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Résumé. Soit l un nombre premier. Dans cet article, nous étudions la pro-l action galoisienne extérieure associée à une courbe modulaire de niveau une puissance de l. En particulier, nous discutons de la question de savoir si cette action se factorise à travers d'un pro-l quotient du groupe de Galois absolu d'un certain corps de nombres. Comme application, nous établissons aussi une relation entre les variétés Jacobiennes de courbes modulaires de niveau puissance d'un nombre premier et l'ensemble défini par Rasmussen et Tamagawa.

Abstract. Let l be a prime number. In the present paper, we study the pro-l outer Galois action associated to a modular curve of level a power of l. In particular, we discuss the issue of whether or not the pro-l outer Galois action factors through a pro-l quotient of the absolute Galois group of a certain number field. Moreover, as an application, we also obtain a result concerning the relationship between the Jacobian varieties of modular curves of prime power level and a set defined by Rasmussen and Tamagawa.

Introduction

Throughout the present paper, let l be a prime number and $\overline{\mathbb{Q}}$ an algebraic closure of the field of rational numbers $\mathbb{Q}$. For a positive integer N, let $\zeta_N \in \overline{\mathbb{Q}}$ be a primitive N-th root of unity; write $G_{\mathbb{Q}(\zeta_N)} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$.

Let us recall that the pro-l outer Galois action associated to a hyperbolic curve is one important object in the study of arithmetic geometry via étale fundamental groups. Some researchers have studied the arithmetic and group-theoretic properties of the kernel/image of the pro-l outer Galois action associated to a hyperbolic curve. For instance, in [7], Ihara studied the arithmetic properties of the Galois extension of $\mathbb{Q}$ corresponding to the kernel of the pro-l outer Galois action associated to the projective line minus three rational points over $\mathbb{Q}$ and posed an interesting problem concerning this Galois extension (cf. [7, Lecture I, §2], [14, Introduction]).

In the present paper, we study the arithmetic and group-theoretic properties of the kernel/image of the pro-l outer Galois action associated to a

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modular curve of level a power of l. In particular, we discuss the issue of whether or not the Galois extension of a certain number field corresponding to the kernel of the pro-l outer Galois action associated to a modular curve of level a power of l is a *pro-l extension*.

For a positive integer N, let us write $Y(N)$ for the *modular curve* over $\mathbb{Q}(\zeta_N)$ with respect to the congruence subgroup $\Gamma(N)$, i.e., the coarse moduli scheme of elliptic curves with $\Gamma(N)$-structures (cf., e.g., [9, Chapter 3]); $X(N)$ for the *smooth compactification* of $Y(N)$; $J(N)$ for the *Jacobian variety* of $X(N)$. The portion of the main result of the present paper related to the modular curve "$X(N)$" is as follows (cf. Theorem 4.1). Note that the portions of the main result related to the modular curves with respect to the congruence subgroups "Γ_1", "Γ_0" may be found in Theorems 4.1 and 4.2, respectively.

Theorem A. *Let m be a positive integer. Then the following four conditions are equivalent:*

(P) $l \in \{2, 3, 5, 7\}$.

(Y) *The Galois extension of $\mathbb{Q}(\zeta_{l^m})$ corresponding to the kernel of the pro-l outer action associated to $Y(l^m)$ is a pro-l extension of $\mathbb{Q}(\zeta_{l^m})$.*

(X) *The Galois extension of $\mathbb{Q}(\zeta_{l^m})$ corresponding to the kernel of the pro-l outer action associated to $X(l^m)$ is a pro-l extension of $\mathbb{Q}(\zeta_{l^m})$.*

(J) *The Galois extension of $\mathbb{Q}(\zeta_{l^m})$ corresponding to the kernel of the l-adic representation associated to $J(l^m)$ is a pro-l extension of $\mathbb{Q}(\zeta_{l^m})$.*

The proof of Theorem A may be summarized as follows.

The first step is proved in a group-theoretic and formal fashion. First, one may prove that, for a curve C whose Jacobian variety we denote by J_C, the pro-l-ness of the image of the pro-l outer action associated to C implies the pro-l-ness of the image of the l-adic representation associated to J_C; moreover, one may also prove the converse implication under an additional assumption, i.e., concerning the fields of definition of the cusps of C (cf. Lemma 2.4). Furthermore, one may prove that, for hyperbolic curves C_1 and C_2 over a field F, if there exists a dominant morphism $C_1 \to C_2$ over F (e.g., the natural open immersion $Y(l^m) \hookrightarrow X(l^m)$), then the pro-$l$-ness of the image of the pro-l outer action associated to C_1 implies the pro-l-ness of the image of the pro-l outer action associated to C_2; moreover, one may also prove the converse implication under an additional assumption (cf. Lemma 2.5). By means of these implications, we obtain the equivalences between the three conditions in Theorem A

$$(\mathrm{Y}) \iff (\mathrm{X}) \iff (\mathrm{J}).$$

In the next step, we apply a deep arithmetic result due to Mazur concerning Eisenstein ideals of Hecke algebras obtained in [9], which is the most important ingredient of the proof. Suppose that the modular curve $Y_0(l)$ with respect to the congruence subgroup $\Gamma_0(l)$ is of positive genus, or, equivalently, that $l \notin \{2, 3, 5, 7, 13\}$. Then the result due to Mazur asserts that there is no Eisenstein prime of the Hecke algebra associated to the Jacobian variety $J_0(l)$ of $Y_0(l)$ whose residue characteristic is equal to l. On the other hand, by applying a technique in [14], one may prove that the pro-l-ness of the image of the l-adic representation associated to $J_0(l)$ implies the existence of such an Eisenstein prime. By this argument, we conclude (cf. Lemma 3.1) that:

A similar condition to condition (J) *in Theorem A for "$J_0(l)$" implies that* $l \in \{2, 3, 5, 7, 13\}$.

This is the most important step of the proof. By this conclusion, together with the argument preceding the above displayed equivalence "(Y) $\iff$ (X) $\iff$ (J)" (here, recall the existence of the natural dominant morphisms $Y(l^m) \to Y_0(l^m) \to Y_0(l)$), we obtain that:

Each of three conditions (Y), (X)*, and* (J) *in Theorem A implies that* $l \in \{2, 3, 5, 7, 13\}$.

Finally, we deal individually the remaining portions. (For instance, by applying the Eichler–Shimura relation and considering the traces of the actions of suitable Frobenius elements on the 13-adic Tate module of $J_0(13^2)$, one may prove that condition (J) in the case where $l = 13$ does not hold — cf. Lemma 3.3.)

