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# One-class genera of exceptional groups over number fields

par MARKUS KIRSCHMER

RÉSUMÉ. Nous montrons que les groupes algébriques exceptionnels sur un corps de nombres n'admettent pas de genres de groupes parahoriques à une seule classe, sauf dans le cas de  $G_2$ . Pour le groupe  $G_2$ , nous énumérons tous les genres à une seule classe pour la représentation usuelle en dimension 7.

ABSTRACT. We show that exceptional algebraic groups over number fields do not admit one-class genera of parahoric groups, except in the case  $G_2$ . For the group  $G_2$ , we enumerate all such one-class genera for the usual seven-dimensional representation.

## 1. Introduction

The enumeration of all one-class genera of definite quadratic forms has a long history. Over the rationals, Watson classified these genera in a long series of papers with some exceptions in dimensions 4 and 5 using some transformations which do not increase class numbers, see [17]. His classification has recently been completed by D. Lorch and the author [8] using Watson's transformations and the explicit mass formula of Minkowski, Siegel and Smith. Recently, the author worked out the one-class genera of definite quadratic and hermitian forms over number fields, see [7].

The purpose of this note is to enumerate the one-class genera of parahoric subgroups of the exceptional algebraic groups. This yields a new proof of the result of Kantor, Liebler and Tits [6] for exceptional groups in characteristic 0. Instead of requiring chamber-transitivity on the associated affine building, our one-class hypothesis allows for significantly less transitivity. For groups of type  $G_2$ , we find several examples in addition to the one in [6]. For the remaining exceptional groups, as in [6], we prove that there are no examples even with our weaker hypothesis.

The paper is organized as follows. In Section 2, we recall some basic facts on parahoric subgroups of algebraic groups. In Section 3, we state Prasad's mass formula. In the last Section, we use his mass formula and obtain a

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list of all one-class genera of parahoric families in exceptional groups over number fields.

### 2. Preliminaries

Let  $k$  be a number field of degree  $n$  and let  $\mathfrak{o}_k$  be its ring of integers. The set of all finite (infinite) places of  $k$  will be denoted by  $V_f$  ( $V_\infty$ ). For any  $v \in V := V_f \cup V_\infty$ , let  $k_v$  be the completion of  $k$  at  $v$  and let  $\mathfrak{o}_{k_v}$  its ring of integers. Further, we will write  $\mathfrak{f}_v$  for the residue class field of  $k_v$  and we set  $q_v = \#\mathfrak{f}_v$ .

Let  $G$  be an absolutely quasi-simple, simply connected algebraic group defined over  $k$ . We always assume that  $\prod_{v \in V_\infty} G(k_v)$  is compact. Then  $k$  is totally real.

We are mostly interested in the exceptional groups, i.e.  $G$  will be a  $k$ -form of  $G_2, F_4, E_6, E_7, E_8$  or a triality form of  $D_4$ , (cf. [13, Chapter 17]).

We will also assume that  $G$  is a subgroup of  $GL_m$  for some  $m$ . Let  $L$  be an  $\mathfrak{o}_k$ -lattice in  $k^m$ , i.e. a finitely generated  $\mathfrak{o}_k$ -submodule of  $k^m$  of full rank. The schematic closure of  $G$  in the group scheme  $\underline{GL}(L)$  yields an integral group scheme  $\underline{G}$ , cf. [2, 3].

Let  $A = \{(\alpha_v)_{v \in V} \mid \alpha_v \notin \mathfrak{o}_{k_v} \text{ for only finitely many } v \in V_f\}$  be the adèle ring of  $k$ . Suppose  $\alpha \in G(A)$ . Then  $L \cdot \alpha$  denotes the  $\mathfrak{o}_k$ -lattice  $L'$  with  $L'_v = L_v \alpha_v$  for all  $v \in V_f$ . Similarly, we define  $\underline{G} \cdot \alpha$  to be the stabilizer of  $L \cdot \alpha$  in  $G(k)$ . Then  $(\underline{G} \cdot \alpha)(\mathfrak{o}_{k_v}) = \alpha_v^{-1} \underline{G}(\mathfrak{o}_{k_v}) \alpha_v$  for all  $v \in V_f$ .

