}}). \tag{1.3}$$

Manuscrit reçu le 6 octobre 2016, accepté le 15 juillet 2019.
2010 *Mathematics Subject Classification.* 11N05, 11A41.
Mots-clefs. Asymptotic expansion, Mandl's inequality, Sum of prime numbers.

Using (1.3) and integration by parts in (1.1), we get the asymptotic expansion

$$S(x) = \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{3x^2}{8\log^4 x} + O\left(\frac{x^2}{\log^5 x}\right). \tag{1.4}$$

The first aim of this paper is to find explicit estimates for $S(x)$ in the direction of (1.4). The current best such upper bound for $S(x)$ is due to Massias and Robin [16, Théorème D (v)]. They found that $S(x) \leq x^2/(2\log x) + 3x^2/(10\log^2 x)$ for every $x \geq 24\,281$. We start with the following result which improves the last inequality.

Theorem 1.1. *For every* $x \geq 110\,118\,925$*, we have*

$$S(x) < \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{5.3x^2}{8\log^4 x}.$$

The current best lower bound for $S(x)$ concerning (1.4) is also due to Massias and Robin [16, Théorème D (ii)]. We find the following improvement.

Theorem 1.2. *For every* $x \geq 905\,238\,547$*, we have*

$$S(x) > \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{1.2x^2}{8\log^4 x}.$$

Using an explicit estimate for $\mathrm{li}(x^2)$, we find for the first time explicit bounds for the difference $S(x) - \mathrm{li}(x^2)$ concerning (1.3) by establishing the following result.

Theorem 1.3. *We have*

$$-\frac{0.25x^2}{\log^4 x} < S(x) - \mathrm{li}(x^2) < \frac{0.25x^2}{\log^4 x},$$

where the left-hand side inequality is valid for every $x \geq 906\,484\,877$ *and the right-hand side inequality holds for every* $x \geq 110\,117\,797$*.*

The case $x = p_n$, where p_n denotes the nth prime number, is of particular interest. Here, $S(x) = \sum_{k \leq n} p_k$ is equal to the sum of the first n prime numbers. Massias and Robin [16, p. 217] found that

$$\sum_{k \leq n} p_k = \mathrm{li}((\mathrm{li}^{-1}(n))^2) + O(n^2 e^{-c\sqrt{\log n}}), \tag{1.5}$$

where c is a positive absolute constant and $\mathrm{li}^{-1}(x)$ is the inverse function of $\mathrm{li}(x)$. Then they [16, p. 217] used (1.5) and a result of Robin [17] to derive the asymptotic expansion

$$\sum_{k \leq n} p_k = \frac{n^2}{2}\left(\log n + \sum_{i=0}^{m} \frac{A_{i+1}(\log\log n)}{\log^i n} + O\left(\frac{(\log\log n)^{m+1}}{\log^{m+1} n}\right)\right), \tag{1.6}$$

where m is a positive integer and the polynomials A_k satisfy the formulas $A_0(x) = 1$ and $A'_{k+1} = A'_k - (k-1)A_k$. Unfortunately, this recursive formula for derivatives does not yield a description of the polynomials A_k, since the constant coefficient of the polynomials A_k remains undetermined by this equation. We fix this problem by applying a method developed by Salvy [21, Theorem 2] and get the following theorem.

Theorem 1.4. *Let N be a positive integer. Then there exist uniquely determined monic polynomials $T_1, \ldots, T_{N-1}$ with real coefficients and $\deg(T_i) = i$, such that*

$$\sum_{k \leq n} p_k = \frac{n^2}{2} \left(\log n + \log\log n - \frac{3}{2} + \sum_{i=1}^{N-1} \frac{(-1)^{i+1} T_i(\log\log n)}{i \log^i n} \right) + O\left(\frac{n^2 (\log\log n)^N}{\log^N n} \right).$$

The polynomials T_i can be computed explicitly. In particular,

- $T_1(x) = x - 5/2,$
- $T_2(x) = x^2 - 7x + 29/2,$
- $T_3(x) = x^3 - 12x^2 + 54x - 185/2,$
- $T_4(x) = x^4 - 52x^3/3 + 124x^2 - 442x + 1996/3.$

For $i \geq 1$, the polynomials A_{i+1} given in (1.6) *and polynomials T_i are connected by the formula $T_i = (-1)^{i+1} i A_{i+1}$.*

The proof of Theorem 1.4 is given in Section 5. The initial motivation for writing this paper was an inequality conjectured by Mandl concerning an upper bound for the sum of the first n prime numbers, namely

$$\frac{np_n}{2} - \sum_{k \leq n} p_k \geq 0 \tag{1.7}$$

for every integer $n \geq 9$. This inequality originally appeared in [20] without proof. In his thesis, Dusart [10] used the identity

$$\sum_{k \leq n} p_k = np_n - \int_2^{p_n} \pi(x)\,\mathrm{d}x$$

and explicit estimates for the prime counting function $\pi(x)$ to prove that (1.7) indeed holds for every integer $n \geq 9$. The second goal of this paper is to study the sequence $(B_n)_{n \in \mathbb{N}}$, where B_n denotes the left-hand side of (1.7), in more detail. For this purpose, we first derive an asymptotic expansion for B_n by using a result of Cipolla [7] concerning an asymptotic expansion for the nth prime number. He proved that for every positive integer N there exist uniquely determined monic polynomials $R_1, \ldots, R_{N-1}$ with real coefficients and $\deg(R_i) = i$, such that

$$p_n = n\left(\log n + \log\log n - 1 + \sum_{i=1}^{N-1} \frac{(-1)^{i+1}R_i(\log\log n)}{i\log^i n}\right) + O\left(\frac{n(\log\log n)^N}{\log^N n}\right).$$

The polynomials R_i are computed explicitly in [21, p. 235]. Setting $V_i = R_i - T_i$, where the polynomials T_i are given by Theorem 1.4, we get the following asymptotic expansion for B_n.