Here, let us observe that if one works with only condition (J), then it is *difficult* to compare the pro-l-ness of the images of the l-adic representations associated to $J(l^m)$ for the various m. On the other hand, since the natural morphism $Y(l^m) \to Y(l^{m'})$ (where $1 \leq m' < m$) is a *finite étale l-covering* if $l \neq 2$ (cf. Lemma 2.6 (2)), it is *not difficult* to compare the pro-l-ness of the images of the pro-l outer actions associated to $Y(l^m)$ for the various m. In our arguments, by relating the pro-l-ness for $Y(l^m)$ to the pro-l-ness for $J(l^m)$ and applying the comparison between the pro-l-ness for the $Y(l^m)$'s, we obtain a comparison between the pro-l-ness for the $J(l^m)$'s. This is the significance of working with three conditions (Y), (X), and (J) (i.e., as opposed to the situation in which we work with only condition (J)).

The main result of the present paper has the following two *applications*.

The first application of the main result is related to the study of Rasmussen and Tamagawa in [14]. Let g be a positive integer and $F \subseteq \overline{\mathbb{Q}}$ a finite extension of $\mathbb{Q}$. In [14], Rasmussen and Tamagawa introduced and studied the set $\mathcal{A}(F, g)$ of pairs $([A], p)$ consisting of the F-isomorphism classes $[A]$ of abelian varieties A over F of dimension g and prime numbers p such that the abelian variety $A \otimes_F F(\zeta_p)$ over $F(\zeta_p)$ has good reduction outside p, and, moreover, the Galois extension of $F(\zeta_p)$ corresponding to the kernel of the p-adic representation associated to $A \otimes_F F(\zeta_p)$ is pro-p (cf. [14, §1]). The main result of the present paper leads naturally to the following result (cf. Corollary 4.4 (3)). Note that similar results to the following result related to the modular curves with respect to the congruence subgroups "Γ_1", "Γ_0" may be found in Corollaries 4.4 (1) and (2) respectively.

Theorem B. *Let m be a positive integer. Suppose that $J(l^m)$ is of positive dimension. Then the following two conditions are equivalent:*

- $l \in \{2, 3, 5, 7\}$.
- $([J(l^m)], l) \in \mathcal{A}(\mathbb{Q}(\zeta_{l^m}), \dim(J(l^m)))$.

The second application of the main result is related to a pro-l version of the congruence subgroup problem for mapping class groups. In [6], the authors of the present paper proved the assertion that a pro-l version of the congruence subgroup problem for mapping class groups of genus one in the case where $l \geq 11$ has a negative answer. In the proof of this assertion, a portion of the main result of the present paper (i.e., Theorem A) plays an essential role.

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1. Preliminaries

Throughout the present paper, let l be a prime number, $\overline{\mathbb{Q}}$ an algebraic closure of the field of rational numbers $\mathbb{Q}$, and $F \subseteq \overline{\mathbb{Q}}$ a subfield of $\overline{\mathbb{Q}}$. Write $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ for the absolute Galois group of F. For a positive integer N, let $\zeta_N \in \overline{\mathbb{Q}}$ be a primitive N-th root of unity.

First, let us recall the definition of a *hyperbolic curve* over F.

Definition 1.1.

(1) We shall say that a scheme X over F is a *curve* over F if X is isomorphic, over F, to a nonempty open subscheme of a scheme X^{cpt} which is of dimension one, smooth, proper, and geometrically connected over F.

(2) Let X be a curve over F. Then it follows from elementary algebraic geometry that the scheme "X^{cpt}" as in (1) is uniquely determined by X up to canonical isomorphism. We shall refer to the scheme X^{cpt} as the *smooth compactification* of X and to an element of the set $X^{\mathrm{cpt}}(\overline{\mathbb{Q}}) \setminus X(\overline{\mathbb{Q}})$ as a *cusp* of X.

(3) We shall say that a curve X over F is *hyperbolic* if $2g-2+r>0$, where we write g for the genus of (the smooth compactification of) X and r for the cardinality of the set of cusps of X.

Next, let us recall the *pro-l outer Galois action* associated to a suitable scheme over F. This notion is one of the main objects studied in the present paper.

Definition 1.2. Let X be a scheme which is geometrically connected and of finite type over F.

(1) We shall write Δ^l_X for the *pro-l geometric fundamental group* of X, i.e., the maximal pro-l quotient of the étale fundamental group $\pi_1(X \otimes_F \overline{\mathbb{Q}})$ of $X \otimes_F \overline{\mathbb{Q}}$.

(2) We shall write Π^l_X for the *geometrically pro-l fundamental group* of X, i.e., the quotient of the étale fundamental group $\pi_1(X)$ of X by the kernel of the natural surjection $\pi_1(X \otimes_F \overline{\mathbb{Q}}) \to \Delta^l_X$.

(3) For a profinite group G, we shall denote by $\mathrm{Out}(G)$ the group of outer automorphisms of G. We shall write

$$\rho^l_X \colon G_F \longrightarrow \mathrm{Out}(\Delta^l_X)$$

for the outer action determined by the natural exact sequence

$$1 \longrightarrow \Delta^l_X \longrightarrow \Pi^l_X \longrightarrow G_F \longrightarrow 1.$$

We shall refer to ρ^l_X as the *pro-l outer Galois action* associated to X.

Remark 1.3. In the situation of Definition 1.2, suppose that X is an *abelian variety* over F. Then one verifies easily that Δ^l_X may be naturally identified with the l-adic Tate module of X, and that ρ^l_X coincides with the (usual) l-adic representation of G_F associated to X.

Next, let us introduce some notational conventions related to *modular curves*.

Definition 1.4. Let N be a positive integer. Then we shall write

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1,\ c \equiv b \equiv 0 \pmod N \right\};$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1,\ c \equiv 0 \pmod N \right\};$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod N \right\}.$$

Definition 1.5. Let N be a positive integer.

(1) We shall write

$$Y_1(N), \quad Y_0(N)$$

for the respective *modular curves* over $\mathbb{Q}$ with respect to $\Gamma_1(N)$, $\Gamma_0(N)$, i.e., the respective coarse moduli schemes of elliptic curves with $\Gamma_1(N)$-, $\Gamma_0(N)$-structures (cf., e.g., [9, Chapter 3]);

$$X_1(N), \quad X_0(N)$$

for the respective smooth compactifications of $Y_1(N)$, $Y_0(N)$;

$$J_1(N), \quad J_0(N)$$

for the respective Jacobian varieties of $X_1(N)$, $X_0(N)$.