**Definition 2.1.** Two integral forms  $\underline{G}$  and  $\underline{G}'$  of  $G$  are isomorphic if  $\underline{G} \cdot \alpha = \underline{G}'$  for some  $\alpha \in G(k)$ . Similarly, they are said to be in the same genus if  $\underline{G} \cdot \alpha = \underline{G}'$  for some  $\alpha \in G(A)$ .

Let  $C = \prod_{v \in V_\infty} G(k_v) \times \prod_{v \in V_f} \underline{G}(\mathfrak{o}_{k_v})$ . Then  $\alpha^{-1} \underline{G} \alpha$  is the stabilizer of  $\underline{G} \cdot \alpha$  in  $G(A)$ . Thus

$$C \alpha G(k) \mapsto \underline{G} \cdot \alpha$$

induces a bijection between the double cosets  $C \backslash G(A) / G(k)$  and the isomorphism classes in the genus of  $\underline{G}$ .

**Lemma 2.2** ([3, Proposition 3.3]). *Let  $\underline{G}$  be an integral group scheme as above. Then  $\underline{G}(\mathfrak{o}_{k_v})$  is a subgroup of finite index in a maximal compact subgroup of  $G(k_v)$  and  $\underline{G}(\mathfrak{o}_{k_v})$  is a hyperspecial maximal compact subgroup at all but finitely many places  $v \in V_f$ .*

The most important integral group schemes  $\underline{G}$  are those for which  $\underline{G}(\mathfrak{o}_{k_v})$  is a parahoric subgroup  $P_v$  of  $G(k_v)$  at each finite place  $v$ . The genus of such a scheme is uniquely determined by the family  $P = (P_v)_{v \in V_f}$ . By the previous remark,  $P_v$  is hyperspecial almost everywhere. Such a family  $P$  is called *coherent* in [11].

It is well known ([2, Theorem 5.1]) that the genus of integral forms corresponding to  $P$  decomposes into finitely many isomorphism classes represented by  $\underline{G}_1, \dots, \underline{G}_{\mathfrak{c}(P)}$  say.

Then the rational number  $\mathfrak{M}(P) = \sum_{i=1}^{\mathfrak{c}(P)} (\#\underline{G}_i(\mathfrak{o}_k))^{-1}$  is called the *mass* of  $P$ . We clearly have  $\mathfrak{c}(P) \geq \mathfrak{M}(P)$  and  $\mathfrak{c}(P) = 1$  implies  $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$ .

### 3. The mass formula

Let  $P$  be a coherent family of parahoric subgroups of  $G$  and let  $\mathcal{G}$  be the unique quasi-split inner  $k$ -form of  $G$ . If  $\mathcal{G}$  is of type  ${}^6D_4$  (cf. [13, Chapter 17.9]), let  $\ell/k$  be a cubic extension contained in a Galois extension of  $k$  of degree 6 over which  $\mathcal{G}$  splits. In all other cases let  $\ell$  be the minimal extension of  $k$  over which  $\mathcal{G}$  splits. The absolute values of the absolute discriminants of  $k$  and  $\ell$  will be denoted by  $D_k$  and  $D_\ell$  respectively. If  $\mathcal{G}$  splits over  $k$ , let  $s(\mathcal{G}) = 0$ . Otherwise let  $s(\mathcal{G})$  be the sum of the number of short roots and the number of short simple roots of the relative roots system of  $\mathcal{G}$  over  $k$ . In particular, if  $\mathcal{G}$  is a triality form of  $D_4$ , then  $s(\mathcal{G}) = 7$  and if  $\mathcal{G}$  is an outer form of  $E_6$  then  $s(\mathcal{G}) = 6$ . For more details, see Section 0.4 of [11].

We fix a family  $\mathcal{P} = (\mathcal{P}_v)_{v \in V_f}$  of maximal parahoric subgroups of  $\mathcal{G}$  such that  $\mathcal{P}_v$  is hyperspecial (special) if  $\mathcal{G}$  splits (does not split) over the maximal unramified extension of  $k_v$  and  $\prod_{v \in V_\infty} \mathcal{G}(k_v) \times \prod_{v \in V_f} \mathcal{P}_v$  is an open subgroup of  $G(A)$ . See [11, Section 1.2] for more details.