Theorem 1.5. *Let N be a positive integer. Then,*

$$B_n = \frac{n^2}{2}\left(\frac{1}{2} + \sum_{i=1}^{N-1} \frac{(-1)^{i+1}V_i(\log\log n)}{i\log^i n} + O\left(\frac{(\log\log n)^N}{\log^N n}\right)\right).$$

The polynomials V_s can be computed explicitly. In particular,

- $V_1(x) = 1/2$,
- $V_2(x) = x - 7/2$,
- $V_3(x) = 3x^2/2 - 12x + 27$,
- $V_4(x) = 2x^3 - 26x^2 + 124x - 221$.

Since it is still difficult to compute B_n for large n, we are interested in explicit estimates for B_n. From (1.7), we get that $B_n \geq 0$ for every integer $n \geq 9$. Hassani [14, Corollary 1.5] has found that the inequality $B_n > n^2/12$ holds for every integer $n \geq 10$. Up to now, the sharpest lower bound for B_n is due to Sun [23, Theorem 1.1]. He proved that $B_n > n^2/4$ for every integer $n \geq 417$. We improve Sun's result as follows.

Theorem 1.6. *For every integer $n \geq 6\,309\,751$, we have*

$$B_n > \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2(\log\log n - 2.9)}{4\log^2 n}.$$

In the other direction, we give the following explicit estimate for B_n, which improves the only known upper bound $B_n < 9n^2/4$, which holds for every integer $n \geq 2$, found by Hassani [14, Corollary 1.5].

Theorem 1.7. *For every integer $n \geq 256\,376$, we have*

$$B_n < \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2(\log\log n - 4.42)}{4\log^2 n}.$$

Theorem 1.4 implies that

$$\sum_{k\le n} p_k = \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n} - \frac{T_2(\log\log n)}{2\log^2 n}\right) + O\left(\frac{n^2(\log\log n)^3}{\log^3 n}\right), \tag{1.8}$$

where $T_2(x) = x^2 - 7x + 29/2$. We use the inequalities found in Theorems 1.6 and 1.7, and combine them with some estimates for the nth prime number given in [6, Theorems 1 and 4] to derive the following estimates for the sum of the first n prime numbers, which refine the ones previously known.

Theorem 1.8. *Let $T_{\mathrm{up}}(x) = x^2 - 7x + 13.567$. Then, for every integer $n \ge 1\,897\,700$, we have*

$$\sum_{k\le n} p_k < \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n} - \frac{T_{\mathrm{up}}(\log\log n)}{2\log^2 n}\right).$$

Theorem 1.9. *Let $T_{\mathrm{low}}(x) = x^2 - 7x + 15.741$. Then, for every integer $n \ge 2$, we have*

$$\sum_{k\le n} p_k > \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n} - \frac{T_{\mathrm{low}}(\log\log n)}{2\log^2 n}\right).$$

2. Proof of Theorem 1.1

The following proof of Theorem 1.1 is based on a recent obtained estimate for $\pi(x)$.

Proof of Theorem 1.1. We denote the right-hand side of the required inequality by $f(x)$ and let $x_0 = 10\,166\,443\,802$. First, we consider the case $x \ge x_0$ and let $n = \pi(x) \ge 462\,277\,798$. We have

$$S(x) = \pi(p_n)p_n - np_n + \sum_{k\le n} p_k. \tag{2.1}$$

Applying the upper bound for the prime counting function $\pi(x)$ given in [4, Proposition 3.6] and the lower bound for $np_n - \sum_{k\le n} p_k$ found in [5, Theorem 1] to (2.1), we get $S(x) < g(p_n)$, where

$$g(t) = \frac{t^2}{2\log t} + \frac{t^2}{4\log^2 t} + \frac{t^2}{4\log^3 t} + \frac{4.8t^2}{8\log^4 t} + \frac{4.5t^2}{4\log^5 t} + \frac{28.5t^2}{8\log^6 t} + \frac{121.5t^2}{8\log^7 t} + \frac{25826.5t^2}{16\log^8 t}.$$

Note that $g(t)$ is an increasing function for every $t \ge 17$. So we conclude that $S(x) < g(x)$ for every $x \ge x_0$. Since $g(t) < f(t)$ for every $t \ge x_0$, the theorem is proved for every $x \ge x_0$. A computer check shows that $f(p_i) \ge S(p_i)$

for every integer i such that $\pi(110\,119\,007) \le i \le \pi(10\,166\,443\,802)$. Hence, $f(x) \ge S(x)$ for every x with $110\,119\,007 \le x \le x_0$. Finally, we notice that $f(x) \ge S(x)$ for every x satisfying $110\,118\,925 \le x < 110\,119\,007$, which completes the proof. □

Remark 2.1. In [9, Corollary 2.7], Deléglise and Nicolas found a slightly weaker version of Theorem 1.1.

3. Proof of Theorem 1.2

The currently best known lower bound for $S(x)$ is also due to Massias and Robin [16, Théorème D (ii)]. They proved that the inequality $S(x) \ge x^2/(2\log x) + 0.954x^2/(4\log^2 x)$ holds for every $x \ge 70\,841$. In order to prove Theorem 1.2, we first note the following lemma, which can be found in [1, Theorem 4.2].

Lemma 3.1 (Abel's identity). *For any function $a : \mathbb{N} \to \mathbb{C}$ let $A(x) = \sum_{n \le x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume g has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then we have*

$$\sum_{y<n\le x} a(n)g(n) = A(x)g(x) - A(y)g(y) - \int_y^x A(t)g'(t)\ \mathrm{d}t.$$

The following proof of Theorem 1.2 is based on the use of Lemma 3.1 and some recently obtained estimates for the prime counting function $\pi(x)$.