(2) We shall write

$$Y(N)$$

for the *modular curve* over $\mathbb{Q}(\zeta_N)$ with respect to $\Gamma(N)$, i.e., the coarse moduli scheme of elliptic curves with $\Gamma(N)$-structures (cf., e.g., [9, Chapter 3]);

$$X(N)$$

for the smooth compactification of $Y(N)$;

$$J(N)$$

for the Jacobian variety of $X(N)$.

Remark 1.6. The following well-known facts will be often applied in the present paper.

(1) It holds that $X_0(l)$ is of genus zero if and only if $l \in \{2, 3, 5, 7, 13\}$ (cf., e.g., [3, §3.1]).
(2) It holds that $X(l)$ (respectively, $X_1(l)$) is of genus zero if and only if $l \in \{2, 3, 5\}$ (respectively, $\in \{2, 3, 5, 7\}$) (cf., e.g., [3, §3.9]).
(3) Every cusp of $Y(l)$, hence also of $Y_1(l)$ and $Y_0(l)$, is defined over $\mathbb{Q}(\zeta_l)$ (cf., e.g., [8, §1.4]).
(4) For a positive integer m, $Y(l^m)$ is a hyperbolic curve over $\mathbb{Q}(\zeta_{l^m})$ (cf., e.g., [3, §3.9]). Also, for a positive integer m, it holds that $Y_1(l^m)$ (respectively, $Y_0(l^m)$) is a hyperbolic curve over $\mathbb{Q}$ if and only if either $l \notin \{2, 3\}$ (respectively, $\notin \{2, 3, 5, 7, 13\}$) or $m \geq 2$ (cf., e.g., [3, §3.1, 3.9]).

2. Some group-theoretic aspects

In the present §2, we discuss some elementary facts concerning outer actions of profinite groups and apply these facts to pro-l outer Galois actions.

First, let us introduce some notational conventions related to profinite groups.

Definition 2.1. Let G be a profinite group.

(1) We shall write G^{ab} for the *abelianization* of G (i.e., the quotient of G by the closure of the commutator subgroup of G).
(2) Let $H \subseteq G$ be a closed subgroup. Then we shall write $Z_G(H)$ for the *centralizer* of H in G.
(3) We shall say that a profinite group G is *slim* if $Z_G(H) = \{1\}$ for every open subgroup $H \subseteq G$.

Next, in the following two lemmas, let us discuss some elementary facts concerning outer actions of profinite groups.

Lemma 2.2. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varDelta_1 & \longrightarrow & \varPi_1 & \longrightarrow & G & \longrightarrow & 1 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \varDelta_2 & \longrightarrow & \varPi_2 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups, where the horizontal sequences are exact, and the right-hand vertical arrow is the identity automorphism of G. Write

$$\rho_1 \colon G \longrightarrow \mathrm{Out}(\varDelta_1), \quad \rho_2 \colon G \longrightarrow \mathrm{Out}(\varDelta_2)$$

for the outer actions of G on $\varDelta_1$, $\varDelta_2$ determined by the upper, lower horizontal sequences of the above diagram, respectively. Suppose that one of the following conditions is satisfied:

(1) *The homomorphism α is surjective.*
(2) *The profinite group $\varDelta_2$ is slim, and the homomorphism α is open.*

Then it holds that $\ker(\rho_1) \subseteq \ker(\rho_2)$.

Proof. First, let us observe that if condition (1) is satisfied, then Lemma 2.2 is immediate. Next, we verify Lemma 2.2 in the case where condition (2) is satisfied. Now let us observe that it follows immediately from Lemma 2.2 in the case where condition (1) is satisfied that we may assume without loss of generality — by replacing $\varDelta_1$ by the image of α — that α is an open injection. Thus, Lemma 2.2 in the case where condition (2) is satisfied follows from a similar argument to the argument applied in the proof of [4, Lemma 23 (iii)]. This completes the proof of Lemma 2.2. □

Lemma 2.3. *In the notation and the assumption of Lemma 2.2, suppose, moreover, that $\varDelta_2$ is pro-l, and that the homomorphism α is an open injection. Then it holds that ρ_1 factors through a pro-l quotient of G if and only if ρ_2 factors through a pro-l quotient of G.*

Proof. Lemma 2.3 follows immediately from Lemma 2.2, together with a similar argument to the argument applied in the proof of [4, Lemma 23 (i)]. □

Next, in the following two lemmas, let us apply the above two lemmas to pro-l outer Galois actions.

Lemma 2.4. *Let Y be a curve over F. Write X for the smooth compactification of Y and J for the Jacobian variety of X. Suppose that $\zeta_l \in F$. Then the following hold:*

(1) *It holds that*

$$\ker(\rho^l_Y) \subseteq \ker(\rho^l_X) \subseteq \ker(\rho^l_J).$$

Moreover, the quotient $\ker(\rho^l_J)/\ker(\rho^l_X)$ is pro-l.

(2) *Suppose, moreover, that the natural action of G_F on the set of cusps of Y factors through a pro-l quotient of G_F. Then the quotient $\ker(\rho^l_X)/\ker(\rho^l_Y)$ (cf. (1)), hence also $\ker(\rho^l_J)/\ker(\rho^l_Y)$ (cf. (1)), is pro-l.*

(3) *In the situation of (2), if, moreover, X is of genus zero, then the outer Galois action ρ^l_Y of G_F on Δ^l_Y factors through a pro-l quotient of G_F.*

Proof. Assertion (1) follows immediately from Lemma 2.2 (in the case where condition (1) is satisfied) and [2, Corollary 7], together with the fact that the natural morphism $X \otimes_F \overline{\mathbb{Q}} \to J \otimes_F \overline{\mathbb{Q}}$ determined by a $\overline{\mathbb{Q}}$-valued point of X induces an isomorphism $(\Delta^l_X)^{\mathrm{ab}} \xrightarrow{\sim} \Delta^l_J$.