Let  $\overline{\mathcal{G}}_v$  and  $\overline{G}_v$  be the groups  $\mathcal{G} \otimes_{\mathfrak{o}_k} \mathfrak{f}_v$  and  $G_v \otimes_{\mathfrak{o}_k} \mathfrak{f}_v$ . By [15, Section 3.5], both these groups admit a Levi decomposition over  $\mathfrak{f}_v$ . Hence we may fix some maximal connected reductive  $\mathfrak{f}_v$ -subgroups  $\overline{\mathcal{M}}_v$  and  $\overline{M}_v$  such that  $\overline{\mathcal{G}}_v = \overline{\mathcal{M}}_v.R_u(\overline{\mathcal{G}}_v)$  and  $\overline{G}_v = \overline{M}_v.R_u(\overline{G}_v)$ . Here  $R_u$  denotes the unipotent radical.

In his seminal paper [11], Prasad gave the following explicit formula for  $\mathfrak{M}(P)$ .

**Theorem 3.1** ([11]).

$$\mathfrak{M}(P) = D_k^{\frac{1}{2} \dim G} \left( D_\ell / D_k^{[\ell:k]} \right)^{s(\mathcal{G})/2} \left( \prod_{i=1}^r \frac{m_i!}{(2\pi)^{m_i+1}} \right)^n \zeta(P)$$

where  $m_1, \dots, m_r$  are the exponents of the simple, simply-connected compact real-analytic Lie group of the same type as  $\mathcal{G}$  and  $\zeta(P) = \prod_{v \in V_f} \zeta(P_v)$  with

$$\zeta(P_v) := \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathcal{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)} > 1.$$

For computational purposes, it is usually more convenient to express  $\mathfrak{M}(P)$  in terms of  $\mathfrak{M}(\mathcal{P})$  which is a product of special values of certain

$L$ -series of  $k$ . For  $v \in V_f$  let

$$z(P_v) := \zeta(P_v)/\zeta(\mathcal{P}_v) = q_v^{(\dim \overline{\mathcal{M}}_v - \dim \overline{\mathcal{M}}_v)/2} \frac{\#\overline{\mathcal{M}}_v(\mathfrak{f}_v)}{\#\overline{\mathcal{M}}_v(\mathfrak{f}_v)}.$$

Then  $\mathfrak{M}(P) = \mathfrak{M}(\mathcal{P}) \cdot \prod_{v \in V_f} z(P_v)$ . Moreover, we have the following empirical fact.

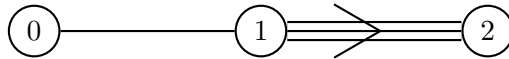
**Lemma 3.2** ([12, 2.5]). *The correction factors  $z(P_v)$  are integral.*

*Proof.* This follows from explicit computations using Bruhat–Tits theory. The groups of type  $A_n$  are discussed in [9, Lemma 2]. Without loss of generality,  $P_v$  is maximal parahoric. Since we are only interested in exceptional groups, we discuss the case  ${}^3D_4$ . The other cases are handled similarly. The comment after [12, 2.5] shows that the result holds whenever  $G(k_v)$  contains a hyperspecial parahoric subgroup. So only the case that  $v$  ramifies in  $\ell$  remains. Then  $G$  is of type  $G_2^1$  (using the notation of [15, Tables 4.2 and 4.3]). From [10, Table 1] and [11, (1.5)] we see that  $q_v^{-\dim \overline{\mathcal{M}}_v/2} \overline{\mathcal{M}}_v(\mathfrak{f}_v) = q_v^{-1}(q_v^2 - 1)(q_v^6 - 1)$ . The theory of Bruhat–Tits shows that every maximal parahoric subgroup of  $G(k_v)$  is of type  $G_2, A_2$  or  $A_1 \times A_1$ . Hence  $q_v^{-\dim \overline{\mathcal{M}}_v/2} \overline{\mathcal{M}}_v(\mathfrak{f}_v)$  is either  $q_v^{-1}(q_v^2 - 1)(q_v^6 - 1), q_v^{-1}(q_v^2 - 1)^2$  or  $q_v^{-1}(q_v^2 - 1)(q_v^3 - 1)$ . In particular,  $z(P_v)$  is integral.  $\square$

