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# Class field theory for open curves over local fields

par Toshiro HIRANOUCHI

Résumé. Nous étudions la théorie des corps de classes des courbes ouvertes sur un corps local. Après avoir introduit l'application de réciprocité nous déterminons son noyau et son conoyau. La duale de Pontrjagin de l'application de réciprocitIé est également étudiée. Cela nous donne, sous certaines hypothèses, une correspondance bijective entre l'ensemble des revêtements étales abéliens et l'ensemble des sous-groupes ouverts d'indice fini du groupe des classes d'idèles.

Abstract. We study the class field theory for open curves over a local field. After introducing the reciprocity map, we determine the kernel and the cokernel of this map. In addition to this, the Pontrjagin dual of the reciprocity map is also investigated. This gives the one to one correspondence between the set of abelian étale coverings and the set of finite index open subgroups of the idèle class group as in the classical class field theory under some assumptions.

## 1. Introduction

In this note, we present the class field theory for open (=non proper) curves over a local field with arbitrary characteristic. Here, a *local field* means a complete discrete valuation field with finite residue field. For a local field with characteristic 0, a large number of studies have been made even for higher dimensional open varieties over the local field (e.g. [11, 14, 32, 33]). Accordingly, our main interest is in the case of positive characteristic local fields.

To state our results precisely, let $k$ be a local field with $\mathrm{char}(k) = p > 0$. Let $\overline{X}$ be a proper, smooth and geometrically connected curve over $k$ and $X$ a nonempty open subscheme in $\overline{X}$. We often say that the pair $X \subset \overline{X}$ is an *open curve* (cf. Definition 3.1). A topological group $C(X)$ which is called the idèle class group, and the reciprocity map $\rho_X : C(X) \to \pi_1^{\mathrm{ab}}(X)$ are introduced as in [11] (Definition 3.2 and Definition 3.3). In this note, we determine the kernel $\mathrm{Ker}(\rho_X)$ and the cokernel $\mathrm{Coker}_{\mathrm{top}}(\rho_X) := \pi_1^{\mathrm{ab}}(X)/\overline{\mathrm{Im}(\rho_X)}$ of $\rho_X$ as (Hausdorff) topological groups, where $\overline{\mathrm{Im}(\rho_X)}$ is

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the topological closure of the image $\mathrm{Im}(\rho_X)$. One of the main results in this note is the following theorem.

**Theorem 1.1** (Theorem 4.2 and Theorem 4.6)**.** *Let $X \subset \overline{X}$ be as above. For the reciprocity map $\rho_X : C(X) \to \pi_1^{\mathrm{ab}}(X)$, we have*

(1) $\mathrm{Coker}_{\mathrm{top}}(\rho_X) \simeq \widehat{\mathbb{Z}}^r$ *for some $r \in \mathbb{Z}_{\geq 0}$, and*
(2) $\mathrm{Ker}(\rho_X)$ *is the maximal $l$-divisible subgroup of $C(X)$ for all prime number $l \neq p$.*

The theorem above is known for $X = \overline{X}$ ([29]) which corresponds to the unramified class field theory. Here, the invariant $r = r(\overline{X})$ is called the *rank* of $\overline{X}$ ([29, Chap. II, Def. 2.5]). For the special fiber $J$ of the Néron model of the Jacobian variety of $\overline{X}$, the rank $r$ equals the dimension of the maximal split subtorus (over the residue field of $k$) of $J$ ([29, Chap. II, Thm. 6.2], see also [34, Thm. 1.1]). For example, we have $r = 0$ if $\overline{X}$ has good reduction. Theorem 1.1 is essentially followed from the arguments used in [11]. It is known that the quotient group $\mathrm{Coker}_{\mathrm{top}}(\rho_X) = \pi_1^{\mathrm{ab}}(X)/\overline{\mathrm{Im}(\rho_X)}$ classifies *completely split coverings* of $X$, that is, finite abelian étale coverings of $X$ in which any closed point $x \in X$ splits completely ([29, Chap. II, Def. 2.1]). In the case of $X = \overline{X}$, it is known that $\mathrm{Ker}(\rho_X)$ is the maximal divisible subgroup of $C(X)$ ([34]). At the moment, we have no examples of an open curve $X$ with $\mathrm{Ker}(\rho_X)/p \neq 0$.

Our main contribution is the following theorem on the Pontrjagin dual of the reciprocity map: For a topological abelian group $G$, we define the *Pontrjagin dual group* of $G$ by

$$G^{\vee} := \{\text{continuous homomorphism } G \to \mathbb{Q}/\mathbb{Z} \text{ with finite order}\}$$

(cf. Notation). Using this, the reciprocity map $\rho_X$ induces $\rho_X^{\vee} : \pi_1^{\mathrm{ab}}(X)^{\vee} \to C(X)^{\vee}$.