Next, we verify assertion (2). First, let us observe that since (we have assumed that) the natural action of G_F on the set of cusps of Y factors through a pro-l quotient of G_F, we may assume without loss of generality, by replacing F by a suitable finite extension of F, that every cusp of Y is defined over F. Write $V_{Y/X}$ for the kernel of the natural surjection $(\Delta^l_Y)^{\mathrm{ab}} \twoheadrightarrow (\Delta^l_X)^{\mathrm{ab}}$ of free $\mathbb{Z}_l$-modules of finite rank (cf., e.g., [11, Remark 1.2.2]). Then one verifies immediately from [2, Corollary 7] that (since the module $\mathrm{Hom}_{\mathbb{Z}_l}((\Delta^l_X)^{\mathrm{ab}}, V_{Y/X})$ of $\mathbb{Z}_l$-linear homomorphisms from $(\Delta^l_X)^{\mathrm{ab}}$ to $V_{Y/X}$ is a pro-l group), to complete the verification of assertion (2), it suffices to verify that the natural action of $\ker(\rho^l_X)$ on $V_{Y/X}$ factors through a pro-l quotient of $\ker(\rho^l_X)$. On the other hand, this follows immediately — in light of our assumption that $\zeta_l \in F$ — from the easily verified fact that $V_{Y/X}$ is isomorphic, as a $\mathbb{Z}_l$-module equipped with an action of G_F, to the direct sum of finitely many copies of the l-adic cyclotomic character of G_F (cf. our assumption that every cusp of Y is defined over F). This completes the proof of assertion (2). Assertion (3) follows immediately from assertion (2), together with the easily verified fact that $\Delta^l_X = \{1\}$ (cf. our assumption that X is of genus zero). This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let C_1, C_2 be hyperbolic curves over F, and $f\colon C_1 \to C_2$ a dominant morphism over F. Then the following hold:*

(1) *If $\rho^l_{C_1}$ factors through a pro-l quotient of G_F, then $\rho^l_{C_2}$ also factors through a pro-l quotient of G_F.*

(2) *Suppose, moreover, that the outer homomorphism $\Pi^l_{C_1} \to \Pi^l_{C_2}$ induced by f is injective. Then it holds that $\rho^l_{C_1}$ factors through a pro-l quotient of G_F if and only if $\rho^l_{C_2}$ factors through a pro-l quotient of G_F.*

Proof. Assertion (1) (respectively, assertion (2)) follows from Lemma 2.2 (respectively, Lemma 2.3) in the case where condition (2) is satisfied, together with the well-known fact that $\Delta^l_{C_2}$ is slim (cf., e.g., [11, Proposition 1.4]). This completes the proof of Lemma 2.5. $\square$

Finally, in the following lemma, let us discuss some elementary facts concerning modular curves.

Lemma 2.6. *Let m be a positive integer. Then the following hold:*

(1) *Let us write $H \subseteq \Gamma_1(l)$ (respectively, $\subseteq \Gamma(l)$) for the normal subgroup obtained by forming the intersection of all $\Gamma_1(l)$- (respectively, $\Gamma(l)$-) conjugates of $\Gamma_1(l^m) \subseteq \Gamma_1(l)$ (respectively, $\Gamma(l^m) \subseteq \Gamma(l)$). Then the index of H in $\Gamma_1(l)$ (respectively, $\Gamma(l)$) is a power of l.*

(2) *Suppose that $l \notin \{2,3\}$ (respectively, $\neq 2$). Then the natural morphism $Y_1(l^m) \to Y_1(l)$ (respectively, $Y(l^m) \to Y(l)$) induces an outer open injection $\Pi^l_{Y_1(l^m)} \hookrightarrow \Pi^l_{Y_1(l)}$ (respectively, $\Pi^l_{Y(l^m)} \hookrightarrow \Pi^l_{Y(l)}$).*

Proof. First, we verify assertion (1). Let us first observe that, to verify assertion (1), it suffices to verify that the finite group $\Gamma_1(l)/\Gamma(l^m)$ is an l-group. On the other hand, one verifies easily that $\Gamma(l)/\Gamma(l^m) \simeq \ker(SL_2(\mathbb{Z}/l^m) \to SL_2(\mathbb{Z}/l))$ is an l-group. Thus, assertion (1) follows from the easily verified fact that the index of $\Gamma_1(l)$ in $\Gamma(l)$ is equal to l. This completes the proof of assertion (1). Assertion (2) follows immediately from assertion (1), together with the well-known fact that if $l \notin \{2,3\}$ (respectively, $\neq 2$), then the pro-l group $\Delta^l_{Y_1(l^m)}$ (respectively, $\Delta^l_{Y(l^m)}$) is obtained by forming the pro-l completion of the discrete group $\Gamma_1(l^m)$ (respectively, $\Gamma(l^m)$). $\square$

Remark 2.7. Let m be a positive integer, p a prime factor of $l-1$, p^ν the largest power of p that divides $l-1$, and $a \in (\mathbb{Z}/l^m)^\times$ an element of order p^ν. Then one verifies easily that the subgroup generated by the matrix

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2(\mathbb{Z}/l^m) \xleftarrow{\sim} SL_2(\mathbb{Z})/\Gamma(l^m)$$

is a *p-Sylow subgroup* of $\Gamma_0(l^m)/\Gamma(l^m) \subseteq SL_2(\mathbb{Z})/\Gamma(l^m)$. Thus, by considering the conjugate of the above matrix by the matrix

$$\begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \in \Gamma_0(l)/\Gamma(l^m) \subseteq SL_2(\mathbb{Z})/\Gamma(l^m),$$

one verifies immediately that

If $l \geq 5$ and $m \geq 2$, then the assertion obtained by replacing "Γ_1" or "Γ" in Lemma 2.6 (1), by "Γ_0" does not hold*.*

3. Cases of prime level

In the present Section 3, we discuss the issue of whether or not the pro-l outer Galois action associated to a *modular curve of level l* factors through a pro-l quotient of the absolute Galois group of a certain number field (cf. Lemma 3.4 below).

Lemma 3.1. *Suppose that $l \notin \{2, 3, 5, 7, 13\}$. Then the restriction of the action of $G_{\mathbb{Q}}$ on the l-adic Tate module $T_l(J_0(l))$ of $J_0(l)$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ does not factor through any pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

Proof. Write $\chi\colon G_{\mathbb{Q}} \to \mathbb{F}_l^{\times}$ for the mod l cyclotomic character of $G_{\mathbb{Q}}$, $\mathrm{End}(J_0(l))$ for the ring of endomorphisms of the abelian variety $J_0(l)$ over $\mathbb{Q}$, $\mathbb{T} \subseteq \mathrm{End}(J_0(l))$ for the *Hecke algebra* (where we refer to [10, p. 90, Definition]), and

$$V := \Delta^l_{J_0(l)} \otimes_{\mathbb{Z}_l} \mathbb{F}_l.$$

Thus, we have a natural homomorphism of groups (respectively, rings)

$$G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathbb{F}_l}(V) \ \ (\text{respectively, } \mathbb{T} \to \mathrm{End}_{\mathbb{F}_l}(V)).$$

Note that since the action of $\mathbb{T} \subseteq \mathrm{End}(J_0(l))$ on $J_0(l)$ is defined over $\mathbb{Q}$, the action of $G_{\mathbb{Q}}$ on V commutes with the action of $\mathbb{T}$ on V. Write $\mathbb{T}[G_{\mathbb{Q}}] \subseteq \mathrm{End}_{\mathbb{F}_l}(V)$ for the subring of $\mathrm{End}_{\mathbb{F}_l}(V)$ generated by the images of $G_{\mathbb{Q}} \to \mathrm{Aut}_{\mathbb{F}_l}(V)$ and $\mathbb{T} \to \mathrm{End}_{\mathbb{F}_l}(V)$. Here, let us recall that since $l \notin \{2, 3, 5, 7, 13\}$, the dimension of V over $\mathbb{F}_l$ is positive (cf. Remark 1.6 (1)).