### 4. The exceptional groups

**4.1. The case  $G_2$ .** Let  $\mathbb{O}$  be the octonion algebra over  $k$  with totally definite norm form and denote by  $\mathbb{O}^0$  its trace zero subspace. The automorphism group  $\text{Aut}(\mathbb{O})$  of  $\mathbb{O}$ , i.e. the stabilizer of the octonion multiplication in the special orthogonal group of  $\mathbb{O}$  yields an algebraic group of type  $G_2$  and  $\mathbb{O}^0$  is an invariant subspace (cf. [14, Chapter 2]). Thus we obtain an algebraic group  $G < \text{GL}_7$  of type  $G_2$ . Further, the construction shows that  $G(k_v)$  is of type  $G_2$  for all finite places  $v$ .

The extended Dynkin diagram of  $G_2$  is as follows.



By [15, 3.5.2], the parahoric subgroups  $P_v$  of  $G(k_v)$  are in one-to-one correspondence with the non-empty subsets of  $\{0, 1, 2\}$ . For any non-empty subset  $T$  of  $\{0, 1, 2\}$  let  $P_v^T$  be the parahoric subgroup of  $G(k_v)$  whose Dynkin diagram is obtained from the extended Dynkin diagram of  $G_2$  by omitting the vertices in  $T$ . For example,  $P_v^{\{0\}}$  is hyperspecial and  $P_v^{\{2\}}$  is of type  $A_2$ .

**Theorem 4.1.** *Suppose  $P$  is a coherent family of parahoric subgroups of  $G$  such that  $\mathfrak{c}(P) = 1$ . Then  $k = \mathbb{Q}$  and  $P_p$  is hyperspecial for all primes*

$p \notin \{2, 3, 5\}$ . The possible combinations  $(T_2, T_3, T_5)$  such that  $P_p = P_p^{T_p}$  for  $p \in \{2, 3, 5\}$  are given in Table 4.1.

$T_2$	$T_3$	$T_5$	$\mathfrak{M}(P)^{-1}$	$\underline{G}(\mathbb{Z})$	$sgdb$
{0}	{0}	{0}	$2^6 \cdot 3^3 \cdot 7$	$G_2(2)$	–
{0}	{0}	{2}	$2^5 \cdot 3$	$(C_4 \times C_4).S_3$	64
{0}	{2}	{0}	$2^4 \cdot 3^3$	$3_+^{1+2}.QD_{16}$	520
{2}	{0}	{0}	$2^6 \cdot 3 \cdot 7$	$2^3.GL_3(2)$	814
{2}	{2}	{0}	$2^4 \cdot 3$	$GL_2(3)$	29
{1}	{0}	{0}	$2^6 \cdot 3^2$	$2_+^{1+4}.((C_3 \times C_3).2)$	8282 <sup>1</sup>
{1, 2}	{0}	{0}	$2^6 \cdot 3$	$2_+^{1+4}.S_3$	1494
{0, 2}	{0}	{0}	$2^6 \cdot 3$	$((C_4 \times C_4).2).S_3$	956
{0, 1}	{0}	{0}	$2^6 \cdot 3$	$2_+^{1+4}.S_3$	988
{0, 1, 2}	{0}	{0}	$2^6$	$Syl_2(G_2(2))$	134

TABLE 4.1. The one-class genera of  $G_2$ .

The last column of Table 4.1 gives the label of the group  $\underline{G}(\mathbb{Z})$  in the list of all groups of order  $\mathfrak{M}(P)^{-1} = \#\underline{G}(\mathbb{Z})$  as defined by the small group database [1].