**Theorem 1.2** (Theorem 5.5)**.** *Let $X \subset \overline{X}$ be as above. We assume $r(\overline{X}) = 0$. Then, the map $\rho_X^{\vee} : \pi_1^{\mathrm{ab}}(X)^{\vee} \to C(X)^{\vee}$ is bijective.*

Since $\pi_1^{\mathrm{ab}}(X)$ is compact, the injectivity of $\rho_X^{\vee}$ in Theorem 1.2 is deduced from Theorem 1.1 (1). However, our idèle class group $C(X)$ may not be locally compact. We have to determine $\mathrm{Ker}(\rho_X)$ and $\mathrm{Coker}(\rho_X^{\vee})$ independently. The assumption on the rank: $r(\overline{X}) = 0$ in this theorem is purely technical. For the case $r(\overline{X}) > 0$, although we have $\mathrm{Ker}(\rho_X^{\vee}) \simeq (\mathbb{Q}/\mathbb{Z})^r$ from Theorem 1.1, the author does not know if $\rho_X^{\vee}$ is still surjective. From this theorem, under the assumption $r(\overline{X}) = 0$, we have the following one to one correspondence as in the classical class field theory:

$$\{\text{abelian étale covering of } X\} \xleftrightarrow{1:1} \{\text{finite index open subgroup of } C(X)\}.$$

**Contents.** The contents of this note is the following:

- Section 2: We review some definitions and results of class field theory for 2-dimensional local fields following [15] and [17].
- Section 3: For an open curve $X \subset \overline{X}$ over a local field, the idèle class group $C(X)$ and the reciprocity map $\rho_X : C(X) \to \pi_1^{\mathrm{ab}}(X)$ are introduced (Definition 3.2 and Definition 3.3). We also define the fundamental group $\pi_1^{\mathrm{ab}}(X, D)$ as a quotient of $\pi_1^{\mathrm{ab}}(X)$ which classifies abelian étale coverings of $X$ with bounded ramification along a given effective Weil divisor $D$ on $\overline{X}$ (Definition 3.4).
- Section 4: After recalling the unramified class field theory (Theorem 4.1), we study the structure of the tame fundamental group $\pi_1^{\mathrm{t,ab}}(X) = \pi_1^{\mathrm{ab}}(X, X_\infty)$, where $X_\infty = \sum_{x \in \overline{X}\setminus X} 1[x]$ considering as a Weil divisor on $\overline{X}$. Using this structure theorem, we prove Theorem 1.1 (=Theorem 4.6).
- Section 5: Following the proof of the class field theory for curves over *global fields* ([19, Lem. 3], [20, Thm. 9.1]) basically, we show Theorem 1.2 (=Theorem 5.5). By using results in Section 4, the proof is simpler than that of Kato and Saito's.

**Notation.** In this note, a *local field* we mean a complete discrete valuation field with finite residue field. Throughout this note, we use the following notation:

- $p$: a fixed prime number, and
- $\mathbb{N}'$: the set of $m \in \mathbb{Z}_{\geq 1}$ which is prime to $p$.

For a field $F$,

- $\mathrm{char}(F)$: the characteristic of $F$,
- $\overline{F}$: a separable closure of $F$,
- $G_F := \mathrm{Gal}(\overline{F}/F)$: the Galois group of the extension $\overline{F}/F$,
- $F^{\mathrm{ab}}$: the maximal abelian extension of $F$ in $\overline{F}$,
- $G_F^{\mathrm{ab}} := \mathrm{Gal}(F^{\mathrm{ab}}/F)$: the Galois group of $F^{\mathrm{ab}}/F$,
- $H^n_{\mathrm{Gal}}(F, M)$: the Galois cohomology group of $G_F$ with coefficients in a $G_F$-module $M$ (cf. [16]), and
- $K_2(F)$: the Milnor $K$-group of degree 2 which is defined by

$$K_2(F) = \left(F^\times \otimes_{\mathbb{Z}} F^\times\right)/J,$$

  where $J$ is the subgroup generated by elements of the form $a \otimes (1-a)$ ($a \in F^\times$). The element in $K_2(F)$ represented by $a \otimes b \in F^\times \otimes_{\mathbb{Z}} F^\times$ is denoted by $\{a, b\}$ (cf. [24]).

Let $A$ be an abelian group whose operation is written additively. The abelian group $A$ is said to be *divisible* if, for every $n \in \mathbb{Z}_{\geq 1}$ and every

$x \in A$, there exists $y \in A$ such that $ny = x$. The abelian group $A$ is $l$-*divisible* for a prime $l$, if for all $n \in \mathbb{Z}_{\geq 1}$ and every $x \in A$, there exists $y \in A$ such that $l^n y = x$. For $n \in \mathbb{Z}_{\geq 1}$, we use the following notation on $A$:

- $A/n :=$ the cokernel of the map $n : A \to A$ defined by $x \mapsto nx$, and
- $A_{\mathrm{tor}}$: the torsion part of $A$.

When $A$ is a topological abelian group, define

- $A^{\vee}$: the set of all continuous homomorphisms $A \to \mathbb{Q}/\mathbb{Z}$ of *finite order*, where $\mathbb{Q}/\mathbb{Z}$ is given the discrete topology.

A *curve* over a field $F$ means an integral separated scheme of dimension 1 over $\mathrm{Spec}(F)$. For a connected Noetherian scheme $X$, we denote by

- $\pi_1^{\mathrm{ab}}(X)$: the abelianlization of the étale fundamental group of $X$ ([10]) omitting the base point,
- $H^n(X, \mathscr{F})$: the étale cohomology group of an étale sheaf $\mathscr{F}$ on $X$, and
- $H^n_Z(X, \mathscr{F})$: the étale cohomology group of an étale sheaf $\mathscr{F}$ on $X$ with support in $Z$.