Proof of Theorem 1.2. First, we consider the case $x \ge 19\,027\,490\,297$. We denote the right-hand side of the required inequality by $f(x)$. Further, let $y = 1$, $g(t) = t$, and

$$a(n) = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

We use Lemma 3.1 to get

$$S(x) = \sum_{1<n\le x} a(n)g(n) = x\pi(x) - \int_1^x \pi(t)\,\mathrm{d}t = x\pi(x) - 143 - \int_{27}^x \pi(t)\,\mathrm{d}t.$$

Now we apply the estimates for the prime counting function found in [4, Propositions 3.6 and 3.12] to the last equality, and use [10, Lemme 1.6] and [3, Proposition 9] to see that

$$\begin{aligned}(3.1)\quad S(x) > \frac{26689x^2}{180\log x} &+ \frac{26689x^2}{360\log^2 x} + \frac{26689x^2}{360\log^3 x} + \frac{5327x^2}{48\log^4 x} + \frac{6661x^2}{30\log^5 x} \\ &+ \frac{1663x^2}{3\log^6 x} + \frac{3317x^2}{2\log^7 x} + \frac{10017x^2}{2\log^8 x} - \frac{26599}{90}\ \mathrm{li}(x^2).\end{aligned}$$

By [3, Lemma 19], we have

$$\text{(3.2)} \quad \operatorname{li}(x^2) \leq \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{3x^2}{8\log^4 x} + \frac{3x^2}{4\log^5 x} + \frac{15x^2}{8\log^6 x} + \frac{45x^2}{8\log^7 x} + \frac{1575x^2}{64\log^8 x}$$

for every $x \geq 10^9$. Combined with (3.1), we get

$$\text{(3.3)} \quad S(x) > \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{3x^2}{20\log^4 x} + \frac{3x^2}{8\log^5 x} + \frac{3x^2}{16\log^6 x} - \frac{63x^2}{16\log^7 x} - \frac{289877x^2}{128\log^8 x},$$

which completes the proof for every $x \geq 19\,027\,490\,297$. To deal with the remaining case $905\,238\,547 \leq x < 19\,027\,490\,297$, we check with a computer that $S(p_i) \geq f(p_{i+1})$ for every integer i with $\pi(905\,238\,547) \leq i \leq \pi(19\,027\,490\,297)$. Since $f'(x) > 0$ for every $x \geq 2.8$, we get $S(x) \geq f(x)$ for every $x \geq 905\,238\,547$. □

Remark 3.2. Recently, Theorem 1.2 was independently found by Deléglise and Nicolas [9, Corollary 2.7].

We obtain the following lower bound for $S(x)$, which corresponds to the first three terms of the asymptotic expansion (1.4).

Corollary 3.3. *For every $x \geq 152\,603\,617$, we have*

$$\text{(3.4)} \qquad S(x) > \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x}.$$

Proof. From Theorem 1.2, it follows that the required inequality holds for every $x \geq 905\,238\,547$. Similar to the proof of Theorem 1.2, we check (3.4) for smaller values of x with a computer. □

The asymptotic formula (1.4) implies that

$$\text{(3.5)} \qquad S(x) \geq \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x}$$

for all sufficiently large values of x. In 1988, Massias, Nicolas, and Robin [15, Lemma 3 (i)] proved that the inequality (3.5) holds for every x such that $302\,791 \leq x \leq e^{90}$. Under the assumption that the Riemann hypothesis is true, Massias and Robin [16, Théorème D (iv)] showed that the inequality (3.5) holds for every $x \geq 302\,971$. Further, they [16, Théorème D (iv)] proved that the inequality (3.5) holds unconditionally for every x such that $302\,971 \leq x \leq e^{98}$ and for every $x \geq e^{63864}$. Using Corollary 3.3, we fill this gap.

Corollary 3.4. *The inequality* (3.5) *holds for every* $x \geq 302\,971$.

Proof. We only need to show that the desired inequality is valid for every x such that $e^{98} < x < e^{63864}$. But this is a consequence of Corollary 3.3. $\square$

4. Proof of Theorem 1.3

So far, we established explicit estimates for $S(x)$ in the direction of (1.4). In the following proof of Theorem 1.3, where we establish for the first time explicit bounds for the difference $S(x) - \mathrm{li}(x^2)$, we use an effective estimate for $\mathrm{li}(x^2)$.

Proof of Theorem 1.3. In the proof of Theorem 1.1 it is shown that

$$S(x) < \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{4.8x^2}{8\log^4 x} + \frac{4.5x^2}{4\log^5 x} + \frac{28.5x^2}{8\log^6 x} + \frac{121.5x^2}{8\log^7 x} + \frac{25826.5x^2}{16\log^8 x}$$

for every $x \geq 10\,166\,443\,802$. By applying the corresponding lower bound for $\mathrm{li}(x^2)$ given in [3, Lemma 15], we establish the correctness of the inequality

$$S(x) - \mathrm{li}(x^2) < \frac{0.225x^2}{\log^4 x} + \frac{0.375x^2}{\log^5 x} + \frac{1.6875x^2}{\log^6 x} + \frac{9.5625x^2}{\log^7 x} + \frac{1594.46875x^2}{\log^8 x}$$

for every $x \geq 10\,166\,443\,802$. This completes the proof of the right-hand side inequality for every $x \geq 15\,884\,423\,625$. Similar to the proof of Theorem 1.2, we check with a computer that this inequality also holds for every x such that $110\,117\,797 \leq x \leq 15\,884\,423\,625$. Analogously, we use (3.2), (3.3) and a computer to verify that the desired left-hand side inequality is valid for every $x \geq 906\,484\,877$. $\square$

Remark 4.1. Under the assumption that the Riemann hypothesis is true, Deléglise and Nicolas [8, Lemma 2.5] improved (1.3) by showing that for every $x \geq 41$,

$$|S(x) - \mathrm{li}(x^2)| \leq \frac{5}{24\pi}\, x^{3/2} \log x.$$

5. Proof of Theorem 1.4

In 1996, Massias and Robin [16, p. 217] found the currently most accurate asymptotic expansion for the sum of the first n primes, namely

$$\sum_{k \leq n} p_k = \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n}\right) + O\left(\frac{n^2(\log\log n)^2}{\log^2 n}\right). \tag{5.1}$$

In Theorem 1.4, we give an asymptotic expansion for the sum of the first n primes, more precise than (5.1). The maintool for the given proof is a result of Salvy [21, Theorem 2].