Assume that the restriction of the action of $G_{\mathbb{Q}}$ on the l-adic Tate module $T_l(J_0(l))$ of $J_0(l)$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through some pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$. Then it follows from [14, Lemma 3] that there exists an integer i such that

$$V^{\chi^i} := \{v \in V \mid g \cdot v = \chi^i(g) \cdot v \, (\text{for all } g \in G_{\mathbb{Q}})\} \subseteq V$$

is a nontrivial subspace of V. Since the action of $G_{\mathbb{Q}}$ on V commutes with the action of $\mathbb{T}$ on V, for each $g \in G_{\mathbb{Q}}$, $t \in \mathbb{T}$, and $v \in V^{\chi^i}$, it holds that

$$g \cdot (t \cdot v) = t \cdot (g \cdot v) = t \cdot (\chi^i(g) \cdot v) = \chi^i(g) \cdot (t \cdot v).$$

Thus, V^{χ^i} is a $\mathbb{T}[G_{\mathbb{Q}}]$-submodule of V. Let W be a constituent of a $\mathbb{T}$-Jordan–Hölder filtration of V^{χ^i}. Then it follows from the definition of V^{χ^i}

that the $\mathbb{T}$-module W is a $\mathbb{T}[G_{\mathbb{Q}}]$-subquotient of V^{χ^i} and, moreover, a constituent of a $\mathbb{T}[G_{\mathbb{Q}}]$-Jordan–Hölder filtration of V, i.e., W is a *constituent* of V in the sense of [10, p. 112]. Thus, the annihilator $\mathfrak{M}$ in $\mathbb{T}$ concerning the action on W is a maximal ideal of $\mathbb{T}$, and the action of $\mathbb{T}$ on W induces an injection

$$\mathbb{T}/\mathfrak{M} \hookrightarrow \mathrm{End}_{\mathbb{F}_l}(W).$$

Since W is a simple $\mathbb{T}$-module by the definition of W, the dimension of W over $\mathbb{T}/\mathfrak{M}$ is equal to one, i.e., the dimension of W is equal to one in the sense of [10, p. 112]. Hence, it follows from [10, Chapter II, Proposition 14.1] that $\mathfrak{M}$ is an *Eisenstein prime* of $\mathbb{T}$ (where we refer to [10, p. 96, Definition]). Thus, by [10, Chapter II, Proposition 9.7], the field $\mathbb{T}/\mathfrak{M}$ is of characteristic prime to l. On the other hand, one verifies easily that $\mathrm{End}_{\mathbb{F}_l}(W)$ is of order a power of l. Thus, we obtain a contradiction. This completes the proof of Lemma 3.1. □

Lemma 3.2. *The following hold:*

1. *The restriction of the action of $G_{\mathbb{Q}}$ on the 7-adic Tate module $T_7(J_0(49))$ of $J_0(49)$ to $G_{\mathbb{Q}(\zeta_7)} \subseteq G_{\mathbb{Q}}$ factors through a pro-7 quotient of $G_{\mathbb{Q}(\zeta_7)}$.*
2. *The action of $G_{\mathbb{Q}(\zeta_7)}$ on the 7-adic Tate module $T_7(J(7))$ of $J(7)$ factors through a pro-7 quotient of $G_{\mathbb{Q}(\zeta_7)}$.*

Proof. Assertion (1) follows immediately from [14, Table 1], together with the well-known fact that $X_0(49)$ admits a structure of elliptic curve over $\mathbb{Q}$ listed as "49a". Next, we verify assertion (2). Let us first observe that the easily verified inclusion

$$\begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \Gamma(7) \cdot \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \subseteq \Gamma_0(49)$$

implies the existence of a dominant morphism $X(7) \to X_0(49) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_7)$ over $\mathbb{Q}(\zeta_7)$, hence also a surjection $J(7) \twoheadrightarrow J_0(49) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_7) = X_0(49) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_7)$ (cf. the proof of assertion (1)) over $\mathbb{Q}(\zeta_7)$. Thus, it follows immediately from [13, Theorem 2] that we have a $G_{\mathbb{Q}(\zeta_7)}$-equivariant isomorphism

$$T_7(J(7)) \otimes_{\mathbb{Z}_7} \mathbb{Q}_7 \xrightarrow{\sim} (T_7(J_0(49))^{\oplus 3}) \otimes_{\mathbb{Z}_7} \mathbb{Q}_7.$$

Thus, it follows immediately from assertion (1) that the restriction of the action of $G_{\mathbb{Q}}$ on $T_7(J(7))$ to $G_{\mathbb{Q}(\zeta_7)} \subseteq G_{\mathbb{Q}}$ factors through a pro-7 quotient of $G_{\mathbb{Q}(\zeta_7)}$. This completes the proof of assertion (2), hence also of Lemma 3.2. □

Lemma 3.3. *The following hold:*

1. *The restriction of the action of $G_{\mathbb{Q}}$ on the 13-adic Tate module $T_{13}(J_0(169))$ of $J_0(169)$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does not factor through any pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$.*

(2) *The restriction of the action of $G_{\mathbb{Q}}$ on the* 13*-adic Tate module* $T_{13}(J_1(13))$ *of* $J_1(13)$ *to* $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ *does not factor through any pro-*13 *quotient of* $G_{\mathbb{Q}(\zeta_{13})}$.