An *étale covering* $Y$ of a scheme $X$ means a finite étale morphism $Y \to X$

## 2. Local class field theory

For a field $F$ with $\mathrm{char}(F) = p$, $n \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 1}$, we define

$$H^q_{\mathrm{Gal}}(F, \mathbb{Z}/p^r(n)) := H^{q-n}_{\mathrm{Gal}}(F, W_r\Omega^n_{F,\log}),$$

where $W_r\Omega^n_{F,\log}$ is the Galois module defined by the étale sheaf of the logarithmic part of the de Rham–Witt complex ([13]). Recall that $\mathbb{N}'$ is the set of $m \in \mathbb{Z}_{\geq 1}$ which is prime to $p$ (cf. Notation). For $m \in \mathbb{N}'$, define $\mathbb{Z}/m(0) := \mathbb{Z}/m$ with the trivial action of $G_F$, and $\mathbb{Z}/m(n) := \mu_m(\overline{F})^{\otimes n}$ for $n \geq 1$, where $\mu_m(\overline{F})$ is the Galois module of $m$-th roots of unity in $\overline{F}$. We define (following [17, §3.2, Def. 1])

$$H^0(F) := \varinjlim_{m \in \mathbb{N}'} H^0_{\mathrm{Gal}}(F, \mathbb{Z}/m(-1)),$$

where $\mathbb{Z}/m(-1) := \mathrm{Hom}(\mu_m(\overline{F}), \mathbb{Q}/\mathbb{Z})$ on which $G_F$ acts by $f \mapsto f \circ \sigma^{-1}$ for $\sigma \in G_F, f \in \mathbb{Z}/m(-1)$ (cf. [17, §1.2]). For $n \in \mathbb{Z}_{\geq 1}$,

$$H^n(F) := \varinjlim_{m \in \mathbb{N}'} H^n_{\mathrm{Gal}}(F, \mathbb{Z}/m(n-1)) \oplus \varinjlim_{r \in \mathbb{Z}_{\geq 1}} H^n_{\mathrm{Gal}}(F, \mathbb{Z}/p^r(n-1)).$$

Using these, it is known that we have

$$H^1(F) \simeq (G_F)^{\vee} \simeq (G_F^{\mathrm{ab}})^{\vee} \tag{2.1}$$

(cf. [17, §3.2]; see also [28, Chap. 2]). From this isomorphism, we identify $H^1(F)$ and $(G_F^{\mathrm{ab}})^{\vee}$ in the following.

**2-dimensional local class field theory.** We recall the 2-dimensional local class field theory following [15] and [17]. For detailed expositions on this section, we also recommend [28, Chap. 2].

**Definition 2.1.** A 2-*dimensional local field* is a complete discrete valuation field whose residue field is a local field.

Throughout this section, we fix such a field and use the following notation:

- $K$ : a 2-dimensional local field of $\mathrm{char}(K) = p$,
- $v_K : K^{\times} \to \mathbb{Z}$: the valuation of $K$,
- $O_K := \{f \in K \mid v_K(f) \geq 0\}$ : the valuation ring of $K$,
- $\mathfrak{m}_K := \{f \in K \mid v_K(f) > 0\}$ : the maximal ideal of $O_K$,
- $k := O_K/\mathfrak{m}_K$ : the residue field of $K$,
- $U_K := O_K^{\times}$ : the group of units in $O_K$, and
- $\partial_K : K_2(K) \to k^{\times}$ : the *boundary map* defined by

$$\partial_K(\{f, g\}) := (-1)^{v_K(f)v_K(g)} f^{v_K(g)} g^{-v_K(f)} \bmod \mathfrak{m}_K, \tag{2.2}$$

for $\{f, g\} \in K_2(K)$.

The class field theory of $K$ describes the abelian Galois group $G_K^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{ab}}/K)$ by a canonical homomorphism $\rho_K : K_2(K) \to G_K^{\mathrm{ab}}$ called the *reciprocity map* (defined in [17, §3.2]).

**Proposition 2.2** ([17, §3.2, Exp. 1 and 2]; see also [28, §2.1])**.**

(1) *We have the following commutative diagram*

$$\begin{tikzcd} K_2(K) \arrow[r, "\rho_K"] \arrow[d, "\partial_K"'] & G_K^{\mathrm{ab}} \arrow[d] \\ k^{\times} \arrow[r, "\rho_k"] & G_k^{\mathrm{ab}}, \end{tikzcd}$$

*where $\rho_k$ is the reciprocity map of $k$, and the right vertical map is the restriction.*

(2) *For a finite extension $L/K$, the following diagram is commutative:*

$$\begin{tikzcd} K_2(K) \arrow[r, "\rho_K"] & G_K^{\mathrm{ab}} \\ K_2(L) \arrow[r, "\rho_L"] \arrow[u, "N_{L/K}"] & G_L^{\mathrm{ab}}, \arrow[u, "\mathrm{Res}_{L/K}"'] \end{tikzcd}$$

*where $N_{L/K}$ is the norm map (defined in [17, §1.7]), and the right vertical map $\mathrm{Res}_{L/K}$ is the restriction.*

(3) *For a finite extension $L/K$, the following diagram is commutative:*

$$\begin{tikzcd} K_2(K) \arrow[r, "\rho_K"] \arrow[d, "i_{L/K}"'] & G_K^{\mathrm{ab}} \arrow[d, "\mathrm{Ver}_{L/K}"] \\ K_2(L) \arrow[r, "\rho_L"] & G_L^{\mathrm{ab}}, \end{tikzcd}$$