Proof of Theorem 1.4. Let N be a positive integer. We define

$$D_N(t) = \sum_{s=0}^{N} s!t^s.$$

First, we note that repeated integration by parts in (1.1) gives

$$\mathrm{li}(x) = \frac{x}{\log x}\left(D_N\left(\frac{1}{\log x}\right) + O\left(\frac{1}{\log^{N+1} x}\right)\right). \tag{5.2}$$

For $x > 1$, the logarithmic integral $\mathrm{li}(x)$ is increasing with $\mathrm{li}((1,\infty)) = \mathbb{R}$. Thus, we can define the inverse function $\mathrm{li}^{-1} : \mathbb{R} \to (1,\infty)$ by

$$\mathrm{li}(\mathrm{li}^{-1}(x)) = x. \tag{5.3}$$

Using (5.2), we get the asymptotic formula

$$\mathrm{li}((\mathrm{li}^{-1}(x))^2) = \frac{e^{2y}}{2y}\left(D_N\left(\frac{1}{2y}\right) + O\left(\frac{1}{y^{N+1}}\right)\right) \tag{5.4}$$

where $y = \log \mathrm{li}^{-1}(x)$. Next, we combine (5.2) and (5.3) to obtain $x = e^y y^{-1} D(1/y)$, where $D(t) = D_N(t) + O(t^{N+1})$. Now we apply Theorem 2 of [21] with $\alpha = 1, \beta = 2$, and $\gamma = -1$ to see that

$$\frac{e^{2y}}{2y} D_N\left(\frac{1}{2y}\right) = \frac{x^2 \log x}{2} \sum_{i=0}^{N} \frac{Q_i(\log\log x)}{\log^i x} + O\left(\frac{x^2(\log\log x)^N}{\log^N x}\right),$$

where the polynomials $Q_i \in \mathbb{R}[x]$ are defined by

$$Q_0 = 1,\ Q'_{i+1} = Q'_i - (i-1)Q_i \tag{5.5}$$

so that $Q_i = A_i$ where A_i is defined by (1.6). Together with (1.5), (5.4), and the fact that $\mathrm{li}^{-1}(x) \sim x \log x$ as $x \to \infty$, we conclude that

$$\sum_{k\le n} p_k = \frac{n^2 \log n}{2} \sum_{i=0}^{N} \frac{Q_i(\log\log n)}{\log^i n} + O\left(\frac{n^2(\log\log n)^N}{\log^N n}\right).$$

By (5.5) and (5.1), we have $Q_0(x) = 1$ and $Q_1(x) = x - 3/2$, respectively. Moreover, Theorem 2 of [21] demonstrates how to compute the value of the constant coefficient of the polynomials Q_i for every integer i satisfying $2 \le i \le N$, which is not given by (5.5). In the appendices of [21], one can find a Maple code for the computation of the polynomials $Q_2, \ldots, Q_N$ and it suffices to write

```
(1/2)*theorem2_part2(1,2,-1,D_N(n),D_N(n/2),n,N);.
```

Finally, we set $T_i = (-1)^{i+1} i Q_{i+1}$ for every integer i with $1 \leq i \leq N-1$. Then, (5.5) implies that the polynomials T_i are monic with $\deg(T_i) = i$, which completes the proof. $\square$

Remark 5.1. The first part of Theorem 1.4 was already proved by Sinha [22, Theorem 2.3].

6. Proof of Theorem 1.5

Recall that $B_n = np_n/2 - \sum_{k \leq n} p_k$. In this section, we use another result of Salvy [21, Corollary 4] (or Cipolla [7]) to give a proof of Theorem 1.5 where we establish an asymptotic expansion for B_n.

Proof of Theorem 1.5. Let N be a positive integer. By Salvy [21, Corollary 4] (or Cipolla [7]) there exist uniquely determined monic polynomials $R_1, \ldots, R_{N-1}$ with real coefficients and $\deg(R_i) = i$, so that

$$p_n = n \left(\log n + \log\log n - 1 + \sum_{i=1}^{N-1} \frac{(-1)^{i+1} R_i(\log\log n)}{i \log^i n} \right) + O\left(\frac{n(\log\log n)^N}{\log^N n} \right). \tag{6.1}$$

Furthermore, in Appendix B.2 of [21], one can find a Maple code for the computation of the polynomials $R_1, \ldots, R_{N-1}$. We set $V_i = R_i - T_i$ for every integer i with $1 \leq i \leq N-1$, where the polynomials T_i are given as in Theorem 1.4. Now it suffices to combine (6.1) and the asymptotic expansion given in Theorem 1.4. $\square$

7. Proof of Theorem 1.6

In order to give a proof of Theorem 1.6, we first note the following proposition. Here, let

$$\gamma(n) = \frac{2.9 \log^2 n}{4 \log^2 p_n} + \frac{\log^2 n}{4 \log p_n} + \frac{16.7 \log^2 n}{4 \log^3 p_n} - \frac{\log n}{4} + \frac{\log\log n}{4}.$$

Proposition 7.1. *For every integer* $n \geq 6\,315\,433$*, we have*

$$B_n > \frac{n^2}{4} + \frac{n^2}{4 \log n} - \frac{n^2 \log\log n}{4 \log^2 n} + \frac{\gamma(n) n^2}{\log^2 n}.$$

Proof. First, we consider the case where $n \geq 440\,200\,309$. By [5, Theorem 1], we have

$$np_n - \sum_{k \leq n} p_k \geq \frac{p_n^2}{2 \log p_n} + \frac{3p_n^2}{4 \log^2 p_n} + \frac{7p_n^2}{4 \log^3 p_n} + L_1(n), \tag{7.1}$$

where $L_1(n) = (44.4p_n^2 \log^2 p_n + 184.2p_n^2 \log p_n + 937.5p_n^2)/(8 \log^6 p_n)$. By (7.1) and the definition of B_n it suffices to prove that

$$\begin{aligned}(7.2)\qquad & \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + L_1(n) \\ & \qquad > \frac{np_n}{2} + \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2 \log\log n}{4\log^2 n} + \frac{\gamma(n)n^2}{\log^2 n}.\end{aligned}$$