Proof. First, we verify assertion (1). Let us observe that it follows immediately from the *Eichler–Shimura relation* (cf., e.g., [10, p. 89]) that the trace of the action of the arithmetic Frobenius element Frob_3 at 3 (respectively, Frob_{29} at 29) on $T_{13}(J_0(169))$ coincides with the trace of the action of the *Hecke operator* T_3 (respectively, T_{29}) (cf., e.g., [10, p. 87]). Now we claim that:

The characteristic polynomial of the action of the Hecke operator T_3 *(respectively,* T_{29}*) on* $T_{13}(J_0(169))$ *is*

$$(t-2)^2(t^3+2t^2-t-1)^2$$
$$\textit{(respectively, } (t-3)^2(t^3+t^2-44t+83)^2\textit{).}$$

In particular, the trace of the action of Frob_3 *(respectively,* Frob_{29}*) on* $T_{13}(J_0(169))$ *is* 0 *(respectively,* 4*).*

Indeed, the above claim follows immediately from [15, `http://wstein.org/Tables/charpoly.html`].

Assume that the restriction of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_0(169))$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ factors through a pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. Then it follows from [14, Lemma 3] that the semi-simplification of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_0(169)) \otimes_{\mathbb{Z}_{13}} \mathbb{F}_{13}$ is isomorphic to the direct product of finitely many powers of mod 13 cyclotomic character of $G_{\mathbb{Q}}$. In particular, since $3 \equiv 29 \pmod{13}$, it follows from *class field theory* that the traces of the actions of Frob_3, Frob_{29} on $T_{13}(J_0(169)) \otimes_{\mathbb{Z}_{13}} \mathbb{F}_{13}$ coincide. Thus, since $0 \not\equiv 4 \pmod{13}$, we obtain a contradiction. This completes the proof of assertion (1).

Finally, we verify assertion (2). Now one verifies immediately from assertion (1), Lemma 2.4 (1) and Lemma 2.5 (1) (cf. also Remark 1.6 (4)), that the restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^{13}_{Y_1(169)}$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does not factor through a pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. Thus, it follows immediately from Lemma 2.5 (2) (cf. also Remark 1.6 (4)), together with Lemma 2.6 (2), that the restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^{13}_{Y_1(13)}$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does not factor through a pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. In particular, it follows from Lemma 2.4 (2) (cf. also Remark 1.6 (3)), that the restriction of the action of $G_{\mathbb{Q}}$ on $T_{13}(J_1(13))$ to $G_{\mathbb{Q}(\zeta_{13})} \subseteq G_{\mathbb{Q}}$ does not factor through any pro-13 quotient of $G_{\mathbb{Q}(\zeta_{13})}$. This completes the proof of assertion (2), hence also of Lemma 3.3. □

Lemma 3.4. *Consider the following conditions:*

(P) $l \in \{2, 3, 5, 7\}$.

(Q) $l \in \{2, 3, 5, 7, 13\}$.

(Y_1) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{Y_1(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(X_1) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{X_1(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(J_1) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the l-adic Tate module* $T_l(J_1(l))$ *of* $J_1(l)$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(Y_0) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{Y_0(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(X_0) *The restriction of the outer action of* $G_{\mathbb{Q}}$ *on* $\Delta^l_{X_0(l)}$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(J_0) *The restriction of the action of* $G_{\mathbb{Q}}$ *on the l-adic Tate module* $T_l(J_0(l))$ *of* $J_0(l)$ *to* $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(Y) *The outer action of* $G_{\mathbb{Q}(\zeta_l)}$ *on* $\Delta^l_{Y(l)}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(X) *The outer action of* $G_{\mathbb{Q}(\zeta_l)}$ *on* $\Delta^l_{X(l)}$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

(J) *The action of* $G_{\mathbb{Q}(\zeta_l)}$ *on the l-adic Tate module* $T_l(J(l))$ *of* $J(l)$ *factors through a pro-l quotient of* $G_{\mathbb{Q}(\zeta_l)}$.

Then the implications

$$\begin{aligned}(\mathrm{P}) \iff (\mathrm{Y}_1) \iff (\mathrm{X}_1) \iff (\mathrm{J}_1) \iff (\mathrm{Y}) \iff (\mathrm{X}) \iff (\mathrm{J})\\ \implies (\mathrm{Q}) \iff (\mathrm{Y}_0) \iff (\mathrm{X}_0) \iff (\mathrm{J}_0)\end{aligned}$$

hold.

Proof. Let us first observe that we have an immediate implication

$$(\mathrm{P}) \implies (\mathrm{Q}).$$

Next, let us observe that the implications

$$\begin{array}{ccccc}(\mathrm{Y}) & \Longrightarrow & (\mathrm{X}) & \Longrightarrow & (\mathrm{J})\\ \Downarrow & & & & \\ (\mathrm{Y}_1) & \Longrightarrow & (\mathrm{X}_1) & \Longrightarrow & (\mathrm{J}_1)\\ \Downarrow & & & & \\ (\mathrm{Y}_0) & \Longrightarrow & (\mathrm{X}_0) & \Longrightarrow & (\mathrm{J}_0)\end{array}$$

follow immediately from Lemma 2.4 (3) (cf. also Remark 1.6 (1), (2), (3)), Lemma 2.5 (1) (cf. also Remark 1.6 (4)), and Lemma 2.4 (1). Next, let us observe that the implications

$$(\mathrm{J}_1) \Longrightarrow (\mathrm{Y}_1), \quad (\mathrm{J}_0) \Longrightarrow (\mathrm{Y}_0), \quad (\mathrm{J}) \Longrightarrow (\mathrm{Y})$$

follow from Lemma 2.4 (2) (cf. also Remark 1.6 (3)).

Next, let us observe that the implication

$$(\mathrm{J}_0) \Longrightarrow (\mathrm{Q}).$$

follows from Lemma 3.1.

Next, we verify the implication

$$(\mathrm{J}_1) \Longrightarrow (\mathrm{P}).$$

Suppose that condition (J_1) is satisfied. Then it follows from the implications $(\mathrm{J}_1) \Rightarrow (\mathrm{J}_0) \Rightarrow (\mathrm{Q})$ already verified that, to complete the verification of condition (P), it suffices to verify that $l \neq 13$. On the other hand, if $l = 13$, then it follows from Lemma 3.3 (2), that condition (J_1) is not satisfied. This completes the proof of the implication $(\mathrm{J}_1) \Rightarrow (\mathrm{P})$.

Next, we verify the implication

$$(\mathrm{P}) \Longrightarrow (\mathrm{J}).$$

Suppose that condition (P) is satisfied. If $l \neq 7$, then condition (Y_1), hence also (cf. the implication $(\mathrm{Y}) \Rightarrow (\mathrm{J})$ already verified) condition (J), follows from Lemma 2.4 (3) (cf. also Remark 1.6 (2), (3)). If $l = 7$, then it follows from Lemma 3.2 (2), that condition (J) is satisfied. This completes the proof of the implication $(\mathrm{P}) \Rightarrow (\mathrm{J})$.