*where $i_{L/K}$ is the map induced from the inclusion $K \hookrightarrow L$, and the right vertical map $\mathrm{Ver}_{L/K}$ is the transfer map ([27, §1.5]).*

The multiplicative group $K^\times$ and the Milnor $K$-group $K_2(K)$ have good topologies (introduced in [15, §7], see also [28, §2.3]). We omit the detailed exposition on the definitions of these topologies. However, under the topologies, the following properties hold:

- The reciprocity map $\rho_K$ is continuous.
- The unit group $U_K = O_K^\times$ is open in $K^\times$.
- The topology on $K_2(K)$ is given by the strongest topology for the so called *symbol map* $K^\times \times K^\times \to K_2(K); f \otimes g \mapsto \{f, g\}$ is continuous.
- For a finite extension $L/K$, the norm map ([17, §1.7]) $N_{L/K} : K_2(L) \to K_2(K)$ is continuous.

Note also that any continuous homomorphism $K_2(K) \to \mathbb{Q}/\mathbb{Z}$ is automatically of finite order with respect to this topology ([17, §3.5, Rem. 4]). Recall that an element $\chi \in H^1(K) = (G_K^{\mathrm{ab}})^\vee$ (cf. (2.1)) is said to be *unramified* if the corresponding cyclic extension of $K$ is unramified.

**Theorem 2.3** ([17, §3.1, 3.5], [29, Chap. I, Thm. 3.1])**.** *The reciprocity map $\rho_K$ satisfies the following:*

(1) *The map $\rho_K$ induces an isomorphism $\rho_K^\vee : H^1(K) \xrightarrow{\simeq} K_2(K)^\vee$.*
(2) *An element $\chi \in H^1(K)$ is unramified if and only if $\rho_K^\vee(\chi)$ annihilates $U^0K_2(K) := \mathrm{Ker}(\partial_K)$.*

We denote by $I_K$ the ineria subgroup of $G_K^{\mathrm{ab}}$ which is defined by the kernel of the restriction $G_K^{\mathrm{ab}} \to G_k^{\mathrm{ab}}$. For any $m \in \mathbb{Z}_{\geq 1}$, the reciprocity map $\rho_K$ induces $\rho_{K,m} : K_2(K)/m \to G_K^{\mathrm{ab}}/m$. Theorem 2.3(1) implies that the dual of this homomorphism

$$\rho_{K,m}^\vee : (G_K^{\mathrm{ab}}/m)^\vee = H^1(K, \mathbb{Z}/m) \xrightarrow{\simeq} (K_2(K)/m)^\vee \tag{2.3}$$

is bijective. The following theorem says that $\rho_{K,m}$ is injective for each $m \in \mathbb{Z}_{\geq 1}$.

**Theorem 2.4** ([6, Thm. 4.5], see also [5, Thm. 2])**.** $\mathrm{Ker}(\rho_K)$ *is divisible.*

**Ramification theory.** For $m \in \mathbb{Z}_{\geq 1}$, let $U_K^m = 1 + \mathfrak{m}_K^m$ be the higher unit groups of $K$. Denote by $U^m K_2(K)$ the subgroup of $K_2(K)$ generated by the image of $U_K^m \times K^\times$ in $K_2(K)$ by the symbol map. We also have an increasing filtration $\{\mathrm{fil}_m H^q(K)\}_{m\in\mathbb{Z}_{\geq 0}}$ on $H^q(K)$ ([18, Def. 2.1]) with $H^q(K) = \cup_{m\in\mathbb{Z}_{\geq 0}} \mathrm{fil}_m H^q(K)$. In particular, we have $\mathrm{fil}_0 H^1(K) \simeq H^1(k) \oplus H^0(k)$ and this subgroup corresponding to tamely ramified abelian extensions of $K$ ([17, Thm. 3], [18, Prop. 6.1]). This filtration on $H^1(K)$ induces the ramification filtration $\{I_K^m\}_{m\in\mathbb{Z}_{\geq 0}}$ on $G_K^{\mathrm{ab}}$, which is defined by $I_K^0 := I_K$ and

$$I_K^m := \{\sigma \in G_K^{\mathrm{ab}} \,|\, \chi(\sigma) = 0 \text{ for all } \chi \in \mathrm{fil}_{m-1} H^1(K)\}$$

for $m \geq 1$. The description of $\mathrm{fil}_0 H^1(K)$ implies that $I_K^m \subset I_K = I_K^0$ for $m \geq 1$ and $I_K^1$ is the wild inertia subgroup of $G_K^{\mathrm{ab}}$, that is, the maximal pro-$p$ subgroup of the inertia subgroup $I_K$.

**Proposition 2.5** ([18, Prop. 6.5, see also Rem. 6.6])**.** *For $\chi \in H^1(K)$, $\chi$ is in $\mathrm{fil}_m H^1(K)$ if and only if $\rho_K^\vee(\chi) \in K_2(K)^\vee$ annihilates $U^{m+1}K_2(K)$.*

From Proposition 2.5, $\rho_K$ induces $U^m K_2(K) \to I_K^m$ for $m \in \mathbb{Z}_{\geq 0}$. In our case of $\mathrm{char}(K) = p$, it is known $I_K^{m+1} = G_{K,\log}^{\mathrm{ab},m+}$ for any $m \in \mathbb{Z}_{\geq 0}$, where the right is the induced group from Abbes–Saito's logarithmic version of ramification subgroups on the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ ([1], see also [2, Cor. 9.12]).