For convenience, in the remaining part of the proof we write $p = p_n, y = \log n$, and $z = \log p$. It is easy to see that $937.5p^2 > 715.32npz + 117.88n^2z^2$. We combine this inequality with the definition of $\gamma(n)$ to get

$$\begin{aligned}& 2n^2z^5y^2 + 5.8n^2z^4y^2 + 55.5n^2z^2y^2(z-1.1) + 937.5p^2y^2 - 56.83n^2z^2y^2 \\ & \qquad > 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6 + 22.1n^2z^3y^2 + 715.32npzy^2.\end{aligned}$$

By Dusart [10, Théorème 1.10], we have $p > n(z - 1.1)$. Hence,

$$\begin{aligned}& 2n^2z^5y^2 + 5.8n^2z^4y^2 + 184.2npzy^2(z-1.1) + 937.5p^2y^2 - 56.83n^2z^2y^2 \\ & \qquad > 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6 + 22.1n^2z^3y^2 \\ & \qquad\qquad + 128.7npz^2y^2 + 512.7npzy^2.\end{aligned}$$

Again, we use the inequality $p > n(z - 1.1)$ to obtain

$$\begin{aligned}(7.3)\quad & 2n^2z^5y^2 + 5.8n^2z^4y^2 + 184.2p^2zy^2 + 937.5p^2y^2 - 56.83n^2z^2y^2 \\ & \qquad > 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6 + 22.1n^2z^3y^2 \\ & \qquad\qquad + 128.7npz^2y^2 + 512.7npzy^2.\end{aligned}$$

Similar, we apply the inequality $p > n(z - 1 - 1.15/z)$ found in [4, Corollary 3.3] to (7.3) and see that

$$\begin{aligned}& 2n^2z^5y^2 + 8L_1(n)z^6y^2 \\ & \qquad > 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6 + 6n^2z^4y^2 + 10.3n^2z^3y^2 \\ & \qquad\qquad + 43.26n^2z^2y^2 + 32.6npz^3y^2 + 84.3npz^2y^2 + 461.64npzy^2.\end{aligned}$$

Analogously, we use the inequality $p > n(z - 1 - 1/z - 3.69/z^2)$ which is valid by [4, Corollary 3.3] to get

$$\begin{aligned}(7.4)\quad & 14p^2z^3y^2 + 8L_1(n)z^6y^2 \\ & \qquad > 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6 + 2n^2z^5y^2 + 2n^2z^4y^2 \\ & \qquad\qquad + 6.3n^2z^3y^2 + 28.5n^2z^2y^2 + 10npz^4y^2 + 18.6npz^3y^2 \\ & \qquad\qquad\qquad + 70.3npz^2y^2 + 409.98npzy^2.\end{aligned}$$

Next, we apply the inequality $p > n(z - 1 - 1/z - 3.15/z^2 - 14.25/z^3)$, see [4, Corollary 3.3], to (7.4) in a similar way to obtain

$$\begin{aligned} &6p^2z^4y^2 + 14p^2z^3y^2 + 8L_1(n)z^6y^2 \\ &\qquad > 2n^2z^6y^2 + 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6 + 4npz^5y^2 \\ &\qquad\qquad + 4npz^4y^2 + 12.6npz^3y^2 + 51.4npz^2y^2 + 324.48npzy^2. \end{aligned}$$

Finally, by applying the inequality $p > n(z-1-1/z-3.15/z^2-12.85/z^3-81.12/z^4)$, which is fulfilled by [4, Corollary 3.3], we get

$$\begin{aligned} &4p^2z^5y^2 + 6p^2z^4y^2 + 14p^2z^3y^2 + 8L_1(n)z^6y^2 \\ &\qquad > 4npz^6y^2 + 2n^2z^6y^2 + 2n^2z^6y - 2n^2z^6\log y + 8\gamma(n)n^2z^6. \end{aligned}$$

We divide the last inequality by $8z^6y^2$ to obtain the inequality (7.2), so the claim follows for every integer $n \geq 440\,200\,309$. We check the remaining cases with a computer. □

Sun [23] proved that the inequality $B_n > n^2/4$ is valid for every integer $n \geq 417$. By proving Theorem 1.6, we improve Sun's lower bound for B_n.

Proof of Theorem 1.6. For convenience, we write again $y = \log n$ and $z = \log p_n$. First, we consider the case where $n \geq 6\,315\,433$. By Proposition 7.1 it suffices to show that $\gamma(n) \geq 2.9/4$. In [2, p. 42], it is shown that for every $m \geq 255$,

$$\log m \geq 0.75\log p_m. \tag{7.5}$$

Furthermore, we have $x^2 - 6.8x + 16.7\cdot 0.75^2 > 0$ for every $x \geq 4.88$. Together with (7.5), we get

$$16.7y^2 + (\log^2 y - 6.8\log y)z^2 + 2.9z\log^2 y - 2.9z(\log y - 1) \geq 0. \tag{7.6}$$

From Dusart [12, Proposition 5.15] and the inequality $\log(1+t) \leq t$, which holds for every $t > -1$, follows that

$$z \leq y + \log y + \frac{\log y - 1}{y} + \frac{\log y - 2}{y^2}. \tag{7.7}$$

Using the result of Rosser [18, Theorem 1] that $p_m > m\log m$ for every positive integer m, we obtain

$$-z + \log y \leq -y. \tag{7.8}$$

Hence, from (7.6), we get

$$16.7y^2 + z^2(\log y - 1 - 2.9)\log y - 2.9zy\log y - 2.9z(\log y - 1) \geq 0. \tag{7.9}$$

Let $f(x) = 3.9(\log\log x - 2)/\log x$. Then it is easy to see that f has a global maximum at $x_0 = 3$. Hence $f(x) \leq f(3) \leq 0.2$ for every $x > 1$. Similary, we get $2.9(\log\log x - 1)/\log x \leq 0.4$ and $2.9(\log\log x - 2)/\log^2 x \leq 0.01$ for $x > 1$. Therefore,

$$\frac{3.9z^2(\log y - 2)}{y} + \frac{2.9z^2(\log y - 1)}{y} + \frac{2.9z^2(\log y - 2)}{y^2} < z^2.$$