*Proof.* Using the notation of Section 3, we have  $\ell = k, r = 2, (m_1, m_2) = (1, 5)$  and  $\dim G = r + 2(m_1 + m_2) = 14$ . Thus Theorem 3.1 shows

$$c(P) \geq D_k^7 \left( \frac{15}{32\pi^8} \right)^n .$$

Hence  $c(P) = 1$  implies

$$D_k^{1/n} \leq \left( \frac{32\pi^8}{15} \right)^{1/7} < 4.123 .$$

Voight’s tables [16] now show that  $k$  is one of  $\mathbb{Q}, \mathbb{Q}(\sqrt{d})$  with  $d \in \{2, 3, 5, 13\}$  or the maximal totally real subfield  $\mathbb{Q}(\theta_7)$  of the seventh cyclotomic field  $\mathbb{Q}(\zeta_7)$ .

The assumption  $c(P) = 1$  forces  $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$ . Hence  $\mathfrak{M}(\mathcal{P})^{-1} \in \mathbb{Z}$  by Lemma 3.2. The exact values of  $\mathfrak{M}(\mathcal{P}) = 2^{-2n}\zeta_k(-1)\zeta_k(-5)$  for the various possible base fields  $k$  is given in the following table.

$k$	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{13})$	$\mathbb{Q}(\theta_7)$
$\mathfrak{M}(\mathcal{P})$	$\frac{1}{2^6 \cdot 3^3 \cdot 7} = \frac{1}{\#G_2(2)}$	$\frac{361}{48384}$	$\frac{1681}{12096}$	$\frac{67}{302400}$	$\frac{33463}{157248}$	$\frac{7393}{84672}$

<sup>1</sup>The group is isomorphic to an index two subgroup of the automorphism group of the root lattice  $\mathbb{F}_4$ .

This shows that  $k = \mathbb{Q}$  as claimed. For any given prime  $p$ , the local correction factor  $z(P_p)$  is given by the following table.

root system of $P_p$	$\emptyset$	$A_1$	$A_2$	$A_1 \times A_1$	$G_2$
$z(P_p)$	$p^8 - p^6 - p^2 + 1$	$p^6 - 1$	$p^3 + 1$	$p^4 + p^2 + 1$	1

If  $p \geq 23$  then  $\#G_2(2) \cdot (p^3 + 1) > 1$  and therefore  $P_p$  is hyperspecial. For  $p < 23$  we can simply check all possible combinations of  $P_p$  which yield  $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$ . This yields precisely the claimed combinations.

Let  $B$  be an Iwahori subgroup of  $G$ . The set of all  $\mathbb{Z}_p B$ -invariant lattices in  $\mathbb{O}_p^0$  have been worked out in [3]. For each candidate  $P$ , Theorem 4.1 of [3] yields a lattice  $L$  in  $\mathbb{O}^0$  such that the stabilizer  $\underline{G}(\mathbb{Z})$  of  $L$  in  $G_2 < \text{GL}(\mathbb{O}^0)$  is of type  $P$ . One checks that  $\mathfrak{M}(P)^{-1} = \#\underline{G}(\mathbb{Z})$  in all cases.  $\square$

Let  $P$  be the parahoric family corresponding to the last entry of Table 4.1. Then  $P_2$  is an Iwahori subgroup of  $G$ , i.e. the stabilizer of a chamber in the affine building  $B$  of  $G_2(\mathbb{Q}_2)$ . This family  $P$  yields the chamber-transitive action of  $G_2(\mathbb{Z}[1/2])$  on  $B$  from [5] and [6, case (iii)].

**4.2. The case  $F_4$ .**

**Proposition 4.2.** *Suppose  $G$  is of type  $F_4$ . Then there exists no coherent family  $P$  of parahoric subgroups of  $G$  with class number one.*

*Proof.* We have  $r = 4$ ,  $(m_1, \dots, m_4) = (1, 5, 7, 11)$  and

$$\dim G = r + 2 \sum_i m_i = 52.$$

Thus Theorem 3.1 shows

$$c(P) \geq D_k^{26} \left( \frac{736745625}{8192\pi^{28}} \right)^n.$$

Hence  $c(P) = 1$  implies

$$D_k^{1/n} \leq \left( \frac{8192\pi^{28}}{736745625} \right)^{1/26} < 2.213 < \sqrt{5}.$$