## 3. Curves over local fields

Let $k$ be a local field of $\mathrm{char}(k) = p$ (cf. Notation).

**Definition 3.1.** We call the pair $X \subset \overline{X}$ of

- $\overline{X}$: a smooth, proper and geometrically connected curve over $k$, and
- $X$: a nonempty open subscheme of $\overline{X}$

an *open curve* over $k$.

Since the smooth compactification $\overline{X}$ of a smooth curve $X$ is unique if it exists by the valuative criterion of properness, we often omit $\overline{X}$ and write $X$ solely as an open curve in the above sense.

For an open curve $X$ over $k$, we also define

- $X_\infty := \overline{X} \setminus X$,
- $X_0$: the set of closed points in $X$, and
- $k(X)$: the function field of $X$.

For a closed point $x \in \overline{X}_0$, we denote by

- $k(x)$ : the residue field at $x$ which is a finite extension of $k$, and
- $k(X)_x$ : the completion of $k(X)$ at $x$ which is a 2-dimensional local field (Definition 2.1) with residue field $k(x)$.

**Idèle class groups.** We fix an open curve $X$ over $k$ and introduce the idèle class group and the reciprocity map for $X$.

**Definition 3.2.** The *idèle class group* $C(X)$ is defined to be the cokernel of

$$\partial : K_2(k(X)) \longrightarrow \bigoplus_{x\in X_0} k(x)^\times \oplus \bigoplus_{x\in X_\infty} K_2(k(X)_x)$$

which is given by the direct sum of the following homomorphisms:

- the boundary map $\partial_x := \partial_{k(X)_x} : K_2(k(X)_x) \to k(x)^\times$ (cf. (2.2)) for $x \in X_0$, and
- $K_2(k(X)) \to K_2(k(X)_x)$ induced by the inclusion $i_x : k(X) \hookrightarrow k(X)_x$ for $x \in X_\infty$.

The restricted product $\prod_{x\in\overline{X}_0} K_2(k(X)_x)$ with respect to the closed subgroup $U^0K_2(k(X)_x) = \ker(\partial_x)$ has a structure of a topological group induced from the topology on $K_2(k(X)_x)$ (cf. Section 2) as in the classical class field theory (cf. [29, Chap. I, §3]). The idèle class group $C(X)$ is a quotient of $\prod_{x\in\overline{X}_0} K_2(k(X)_x)$ and is endowed with the quotient topology.

The abelian fundamental group $\pi_1^{\mathrm{ab}}(X)$ has a description as a Galois group: we have $\pi_1^{\mathrm{ab}}(X) \simeq \mathrm{Gal}(k(X)^{\mathrm{ur}}/k(X))$, where $k(X)^{\mathrm{ur}}$ is generated by all finite separable extensions $E$ of $k(X)$ contained in $k(X)^{\mathrm{ab}}$ satisfying that the normalization $\widetilde{X}^E \to X$ of $X$ in $E$ is unramified (cf. [10, Exp. V, 8.2]). In particular, we have $k(X)^{\mathrm{ur}} \subset k(X)^{\mathrm{ab}}$ so that the restriction gives $G^{\mathrm{ab}}_{k(X)} = \mathrm{Gal}(k(X)^{\mathrm{ab}}/k(X)) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X)$. The 2-dimensional local class field theory $\rho_{k(X)_x} : K_2(k(X)_x) \to G^{\mathrm{ab}}_{k(X)_x}$ and the restriction $G^{\mathrm{ab}}_{k(X)_x} \to G^{\mathrm{ab}}_{k(X)}$ induce a continuous homomorphism

$$\prod_{x\in\overline{X}_0} K_2(k(X)_x) \longrightarrow G^{\mathrm{ab}}_{k(X)} \longrightarrow\!\!\!\!\rightarrow \pi_1^{\mathrm{ab}}(X).$$

By the reciprocity law of $k(X) = k(\overline{X})$ ([29, Chap. II, Prop. 1.2]) and the 2-dimensional local class field theory (Theorem 2.3), this factors through $C(X)$.

**Definition 3.3.** The induced continuous homomorphism $\rho_X : C(X) \to \pi_1^{\mathrm{ab}}(X)$ is called the *reciprocity map* of $X$.

We denote by

$$\mathrm{Coker}_{\mathrm{top}}(\rho_X) := \pi_1^{\mathrm{ab}}(X)/\overline{\mathrm{Im}(\rho_X)}, \tag{3.1}$$

where $\overline{\mathrm{Im}(\rho_X)}$ is the topological closure of $\mathrm{Im}(\rho_X)$.

The norm map $N_{k(x)/k} : k(x)^\times \to k^\times$ for $x \in X_0$ and the composition $N_{k(x)/k} \circ \partial_x : K_2(k(X)_x) \to k^\times$ for $x \in X_\infty$ induce a homomorphism

$N_X : C(X) \to k^\times$. They make the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & C(X)^0 \arrow[r] \arrow[d, "\rho_X^0"] & C(X) \arrow[r, "N_X"] \arrow[d, "\rho_X"] & k^\times \arrow[d, "\rho_k"] & \\ 0 \arrow[r] & \pi_1^{\mathrm{ab}}(X)^0 \arrow[r] & \pi_1^{\mathrm{ab}}(X) \arrow[r, "\varphi"] & G_k^{\mathrm{ab}} \arrow[r] & 0, \end{tikzcd} \tag{3.2}$$

where $\varphi$ is the induced homomorphism from the structure morphism $X \to \mathrm{Spec}(k)$ ([9, §3.3]) and the groups $C(X)^0$ and $\pi_1^{\mathrm{ab}}(X)^0$ are defined by the exactness of the horizontal rows.