We combine this with (7.9) to obtain

$$\begin{aligned} z^2 &+ 16.7y^2 + 2.9zy^2 + z^2(\log y - 1 - 2.9)\log y \\ &\geq 2.9zy\left(y + \log y + \frac{\log y - 1}{y} + \frac{\log y - 2}{y^2}\right) + \frac{z^2(\log y - 2)}{y} \\ &\qquad + \frac{2.9z^2}{y}\left(\log y - 1 + \frac{\log y - 2}{y}\right). \end{aligned}$$

Now we use (7.7) to obtain

$$\begin{aligned} z^2 &+ 16.7y^2 + 2.9zy^2 + z^2(\log y - 1)\log y \\ &\geq 2.9z^2\left(y + \log y + \frac{\log y - 1}{y} + \frac{\log y - 2}{y^2}\right) + \frac{z^2(\log y - 2)}{y}. \end{aligned}$$

Again, by using (7.7), we get

$$z^2 + 16.7y^2 + 2.9zy^2 + z^2(\log y - 1)\log y \geq 2.9z^3 + \frac{z^2(\log y - 2)}{y}.$$

Finally we apply (7.8) and (7.7) to the last inequality and get $4z^3\gamma(n) \geq 2.9z^3$. Hence, the claim follows from Proposition 7.1 for every $n \geq 6\,315\,433$. A computer check for smaller values of n completes the proof. □

8. Proof of Theorem 1.7

We set

$$\kappa(n) = \frac{\log^2 n}{4\log p_n} + \frac{4.1\log^2 n}{4\log^2 p_n} - \frac{\log n}{4} + \frac{\log\log n}{4} + \frac{r(\log p_n)\log^2 n}{8\log^6 p_n},$$

where $r(x)$ is defined by

$$r(x) = 34.6x^3 + 207.1x^2 + 1431.56x + 28972.335, \tag{8.1}$$

to obtain the following proposition.

Proposition 8.1. *For every integer $n \geq 256\,265$, we have*

$$B_n < \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2\log\log n}{4\log^2 n} + \frac{\kappa(n)n^2}{\log^2 n}.$$

Proof. First, let $n \geq 841\,424\,976$, i.e. $p_n \geq 19\,033\,744\,403$. By [5, Theorem 2] and the definition of B_n it suffices to show that

$$\begin{aligned}(8.2)\quad \frac{np_n}{2} + \frac{n^2}{4} + \frac{n^2}{4\log n} - \frac{n^2\log\log n}{4\log^2 n} + \frac{\kappa(n)n^2}{\log^2 n}& \\ > \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + U(n)&,\end{aligned}$$

where

$$(8.3)\quad U(n) = \frac{45.6p_n^2}{8\log^4 p_n} + \frac{93.9p_n^2}{4\log^5 p_n} + \frac{952.5p_n^2}{8\log^6 p_n} + \frac{5755.5p_n^2}{8\log^7 p_n} + \frac{116371p_n^2}{16\log^8 p_n}.$$

For convenience, we denote again $p = p_n, y = \log n$ and $z = \log p$. From the definiton of $\kappa(n)$ and $r(x)$, it follows that

$$\begin{aligned}&2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 3913.24n^2z^2y^2 \\ &\qquad = 2n^2z^7y^2 + 8.2n^2z^6y^2 + 34.6n^2z^5y^2 + 207.1n^2z^4y^2 \\ &\qquad\qquad + 1431.56n^2z^3y^2 + 32885.575n^2z^2y^2.\end{aligned}$$

By Rosser and Schoenfeld [19, Corollary 1], we have $p < nz$. Hence, we obtain the inequality

$$\begin{aligned}&2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 3913.24n^2z^2y^2 + 25299.925npzy^2 \\ &\qquad > 2n^2z^7y^2 + 8.2n^2z^6y^2 + 34.6n^2z^5y^2 + 207.1n^2z^4y^2 \\ &\qquad\qquad + 1431.56n^2z^3y^2 + 58185.5npzy^2.\end{aligned}$$

Again, we use the inequality $p < nz$ to get

$$\begin{aligned}&2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 3913.24n^2z^2y^2 + 25299.925npzy^2 \\ &\qquad > 2n^2z^7y^2 + 8.2n^2z^6y^2 + 34.6n^2z^5y^2 + 207.1n^2z^4y^2 \\ &\qquad\qquad + 1431.56n^2z^3y^2 + 58185.5p^2y^2.\end{aligned}$$

Next, we apply the inequality $p < n(z-1)$, which was found by Dusart [10], in a similar way to get

$$\begin{aligned}(8.4)\quad &2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 610.47n^2z^3y^2 \\ &\qquad + 1871.21n^2z^2y^2 + 3713.47npz^2y^2 + 19544.425npzy^2 \\ &\quad > 2n^2z^7y^2 + 8.2n^2z^6y^2 + 34.6n^2z^5y^2 + 207.1n^2z^4y^2 \\ &\qquad + 5755.5p^2zy^2 + 58185.5p^2y^2.\end{aligned}$$

The double usage of the inequality $p < n(z-1-1/z)$, see [4, Corollary 3.9], to (8.4) gives

$$\begin{aligned} 2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 110.4n^2z^4y^2 + 292.97n^2z^3y^2 \\ + 1553.71n^2z^2y^2 + 635npz^3y^2 + 2760.97npz^2y^2 \\ + 18591.925npzy^2 \\ > 2n^2z^7y^2 + 8.2n^2z^6y^2 + 34.6n^2z^5y^2 + 952.5p^2z^2y^2 \\ + 5755.5p^2zy^2 + 58185.5p^2y^2. \end{aligned}$$

Analogously, we apply the inequality $p < n(z-1-1/z-2.85/z^2)$, which was found in [4, Corollary 3.9], in a similar way to obtain

$$\begin{aligned} 2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 23.9n^2z^5y^2 + 51.9n^2z^4y^2 \\ + 234.47n^2z^3y^2 + 1386.985n^2z^2y^2 + 129.3npz^4y^2 \\ + 447.2npz^3y^2 + 2573.17npz^2y^2 + 18056.695npzy^2 \\ > 2n^2z^7y^2 + 8.2n^2z^6y^2 + 187.8p^2z^3y^2 + 952.5p^2z^2y^2 \\ + 5755.5p^2zy^2 + 58185.5p^2y^2. \end{aligned}$$