Hence  $k = \mathbb{Q}$ . But for  $k = \mathbb{Q}$  we have

$$\mathfrak{M}(P) = \frac{736745625}{8192\pi^{28}} \cdot \prod_{i=1}^4 \zeta_{\mathbb{Q}}(m_i + 1) = \frac{1}{4} \prod_{i=1}^4 \zeta_{\mathbb{Q}}(-m_i) = \frac{691}{2^{15} 3^6 5^2 7^2 13}.$$

In particular,  $\mathfrak{M}(P)^{-1} \notin \mathbb{Z}$ .  $\square$

If  $k = \mathbb{Q}$ , then  $\mathcal{P}$  is the model in the sense of Gross and it actually has class number 2 (see [4, Proposition 5.3]).

**4.3. Triality forms of  $D_4$ .** Let  $G$  be of type  ${}^3D_4$  or  ${}^6D_4$ . The field  $\ell$  is a totally real cubic extension of  $k$ . The extension is normal (and thus cyclic) if and only if  $G$  is of type  ${}^3D_4$ .

**Lemma 4.3.** *Suppose  $G$  is a  $k$ -form of  $D_4$  and  $P$  a parahoric family of  $G$  with class number one. Then the base field  $k$  is either  $\mathbb{Q}$ ,  $\mathbb{Q}(\sqrt{d})$  with  $d \in \{2, 3, 5, 13, 17\}$  or the maximal totally real subfield  $\mathbb{Q}(\theta_e)$  of  $\mathbb{Q}(\zeta_e)$  for  $e \in \{7, 9\}$ .*

*Proof.* If  $G$  is any form of  $D_4$ , then  $r = 4$ ,  $(m_1, \dots, m_4) = (1, 3, 3, 5)$  and  $\dim G = r + 2 \sum_i m_i = 28$ . Thus Theorem 3.1 shows that

$$c(P) \geq \mathfrak{M}(P) \geq D_k^{14} \left( \frac{135}{2^{11}\pi^{16}} \right)^n .$$

Hence  $c(P) = 1$  implies

$$D_k^{1/n} \leq \left( \frac{2^{11}\pi^{16}}{135} \right)^{1/14} < 4.493 .$$

The result follows from Voight’s tables of totally real number fields [16].  $\square$

Given a finite place  $v \in V_f$  let  $\ell_v = \ell \otimes_k k_v$ . By [15, Section 4], the type of  $G$  at  $v$  is (using the notation of [15, Tables 4.2 and 4.3])

$$\left\{ \begin{array}{ll} {}^1D_4 & \text{if } v \text{ is completely split in } \ell, \\ {}^3D_4 & \text{if } \ell_v/k_v \text{ is an unramified cubic field extension,} \\ G_2^1 & \text{if } \ell_v/k_v \text{ is a ramified cubic field extension,} \\ {}^2D_4 & \text{if } \ell_v \cong k_v \oplus m_v \text{ for some unramified quadratic extension } m_v/k_v, \\ B-C_3 & \text{if } \ell_v \cong k_v \oplus m_v \text{ for some ramified quadratic extension } m_v/k_v. \end{array} \right.$$

Therefore  $\zeta(\mathcal{P}_v) = \left(1 - \frac{1}{q_v^2}\right) \left(1 - \frac{1}{q_v^6}\right) \cdot \lambda_v$  where  $\lambda_v$  is given by

$$\left\{ \begin{array}{ll} \left(1 - \frac{1}{q_v^4}\right)^2 & \text{if } v \text{ is completely split in } \ell, \\ 1 + \frac{1}{q_v^4} + \frac{1}{q_v^8} & \text{if } \ell_v/k_v \text{ is an unramified cubic field extension,} \\ 1 & \text{if } \ell_v/k_v \text{ is a ramified cubic field extension,} \\ \left(1 + \frac{1}{q_v^4}\right)\left(1 - \frac{1}{q_v^4}\right) & \text{if } \ell_v = k_v \oplus m_v \text{ for some unramified extension } m_v/k_v, \\ 1 - \frac{1}{q_v^4} & \text{if } \ell_v = k_v \oplus m_v \text{ for some ramified extension } m_v/k_v. \end{array} \right.$$

Using the functional equation for L-series, we obtain

$$(4.1) \quad \mathfrak{M}(\mathcal{P}) = 2^{-4n} \cdot |\zeta_k(-1)\zeta_{\ell/k}(-3)\zeta_k(-5)|$$

(see also [12, Section 2.8]).