**Restricted ramification.** For the open curve $X$, let $D = \sum_{x\in X_\infty} m_x[x]$ be an effective Weil divisor on $\overline{X}$ with support $|D| \subset X_\infty = \overline{X} \setminus X$.

**Definition 3.4.** By putting $m_x = 0$ if $x \notin |D|$, we define the abelian fundamental group $\pi_1^{\mathrm{ab}}(X, D)$ with bounded ramification by

$$\pi_1^{\mathrm{ab}}(X, D) = \mathrm{Coker}\left(\bigoplus_{x\in X_\infty} I^{m_x}_{k(X)_x} \hookrightarrow \bigoplus_{x\in X_\infty} G^{\mathrm{ab}}_{k(X)_x} \longrightarrow \pi_1^{\mathrm{ab}}(X)\right),$$

where $I^{m_x}_{k(X)_x}$ is the ramification subgroup of $G^{\mathrm{ab}}_{k(X)_x} = \mathrm{Gal}(k(X)_x^{\mathrm{ab}}/k(X)_x)$ (Section 2).

By Proposition 2.5, the composition $C(X) \overset{\rho_X}{\to} \pi_1^{\mathrm{ab}}(X) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D)$ factors through

$$C(X, D) := \mathrm{Coker}\left(\bigoplus_{x\in X_\infty} U^{m_x} K_2(k(X)_x) \longrightarrow C(X)\right)$$

and the induced homomorphism is denoted by $\rho_{X,D} : C(X, D) \to \pi_1^{\mathrm{ab}}(X, D)$. Furthermore, the norm maps $N_{k(x)/k} : k(x)^\times \to k^\times$ define $N_{X,D} : C(X, D) \to k^\times$ and the following diagram is commutative as in (3.2):

$$\begin{tikzcd} 0 \arrow[r] & C(X,D)^0 \arrow[r] \arrow[d, "\rho_{X,D}^0"] & C(X,D) \arrow[r, "N_{X,D}"] \arrow[d, "\rho_{X,D}"] & k^\times \arrow[d, "\rho_k"] & \\ 0 \arrow[r] & \pi_1^{\mathrm{ab}}(X,D)^0 \arrow[r] & \pi_1^{\mathrm{ab}}(X,D) \arrow[r] & G_k^{\mathrm{ab}} \arrow[r] & 0. \end{tikzcd} \tag{3.3}$$

Here, the groups $C(X, D)^0$ and $\pi_1^{\mathrm{ab}}(X, D)^0$ are defined by the exactness of the horizontal rows.

Consider $X_\infty = \sum_{x\in X_\infty} 1[x]$ as a Weil divisor. Recalling that $I^1_{k(X)_x}$ is the wild inertia subgroup, the groups

$$\pi_1^{\mathrm{t,ab}}(X) := \pi_1^{\mathrm{ab}}(X, X_\infty), \quad \text{and} \quad \pi_1^{\mathrm{t,ab}}(X)^0 := \pi_1^{\mathrm{ab}}(X, X_\infty)^0 \tag{3.4}$$

classify *tame coverings* of $X$, that is, finite étale coverings over $X$ and ramify at most tamely along the boundary $X_\infty$. We also employ the following notation:

$$\begin{aligned} &\rho_X^{\mathrm{t}} := \rho_{X,X_\infty} : C^{\mathrm{t}}(X) := C(X, X_\infty) \longrightarrow \pi_1^{\mathrm{t,ab}}(X), \\ \text{and} \quad &\rho_X^{\mathrm{t},0} := \rho_{X,X_\infty}^0 : C^{\mathrm{t}}(X)^0 := C(X, X_\infty)^0 \longrightarrow \pi_1^{\mathrm{t,ab}}(X)^0. \end{aligned} \tag{3.5}$$

**Functorial properties.** We define the pullback and the norm homomorphism on the idèle class groups with respect to étale coverings of open curves in the following sense.

**Definition 3.5.** An *étale covering* $f : Y \to X$ of open curves is defined to be the commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "f"'] & \overline{Y} \arrow[d, "\overline{f}"] & Y_\infty \arrow[l, hook'] \arrow[d] \\ X \arrow[r, hook] & \overline{X} & X_\infty, \arrow[l, hook'] \end{tikzcd} \tag{3.6}$$

where the horizontal maps are the inclusions, $\overline{f}$ is a morphism of schemes over $\mathrm{Spec}(k)$ and, $f$ is an étale covering (that is, a finite étale morphism of schemes, cf. Notation in Section 1) over $\mathrm{Spec}(k)$. The right commutative square in (3.6) means $\overline{f}(Y_\infty) \subset X_\infty$.

In the following, we fix an étale covering $f : Y \to X$ of open curves in the above sense.

**Definition 3.6.** We define a canonical homomorphism $i_{Y/X} := f^* : C(X) \to C(Y)$ as follows:

- For $x \in X_0$ and $y \in Y_0$ with $f(y) = x$, the inclusion $k(x) \hookrightarrow k(y)$ gives $i_{k(y)/k(x)} : k(x)^\times \hookrightarrow k(y)^\times$.
- For $x \in X_\infty$, and $y \in Y_\infty$ with $\overline{f}(y) = x$, the inclusion map $k(X)_x \hookrightarrow k(Y)_y$ gives $i_{k(Y)_y/k(X)_x} : K_2(k(X)_x) \to K_2(k(Y)_y)$.