Next, we use that $p < n(z-1-1/z-2.85/z^2-13.15/z^3)$, see [4, Corollary 3.9], to get

$$\begin{aligned} 2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 6n^2z^6y^2 + 9.7n^2z^5y^2 + 37.7n^2z^4y^2 \\ + 194n^2z^3y^2 + 1200.255n^2z^2y^2 + 31.4npz^5y^2 + 83.7npz^4y^2 \\ + 401.6npz^3y^2 + 2443.21npz^2y^2 + 17457.055npzy^2 \\ > 2n^2z^7y^2 + 8U(n)z^8y^2, \end{aligned}$$

where $U(n)$ is defined by (8.3). Similar, we apply the inequality $p < n(z-1-1/z-2.85/z^2-13.15/z^3-70.7/z^4)$, which is valid by [4, Corollary 3.9], to the last inequality and get

$$\begin{aligned} 2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 2n^2z^7y^2 + 2n^2z^6y^2 + 5.7n^2z^5y^2 \\ + 26.3n^2z^4y^2 + 141.4n^2z^3y^2 + 917.455n^2z^2y^2 + 10npz^6y^2 \\ + 17.4npz^5y^2 + 69.7npz^4y^2 + 361.7npz^3y^2 + 2259.11npz^2y^2 \\ + 16467.255npzy^2 \\ > 14p^2z^5y^2 + 8U(n)z^8y^2. \end{aligned}$$

Now, we use $p < n(z-1-1/z-2.85/z^2-13.15/z^3-70.7/z^4-458.7275/z^5)$, see [4, Corollary 3.9], in a analogical way to get

$$
\begin{aligned}
(8.5)\quad 2n^2z^8y^2 + 2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 + 4npz^7y^2 + 4npz^6y^2 \\
+ 11.4npz^5y^2 + 52.6npz^4y^2 + 282.8npz^3y^2 \\
+ 1834.91npz^2y^2 + 13714.89npzy^2 \\
> 6p^2z^6y^2 + 14p^2z^5y^2 + 8U(n)z^8y^2.
\end{aligned}
$$

Finally, we similary apply [4, Theorem 3.8] to the inequality (8.5) to get

$$
\begin{aligned}
4npz^8y^2 + 2n^2z^8y^2 + 2n^2z^8y - 2n^2z^8\log y + 8\kappa(n)n^2z^8 \\
> 4p^2z^7y^2 + 6p^2z^6y^2 + 14p^2z^5y^2 + 8U(n)z^8y^2.
\end{aligned}
$$

We divide both sides of this inequality by $8z^8y^2$ to obtain the inequality (8.2) for every integer $n \geq 841\,424\,976$. We verify the remaining cases by using a computer. □

Now we use Proposition 8.1 to give a proof of Theorem 1.7.

Proof of Theorem 1.7. The proof consists of four steps. In the first step, we set $a_1 = 0.08$ and notice that

$$f(x) = 4a_1(x + \log x) + (x + 4a_1 - \log x)\log\left(1 + \frac{\log x - 1}{x}\right) - \log^2 x$$

is positive for every $x \geq e^{19.63}$. In the following three steps, we write again $y = \log n$ and $z = \log p_n$, and consider the case $y \geq 19.63$. Then, $f(y) \geq 0$, i.e.

$$(8.6)\qquad \left(y + \log y + \log\left(1 + \frac{\log y - 1}{y}\right)\right)(4a_1 + y - \log y) \geq y^2.$$

From Dusart [11] follows that

$$(8.7)\qquad z \geq y + \log y + \log\left(1 + \frac{\log y - 1}{y}\right).$$

We apply this inequality to (8.6) to get

$$(8.8)\qquad 8a_1z^8 \geq 2z^7y^2 - 2z^8y + 2z^8\log y.$$

In the third step, we set $a_2 = 1.025$ and $t(x) = 16a_2x^3\log x + 8a_2x^2\log^2 x - r(x)$, where $r(x)$ is defined by (8.1). Then $t(x) \geq 0$ for every $x \geq 19.71$ and it follows that

$$16a_2z^5y^2\log z + 8a_2z^4y^2\log^2 z - r(z)z^2y^2 + (8a_2 - 8.2)z^6y^2 = z^2y^2t(z) \geq 0.$$

The function $s \mapsto \log s/s$ is decreasing for every $s \geq e$. So, $\log(y)/y \geq \log(z)/z$ and we get

$$8a_2z^6(y + \log y)^2 - r(z)z^2y^2 - 8.2z^6y^2 \geq 0.$$

By (8.7), we obtain $z \geq y + \log y$. Hence $8a_2z^8 \geq r(z)z^2y^2 + 8.2z^6y^2$. Now, in the final step, we combine the last inequality with (8.8) to see that $\kappa(n) \leq 4.42/4$ for every integer $n \geq e^{19.63}$. We apply this to Proposition 8.1, which completes the proof for every integer $n \geq e^{19.63}$. We conclude by a direct computation. □

Remark 8.2. Theorem 1.7 improves the only known upper bound $B_n < 9n^2/4$, which holds for every integer $n \geq 2$, found by Hassani [14, Corollary 1.5].

9. Proof of Theorems 1.8 and 1.9

In 1998, Dusart [10] proved that the inequality

$$\sum_{k \leq n} p_k \leq \frac{n^2}{2} \left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 2.29}{\log n} \right)$$

holds for every integer $n \geq 10\,134$. In this section, we use the identity

$$\sum_{k \leq n} p_k = \frac{np_n}{2} - B_n, \tag{9.1}$$

the inequalities stated in Theorems 1.6 and 1.7, and some explicit estimates for the nth prime number given in [6] to find proofs of Theorems 1.8 and 1.9. We start with the proof of Theorem 1.8.