**Proposition 4.4.** *If  $G$  is of type  ${}^3D_4$  or  ${}^6D_4$  then there exists no coherent parahoric family of class number one.*



*Proof.* If  $G$  admits a one-class parahoric family  $P$ , then

$$1 \geq \mathfrak{M}(P) \geq \mathfrak{M}(\mathcal{P}) > D_k^{7/2} D_\ell^{7/2} \left( \frac{135}{2^{11}\pi^{16}} \right)^n$$

or equivalently,  $D_\ell \leq D_k^{-1} \cdot \left( \frac{2^{11}\pi^{16}}{135} \right)^{2n/7}$ . By Lemma 4.3, there are only finitely many candidates for  $k$ . For each such field  $k$ , [16] lists all possible cubic extensions  $\ell$  that satisfy the previous inequality. We only find the possibility  $k = \mathbb{Q}$  and  $\ell = k[x]/(f(x))$  where  $f(x)$  is one of the ten polynomials given below. In each case, we can now evaluate  $\mathfrak{M}(\mathcal{P})$  explicitly using equation (4.1).

$f(x)$	$\mathfrak{M}(\mathcal{P})$
$x^3 - x^2 - 2x + 1$	79/84672
$x^3 - 3x - 1$	199/36288
$x^3 - x^2 - 3x + 1$	577/12096
$x^3 - x^2 - 4x - 1$	11227/157248
$x^3 - 4x - 1$	1333/6048
$x^3 - x^2 - 4x + 3$	1891/6048
$x^3 - x^2 - 4x + 2$	2185/3024
$x^3 - x^2 - 4x + 1$	925/1344
$x^3 - x^2 - 6x + 7$	4087/4032
$x^3 - x^2 - 5x - 1$	19613/12096

The result now follows from the fact that  $\mathfrak{M}(P)$  is an integral multiple of  $\mathfrak{M}(\mathcal{P})$  and therefore never the reciprocal of an integer. □

**4.4. The case  $E_6$ .** Let  $G$  be a form of  $E_6$ . The assumption that  $G(k_v)$  is anisotropic for all  $v \in V_\infty$  forces  $G$  to be of type  ${}^2E_6$ , see for example [4, Proposition 2.2] for details. Thus the splitting field  $\ell$  of  $G$  is a totally complex quadratic extension of  $k$ .

**Proposition 4.5.** *There exists no coherent family  $P$  of parahoric subgroups of  $G$  with class number one.*

*Proof.* We have  $r = 6$ ,  $(m_1, \dots, m_6) = (1, 4, 5, 7, 8, 11)$ ,  $s(\mathcal{G}) = 26$  and  $\dim G = 78$ . Suppose  $P$  is a parahoric family of class number one. Let

$$\gamma := \prod_{i=1}^6 \frac{m_i!}{(2\pi)^{m_i+1}}.$$

Then Theorem 3.1 implies

$$1 = \mathfrak{c}(P) \geq \mathfrak{M}(P) > D_k^{39} \cdot (D_\ell/D_k^2)^{13} \cdot \gamma^n \geq D_k^{39} \cdot \gamma^n$$

and therefore  $D_k^{1/n} < \gamma^{-1/39} < 2.31$ . Hence  $k$  is either  $\mathbb{Q}$  or  $\mathbb{Q}(\sqrt{5})$ .