Finally, we verify the implication

$$(\mathrm{Q}) \Longrightarrow (\mathrm{J}_0).$$

Suppose that condition (Q) is satisfied. If $l \neq 13$ (i.e., condition (P) is satisfied), then condition (J_0) follows from the implications $(\mathrm{P}) \Rightarrow (\mathrm{J}) \Rightarrow (\mathrm{J}_0)$ already verified. If $l = 13$, then condition (Y_0), hence also (cf. the implication $(\mathrm{Y}_0) \Rightarrow (\mathrm{J}_0)$ already verified) condition (J_0), follows from Lemma 2.4 (3) (cf. also Remark 1.6 (1), (3)). This completes the proof of the implication $(\mathrm{Q}) \Rightarrow (\mathrm{J}_0)$, hence also of Lemma 3.4. □

4. Cases of prime power level

In the present Section 4, we discuss the issue of whether or not the pro-l outer Galois action associated to a *modular curve of level a power of* l factors through a pro-l quotient of the absolute Galois group of a certain number field (cf. Theorems 4.1, 4.2 below). Moreover, as an application, we also obtain a result concerning the relationship between the Jacobian varieties of modular curves of prime power level and a set defined by Rasmussen and Tamagawa (cf. Corollary 4.4, Remark 4.5 below).

Theorem 4.1. *Let l be a prime number and m a positive integer. Then the following conditions are equivalent:*

(P) $l \in \{2, 3, 5, 7\}$.

(Y_1) *The restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^l_{Y_1(l^m)}$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(X_1) *The restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^l_{X_1(l^m)}$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(J_1) *The restriction of the action of $G_{\mathbb{Q}}$ on the l-adic Tate module $T_l(J_1(l^m))$ of $J_1(l^m)$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(Y) *The outer action of $G_{\mathbb{Q}(\zeta_{l^m})}$ on $\Delta^l_{Y(l^m)}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_{l^m})}$.*

(X) *The outer action of $G_{\mathbb{Q}(\zeta_{l^m})}$ on $\Delta^l_{X(l^m)}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_{l^m})}$.*

(J) *The action of $G_{\mathbb{Q}(\zeta_{l^m})}$ on the l-adic Tate module $T_l(J(l^m))$ of $J(l^m)$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_{l^m})}$.*

Proof. Let us first observe that the implications

$$\begin{array}{ccccc} (\mathrm{Y}) & \Longleftrightarrow & (\mathrm{X}) & \Longleftrightarrow & (\mathrm{J}) \\ \Downarrow & & & & \\ (\mathrm{Y}_1) & \Longleftrightarrow & (\mathrm{X}_1) & \Longleftrightarrow & (\mathrm{J}_1) \end{array}$$

follow immediately from similar arguments to the arguments applied in the first paragraph of the proof of Lemma 3.4.

Next, we verify the implication

$$(\mathrm{Y}_1) \Longrightarrow (\mathrm{P}).$$

Suppose that condition (Y_1) is satisfied, and that condition (P) is not satisfied. Then it follows from Lemma 2.5 (1) (cf. also Remark 1.6 (4)) that condition (Y_1) in the case where we take "m" to be 1 is satisfied. In particular, by the implication (Y_1) ⇒ (P) of Lemma 3.4, we conclude that condition (P) is satisfied, a contradiction. This completes the proof of the implication (Y_1) ⇒ (P).

Finally, we verify the implication

$$(\mathrm{P}) \Longrightarrow (\mathrm{Y}).$$

Suppose that condition (P) is satisfied. If $l = 2$, then condition (Y) follows immediately from [1, §0.6]. Thus, we may assume without loss of generality that $l \neq 2$. Then it follows from the implication (P) ⇒ (Y) of Lemma 3.4 that condition (Y) in the case where we take "m" to be 1 is satisfied. Thus, it follows from Lemma 2.5 (1) (cf. also Remark 1.6 (4)), together with Lemma 2.6 (2) (cf. our assumption that $l \neq 2$), that condition (Y) is satisfied. This completes the proof of the implication (P) $\Rightarrow$ (Y), hence also of Theorem 4.1. $\square$

Theorem 4.2. *Let l be a prime number and m a positive integer. Then the following conditions are equivalent:*

(R) *$l \in \{2, 3, 5, 7, 13\}$, and $m = 1$ if $l = 13$.*

(Y_0) *The restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^l_{Y_0(l^m)}$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(X_0) *The restriction of the outer action of $G_{\mathbb{Q}}$ on $\Delta^l_{X_0(l^m)}$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

(J_0) *The restriction of the action of $G_{\mathbb{Q}}$ on the l-adic Tate module $T_l(J_0(l^m))$ of $J_0(l^m)$ to $G_{\mathbb{Q}(\zeta_l)} \subseteq G_{\mathbb{Q}}$ factors through a pro-l quotient of $G_{\mathbb{Q}(\zeta_l)}$.*

Proof. Let us first observe that if $m = 1$, then Theorem 4.2 follows from Lemma 3.4. Thus, to verify Theorem 4.2, we may assume without loss of generality that $m > 1$. Note that the implications

$$(\mathrm{Y}_0) \iff (\mathrm{X}_0) \iff (\mathrm{J}_0)$$

follow immediately from similar arguments to the arguments applied in the first paragraph of the proof of Lemma 3.4.

Next, we verify the implication

$$(\mathrm{Y}_0) \Longrightarrow (\mathrm{R}).$$

First, suppose that $l = 13$. Then it follows from Lemma 3.3 (1), that condition (J_0) in the case where we take "m" to be 2, hence also (cf. the implication (Y_0) $\Rightarrow$ (J_0) already verified) condition (Y_0) in the case where we take "m" to be 2, is not satisfied. Thus, it follows from Lemma 2.5 (1) (cf. also Remark 1.6 (4)) that condition (Y_0) is not satisfied.

Next, suppose that $l \notin \{2, 3, 5, 7, 13\}$. Then it follows from the implication (Y_0) $\Rightarrow$ (Q) of Lemma 3.4 that condition (Y_0) in the case where we take "m" to be 1 is not satisfied. Thus, it follows from Lemma 2.5 (1) (cf. also Remark 1.6 (4)) that condition (Y_0) is not satisfied. This completes the proof of the implication (Y_0) $\Rightarrow$ (R).