Proof of Theorem 1.8. We combine (9.1), [6, Theorem 1], and Theorem 1.6, to get that the required inequality holds for every integer $n \geq 46\,254\,381$. The remaining cases are verified with a computer. □

Based on Theorem 1.8, we obtain the following upper bound for the sum of the first n prime numbers, which corresponds to the first four terms of the asymptotic expansion found in Theorem 1.4.

Corollary 9.1. *For every integer $n \geq 115\,149$, we have*

$$\sum_{k \leq n} p_k < \frac{n^2}{2} \left(\log n + \log\log n - \frac{3}{2} + \frac{\log\log n - 5/2}{\log n} \right). \tag{9.2}$$

Proof. Theorem 1.8 implies the validity of (9.2) for every integer $n \geq 1\,897\,700$. It remains to check the required inequality for smaller values of n with a computer. □

The current best lower bound for the sum of the first n primes in the direction of (1.8) is also due to Dusart [10, Lemme 1.7]. He proved that

$$\sum_{k \leq n} p_k \geq \frac{n^2}{2} \left(\log n + \log\log n - \frac{3}{2} \right)$$

for every integer $n \geq 305\,494$. Using [6, Theorem 4] and Theorem 1.7, we finally give the following proof of Theorem 1.9.

Proof of Theorem 1.9. Applying [6, Theorem 4] and Theorem 1.7 to (9.1), we get that the desired inequality holds for every integer $n \geq 256\,376$. For the remaining cases, we use a computer. $\square$

Acknowledgement

I would like to thank Jean Pierre Massias and Guy Robin whose paper inspired me to deal with this subject. I would also like to thank Raul for being a never-ending inspiration. Further, I would like to thank the anonymous referees for useful comments to improve the quality of this paper.

References

[1] T. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer, 1976.

[2] C. Axler, "Über die Primzahl-Zählfunktion, die n-te Primzahl und verallgemeinerte Ramanujan-Primzahlen", PhD Thesis, Heinrich-Heine-Universität Düsseldorf (Germany), 2013, https://docserv.uni-duesseldorf.de/servlets/DocumentServlet?id=26247.

[3] ———, "On a sequence involving prime numbers", *J. Integer Seq.* **18** (2015), no. 7, article ID 15.7.6 (13 pages).

[4] ———, "New estimates for some functions defined over primes", *Integers* **18** (2018), article ID A52 (21 pages).

[5] ———, "Improving the Estimates for a Sequence Involving Prime Numbers", *Notes Number Theory Discrete Math.* **25** (2019), no. 1, p. 8-13.

[6] ———, "New estimates for the nth prime number", *J. Integer Seq.* **22** (2019), no. 4, article ID 19.4.2 (30 pages).

[7] M. Cipolla, "La determinazione assintotica dell' $n^i mo$ numero primo", *Napoli Rend.* **8** (1902), p. 132-166.

[8] M. Deléglise & J.-L. Nicolas, "Maximal product of primes whose sum is bounded", *Proc. Steklov Inst. Math.* **282** (2013), no. 1, p. 73-102.

[9] ———, "An arithmetic equivalence of the Riemann hypothesis", *J. Aust. Math. Soc.* **106** (2019), no. 2, p. 235-273.

[10] P. Dusart, "Autour de la fonction qui compte le nombre de nombres premiers", PhD Thesis, Université de Limoges (France), 1998, http://www.theses.fr/1998LIMO0007.

[11] ———, "The kth prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$", *Math. Comput.* **68** (1999), no. 225, p. 411-415.

[12] ———, "Explicit estimates of some functions over primes", *Ramanujan J.* **45** (2018), no. 1, p. 227-251.

[13] J. Hadamard, "Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques", *Bull. Soc. Math. Fr.* **24** (1896), p. 199-220.

[14] M. Hassani, "On the ratio of the arithmetic and geometric means of the prime numbers and the number e", *Int. J. Number Theory* **9** (2013), no. 6, p. 1593-1603.

[15] J.-P. Massias, J.-L. Nicolas & G. Robin, "Effective bounds for the maximal order of an element in the symmetric group", *Math. Comput.* **53** (1989), no. 188, p. 665-678.

[16] J.-P. Massias & G. Robin, "Bornes effectives pour certaines fonctions concernant les nombres premiers", *J. Théor. Nombres Bordeaux* **8** (1996), no. 1, p. 213-238.

[17] G. Robin, "Permanence de relations de récurrence dans certains développements asymptotiques", *Publ. Inst. Math., Nouv. Sér.* **43** (1988), p. 17-25.

[18] J. B. Rosser, "The n-th prime is greater than $n \log n$", *Proc. Lond. Math. Soc.* **45** (1939), p. 21-44.

[19] J. B. Rosser & L. Schoenfeld, "Approximate formulas for some functions of prime numbers", *Ill. J. Math.* **6** (1962), p. 64-94.

[20] ———, "Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$", *Math. Comput.* **29** (1975), p. 243-269.

[21] B. Salvy, "Fast computation of some asymptotic functional inverses", *J. Symb. Comput.* **17** (1994), no. 3, p. 227-236.

[22] N. K. Sinha, "On the asymptotic expansion of the sum of the first n primes", `https://arxiv.org/abs/1011.1667`, 2015.

[23] Z.-W. Sun, "Some new inequalities for primes", `https://arxiv.org/abs/1209.3729`, 2012.

[24] M. Szalay, "On the maximal order in S_n and S_n^*", *Acta Arith.* **37** (1980), p. 321-331.

[25] C.-J. de la Vallée Poussin, "Recherches analytiques la théorie des nombres premiers", *Ann. Soc. scient. Bruxelles* **20** (1896), p. 183-256.

[26] ———, "Sur la fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieurs à une limite donnée", *Mem. Couronnés de l'Acad. Roy. Sci. Bruxelles* **59** (1899), p. 1-74.

Christian Axler
Institute of Mathematics
Heinrich-Heine-University Düsseldorf
Building 25.22, Room 02.42
Universitätsstraße 1
40225 Düsseldorf, Germany
E-mail: `christian.axler@hhu.de`