These maps give a canonical homomorphism

$$\bigoplus_{x \in X_0} k(x)^\times \oplus \bigoplus_{x \in X_\infty} K_2(k(X)_x) \longrightarrow \bigoplus_{y \in Y_0} k(y)^\times \oplus \bigoplus_{y \in Y_\infty} K_2(k(Y)_y).$$

Since the homomorphism $K_2(k(X)) \to K_2(k(Y))$ induced from $k(X) \hookrightarrow k(Y)$ is compatible with above homomorphisms, we obtain $i_{Y/X}$.

**Definition 3.7.** We define the *norm map* $N_{Y/X} := f_* : C(Y) \to C(X)$ as follows:

- For $y \in Y_0$ with $x = f(y)$, we have the norm homomorphism $N_{k(y)/k(x)} : k(y)^\times \to k(x)^\times$.
- For $y \in Y_\infty$ with $x = \overline{f}(y)$, we have the norm map $N_{k(Y)_y/k(X)_x} : K_2(k(Y)_y) \to K_2(k(X)_x)$.

These maps give a canonical homomorphism

$$\bigoplus_{y\in Y_0} k(y)^\times \oplus \bigoplus_{y\in Y_\infty} K_2(k(Y)_y) \longrightarrow \bigoplus_{x\in X_0} k(x)^\times \oplus \bigoplus_{x\in X_\infty} K_2(k(X)_x).$$

Since the norm $N_{k(Y)/k(X)} : K_2(k(Y)) \to K_2(k(X))$ is compatible with above norms, we obtain $N_{Y/X}$.

**Lemma 3.8.** *We have $N_{Y/X} \circ i_{Y/X} = [k(Y) : k(X)] \cdot \mathrm{id}_{C(X)}$, where $\mathrm{id}_{C(X)}$ is the identity map of $C(X)$.*

*Proof.* The projection formula of the Milnor $K$-groups (e.g. [25, §14]) gives

$$N_{k(Y)_y/k(X)_x} \circ i_{k(Y)_y/k(X)_x} = [k(Y)_y : k(X)_x] \cdot \mathrm{id}_{K_2(k(X)_x)} .$$

The assertion follows from the equality

$$[k(Y) : k(X)] = \sum_{y\in \overline{f}^{-1}(x)} [k(Y)_y : k(X)_x]$$

for a closed point $x \in \overline{X}_0$ ([31, Chap. I, §4, Prop. 10]). □

From the construction of $\rho_X$ and the properties of $\rho_{K_x}$ for each $x \in \overline{X}_0$ given in Proposition 2.2, we obtain the following commutative diagrams:

$$(3.7)\qquad \begin{array}{ccc} C(X) & \xrightarrow{\rho_X} & \pi_1^{\mathrm{ab}}(X) \\ {\scriptstyle N_{Y/X}}\uparrow & & \uparrow{\scriptstyle \varphi} \\ C(Y) & \xrightarrow{\rho_Y} & \pi_1^{\mathrm{ab}}(Y) \end{array} \quad \text{and} \quad \begin{array}{ccc} C(X) & \xrightarrow{\rho_X} & \pi_1^{\mathrm{ab}}(X) \\ {\scriptstyle i_{Y/X}}\downarrow & & \downarrow{\scriptstyle \psi} \\ C(Y) & \xrightarrow{\rho_Y} & \pi_1^{\mathrm{ab}}(Y) \end{array}$$

where $\varphi$ is the induced homomorphism of the fundamental groups from $f$ and $\psi$ is given by the transfer map.

## 4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using the following notation:

- $k$: a local field of $\mathrm{char}(k) = p > 0$, and
- $X \subset \overline{X}$: an open curve over $k$ in the sense of Definition 3.1.

**Unramified class field theory.** We recall the class field theory for the projective smooth curve $\overline{X}$ following [29] and [34]. Note that the idèle class groups $C(\overline{X})$ and $C(\overline{X})^0$ are denoted by $SK_1(\overline{X})$ and $V(\overline{X})$ respectively in op. cit.

**Theorem 4.1** ([29, Chap. II, Thm. 2.6, 5.1, Prop. 3.5, and Thm. 4.1], [34, Thm. 5.1])**.** *For the reciprocity map $\rho_{\overline{X}} : C(\overline{X}) \to \pi_1^{\mathrm{ab}}(\overline{X})$, we have:*

1. $\mathrm{Coker}_{\mathrm{top}}(\rho_{\overline{X}}) \simeq \widehat{\mathbb{Z}}^{r(\overline{X})}$, *where $r(\overline{X})$ is the rank of $\overline{X}$ (cf. Section 1),*
2. $\mathrm{Ker}(\rho_{\overline{X}})$ *and* $\mathrm{Ker}(\rho^0_{\overline{X}})$ *are the maximal divisible subgroups of $C(\overline{X})$ and $C(\overline{X})^0$ respectively,*

(3) $\#\operatorname{Im}(\rho^0_{\overline{X}}) < \infty$, *and* $\operatorname{Coker}(\rho^0_{\overline{X}}) \simeq \widehat{\mathbb{Z}}^{r(\overline{X})}$.