Suppose  $k = \mathbb{Q}(\sqrt{5})$ . Since the narrow class group of  $k$  is trivial, the extension  $\ell/k$  ramifies at a finite place. Thus  $D_\ell/D_k^2 \geq 4$  and hence

$$c(P) > 5^{39} \cdot 4^{13} \cdot \gamma^2 > 1.$$

So we may suppose that  $k = \mathbb{Q}$ . Then  $1 = c(P) > D_\ell^{13} \cdot \gamma$  implies that  $D_\ell \leq 12$ . Thus  $\ell$  is  $\mathbb{Q}(\sqrt{-d})$  for some  $d \in \{1, 2, 3, 7, 11\}$ . For any  $v \in V_f$ , the group  $G$  is quasi-split over  $k_v$ . Moreover, the type of  $G$  over  $k_v$  is  ${}^1E_6$ ,  ${}^2E_6$  or  $F_4^I$  (using the notation of [15, Section 4]) depending on whether  $v$  is split, inert or ramified in  $\ell$ . Thus  $\zeta(\mathcal{P}_v)^{-1}$  equals

$$(1 - q_v^{-2})(1 - q_v^{-6})(1 - q_v^{-8})(1 - q_v^{-12}) \cdot c_v$$

where

$$c_v = \begin{cases} (1 - q_v^{-5})(1 - q_v^{-9}) & \text{if } v \text{ is split in } \ell, \\ (1 + q_v^{-5})(1 + q_v^{-9}) & \text{if } v \text{ is inert in } \ell, \\ 1 & \text{if } v \text{ is ramified in } \ell. \end{cases}$$

Using the functional equation for zeta functions, we obtain

$$\mathfrak{M}(\mathcal{P}) = 2^{-6} \cdot |\zeta_{\mathbb{Q}}(-1)\zeta_{\ell/\mathbb{Q}}(-4)\zeta_{\mathbb{Q}}(-5)\zeta_{\mathbb{Q}}(-7)\zeta_{\ell/\mathbb{Q}}(-8)\zeta_{\mathbb{Q}}(-11)|.$$

The values for  $\mathfrak{M}(\mathcal{P})$  for all possible fields  $\ell = \mathbb{Q}(\sqrt{-d})$  are

$d$	1	2	3	7	11
$\mathfrak{M}(\mathcal{P})$	$\frac{191407}{243465191424}$	$\frac{1097308691}{169073049600}$	$\frac{559019}{30813563289600}$	$\frac{6102221}{5200977600}$	$\frac{7340406625}{18598035456}$

In particular, there exists no parahoric family  $P$  such that  $\mathfrak{M}(P)^{-1} \in \mathbb{Z}$ .  $\square$

### 4.5. The case $E_7$ .

**Proposition 4.6.** *If  $G$  is of type  $E_7$  then there exists no coherent family  $P$  of parahoric subgroups of  $G$  with class number one.*

*Proof.* If  $G$  is of type  $E_7$  then  $r = 7$ ,  $(m_1, \dots, m_7) = (1, 5, 7, 9, 11, 13, 17)$  and  $\dim G = 133$ . If  $c(P) = 1$ , then Theorem 3.1 implies that

$$D_k^{1/n} < \left( \prod_{i=1}^7 \frac{(2\pi)^{m_i+1}}{m_i!} \right)^{2/133} < 1.547 < \sqrt{5}.$$

Thus  $k = \mathbb{Q}$ . But then

$$\mathfrak{M}(\mathcal{P}) = 2^{-7} \prod_{i=1}^7 |\zeta_{\mathbb{Q}}(-m_i)| = \frac{691 \cdot 43867}{2^{24} 3^{11} 5^2 7^3 11^1 13^1 19^1}$$

shows that  $c(P) > 1$  for all parahoric families  $P$ .  $\square$

#### 4.6. The case $E_8$ .

**Proposition 4.7.** *If  $G$  is of type  $E_8$  and  $P$  is a coherent family of parahoric subgroups of  $G$  then  $\mathfrak{c}(P) \geq 8435$ .*

*Proof.* If  $G$  is of type  $E_8$  then  $r = 8$  and

$$(m_1, \dots, m_8) = (1, 7, 11, 13, 17, 19, 23, 29) .$$

Thus Theorem 3.1 implies that

$$\mathfrak{c}(P) \geq \mathfrak{M}(P) > \prod_{i=1}^8 \frac{m_i!}{(2\pi)^{m_i+1}} > 8434 . \quad \square$$

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