Finally, we verify the implication

$$(\mathrm{R}) \Longrightarrow (\mathrm{Y}_0).$$

Suppose that condition (R) is satisfied (i.e., that $l \in \{2, 3, 5, 7\}$). Then it follows from the implication (P) $\Rightarrow$ (Y) of Theorem 4.1 that condition (Y) of Theorem 4.1 is satisfied. Thus, it follows from Lemma 2.5 (1) (cf. also Remark 1.6 (4)) that condition (Y_0) is satisfied. This completes the proof of the implication (R) $\Rightarrow$ (Y_0), hence also of Theorem 4.2. $\square$

Next, in order to present an application of the above two theorems to the study of Rasmussen and Tamagawa in [14], let us recall an important set defined in [14].

Definition 4.3. Suppose that F is finite over $\mathbb{Q}$. Let g be a positive integer. Then we shall write $\mathcal{A}(F, g)$ for the set of pairs $([A], p)$ consisting of the F-isomorphism classes $[A]$ of abelian varieties A over F of dimension g and prime numbers p such that the abelian variety $A \otimes_F F(\zeta_p)$ over $F(\zeta_p)$ has good reduction outside p, and, moreover, the Galois extension of $F(\zeta_p)$ corresponding to the kernel of the p-adic representation associated to $A \otimes_F F(\zeta_p)$ is pro-p (cf. [14, §1]).

Then we have the following application of the above two theorems.

Corollary 4.4. *Let l be a prime number and m a positive integer. Consider condition* (P) $l \in \{2, 3, 5, 7\}$. *Then the following hold:*

(1) *Suppose that $J_1(l^m)$ is of positive dimension. Then condition* (P) *is equivalent to condition*
 (A_1) $([J_1(l^m)], l) \in \mathcal{A}(\mathbb{Q}, \dim(J_1(l^m)))$.
(2) *Suppose that $J_0(l^m)$ is of positive dimension. Then condition* (P) *is equivalent to condition*
 (A_0) $([J_0(l^m)], l) \in \mathcal{A}(\mathbb{Q}, \dim(J_0(l^m)))$.
(3) *Suppose that $J(l^m)$ is of positive dimension. Then condition* (P) *is equivalent to condition*
 (A) $([J(l^m)], l) \in \mathcal{A}(\mathbb{Q}(\zeta_{l^m}), \dim(J(l^m)))$.

Proof. First, we verify assertion (1). The implication (P) $\Rightarrow$ (A_1) follows from the implication (P) $\Rightarrow$ (J_1) of Theorem 4.1, together with the well-known fact that $J_1(l^m)$ has good reduction outside l. The implication (A_1) $\Rightarrow$ (P) follows immediately from the implication (J_1) $\Rightarrow$ (P) of Theorem 4.1. This completes the proof of assertion (1).

Assertions (2), (3) follow immediately from a similar argument to the argument applied in the proof of assertion (1) (cf. also the fact that if $(l, m) = (13, 1)$, then, by Remark 1.6 (1), the abelian variety $J_0(l^m)$ is of dimension zero). □

Remark 4.5.

(1) In the situation of Corollary 4.4 (1), if $l \neq 13$, then the two conditions discussed in Corollary 4.4 (1), are equivalent to the following condition:
 (A_1') There exists a prime number p such that $([J_1(l^m)], p)$ is contained in $\mathcal{A}(\mathbb{Q}, \dim(J_1(l^m)))$.
 Indeed, the implication (A_1) $\Rightarrow$ (A_1') is immediate. Next, let us verify the implication (A_1') $\Rightarrow$ (P). Suppose that there exists a prime number p such that $([J_1(l^m)], p)$ is contained in $\mathcal{A}(\mathbb{Q}, \dim(J_1(l^m)))$.

Then it follows from the definition of the set $\mathcal{A}(\mathbb{Q}, \dim(J_1(l^m)))$ that:

(G) the abelian variety $J_1(l^m) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_p)$ over $\mathbb{Q}(\zeta_p)$ has good reduction outside p, and

(I) the restriction of the action of $G_{\mathbb{Q}}$ on the p-adic Tate module $T_p(J_1(l^m))$ of $J_1(l^m)$ to $G_{\mathbb{Q}(\zeta_p)} \subseteq G_{\mathbb{Q}}$ factors through a pro-p quotient of $G_{\mathbb{Q}(\zeta_p)}$.

Now assume that condition (P) is not satisfied. Then it is well-known (cf. [10, §1 of Appendix]) that, for every finite extension F of $\mathbb{Q}$ containing ζ_l, the abelian variety $J_0(l^m) \otimes_{\mathbb{Q}} F$, hence also $J_1(l^m) \otimes_{\mathbb{Q}} F$ and $J(l^m) \otimes_{\mathbb{Q}(\zeta_l)} F$, over F does not have good reduction at any prime of F over l (cf. also our assumption that $l \neq 13$). In particular, it follows from condition (G) that $p = l$. Thus, since (we have assumed that) $l \ (= p) \notin \{2, 3, 5, 7\}$, it follows from condition (I), together with the implication $(\mathrm{J}_1) \Rightarrow (\mathrm{P})$ of Theorem 4.1, that we obtain a contradiction. This completes the proof of the implication $(\mathrm{A}'_1) \Rightarrow (\mathrm{P})$.

(2) It follows immediately from a similar argument to the argument applied in (1) that, in the situation of Corollary 4.4 (2) (respectively (3)), if $l \neq 13$, then the two conditions discussed in Corollary 4.4 (2) (respectively (3)), are equivalent to the following condition (A'_0) (respectively, (A')):

(A'_0) There exists a prime number p such that $([J_0(l^m)], p)$ is contained in $\mathcal{A}(\mathbb{Q}, \dim(J_0(l^m)))$.

(A') There exists a prime number p such that $([J(l^m)], p)$ is contained in $\mathcal{A}(\mathbb{Q}(\zeta_{l^m}), \dim(J(l^m)))$.

Remark 4.6.

(1) A result of [1] implies that, for a positive integer m, if the Jacobian variety $J(2^m)$ is of positive dimension, then

$$([J(2^m)], 2) \in \mathcal{A}(\mathbb{Q}(\zeta_{2^m}), \dim(J(2^m)))$$

(cf. [1, §0.6]).

(2) Let us recall that the problem by Ihara recalled in the second paragraph of Introduction was already solved affirmatively in the case where $l = 3$ (cf., e.g., [5, Theorem 1.9]). Thus, the main result of [12] (i.e., [12, Theorem 1]) is equivalent to the assertion that, for a positive integer m, if the Jacobian variety $J(3^m)$ is of positive dimension, then

$$([J(3^m)], 3) \in \mathcal{A}(\mathbb{Q}(\zeta_{3^m}), \dim(J(3^m))).$$

(Note that this assertion also follows from [5, Example 3.4 (v)].)

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