Theorem 4.1 gives the structure of the fundamental group $\pi_1^{\mathrm{ab}}(\overline{X})^0$ as in the following short exact sequence:

$$0 \longrightarrow \pi_1^{\mathrm{ab}}(\overline{X})^0_{\mathrm{tor}} = \operatorname{Im}(\rho^0_{\overline{X}}) \longrightarrow \pi_1^{\mathrm{ab}}(\overline{X})^0 \longrightarrow \widehat{\mathbb{Z}}^{r(\overline{X})} \longrightarrow 0, \tag{4.1}$$

where $\pi_1^{\mathrm{ab}}(\overline{X})^0_{\mathrm{tor}}$ is the torsion part of $\pi_1^{\mathrm{ab}}(\overline{X})^0$ which is finite.

**Tame fundamental groups.** The goal of this paragraph is to determine the structure of the abelian tame fundamental group $\pi_1^{\mathrm{t,ab}}(X) = \pi_1^{\mathrm{ab}}(X, X_\infty)$ (cf. (3.4)) as in (4.1).

**Theorem 4.2.** $\operatorname{Coker}_{\mathrm{top}}(\rho_X) \simeq \widehat{\mathbb{Z}}^{r(\overline{X})}$, *where* $r(\overline{X})$ *is the rank of* $\overline{X}$.

*Proof.* For any $x \in X_\infty = \overline{X} \setminus X$, put $Y := \operatorname{Spec}(\mathscr{O}^\wedge_{\overline{X},x})$, where $\mathscr{O}^\wedge_{\overline{X},x}$ is the completion of the local ring $\mathscr{O}_{\overline{X},x}$. The localization sequence of the étale cohomology groups on $i : x \hookrightarrow Y$ ([9, Prop. 5.6.12]) gives an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(Y, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(\operatorname{Spec}(k(X)_x), \mathbb{Q}/\mathbb{Z}) \\ \longrightarrow H^2_x(Y, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^2(Y, \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

In terms of the Galois cohomology groups ([9, Prop. 5.7.8]), we have

$$H^n(Y, \mathbb{Q}/\mathbb{Z}) \xrightarrow[i^*]{\simeq} H^n(x, \mathbb{Q}/\mathbb{Z}) \simeq H^n_{\mathrm{Gal}}(k(x), \mathbb{Q}/\mathbb{Z})$$

(here, the first isomorphism follows from [12, Thm. 0.1]) and

$$H^n(\operatorname{Spec}(k(X)_x), \mathbb{Q}/\mathbb{Z}) \simeq H^n_{\mathrm{Gal}}(k(X)_x, \mathbb{Q}/\mathbb{Z}).$$

By the Tate duality theorem for local fields ([27, Thm. 7.2.6]) for prime to the $p$-part and the dimension reason ([27, Prop. 6.5.10]) for the $p$-part, we have

$$H^2_{\mathrm{Gal}}(k(x), \mathbb{Q}/\mathbb{Z}) = 0. \tag{4.2}$$

The excision theorem induces an isomorphism $H^2_x(Y, \mathbb{Q}/\mathbb{Z}) \simeq H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z})$ (cf. [9, Prop. 5.6.12]). We also have $H^1_{\mathrm{Gal}}(k(X)_x, \mathbb{Q}/\mathbb{Z}) \simeq H^1(k(X)_x)$ (cf. (2.1)). Thus, we obtain the commutative diagram below:

$$\begin{tikzcd} 0 \arrow[r] & H^1(k(x)) \arrow[r] \arrow[d, "\rho^\vee_{k(x)}", "\simeq"'] & H^1(k(X)_x) \arrow[r] \arrow[d, "\rho^\vee_{k(X)_x}", "\simeq"'] & H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, dashed, "\phi_x"] & 0 \\ 0 \arrow[r] & (k(x)^\times)^\vee \arrow[r, "\partial^\vee_x"] & K_2(k(X)_x)^\vee \arrow[r] & U^0K_2(k(X)_x)^\vee, & \end{tikzcd} \tag{4.3}$$

where $\rho_{k(x)}$ and $\rho_{k(X)_x}$ are the reciprocity maps of $k(x)$ and $k(X)_x$ respectively (Theorem 2.3). Here, the bottom sequence is exact.

Recall that there exist a canonical isomorphism

$$H^1(X, \mathbb{Z}/m) \simeq (\pi_1^{\mathrm{ab}}(X)/m)^\vee \tag{4.4}$$

for each $m \in \mathbb{Z}_{\geq 1}$, and

$$H^1(X, \mathbb{Q}/\mathbb{Z}) \simeq \pi_1^{\mathrm{ab}}(X)^\vee \tag{4.5}$$

([4, Exp. 1, §2.2.1], or [9, Prop. 5.7.20]). Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\rho_{\overline{X}}^\vee"] & H^1(X, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\rho_X^\vee"] & \bigoplus_{x \in X_\infty} H^2_x(\overline{X}, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\oplus \phi_x"] \\ 0 \arrow[r] & C(\overline{X})^\vee \arrow[r] & C(X)^\vee \arrow[r, "i"] & \bigoplus_{x \in X_\infty} U^0 K_2(k(X)_x)^\vee, \end{tikzcd} \tag{4.6}$$

where the map $i$ is induced from the composition

$$U^0 K_2(k(X)_x) \hookrightarrow K_2(k(X)_x) \longrightarrow C(X)$$

for each $x \in X_\infty$. Here, the upper horizontal sequence is the localization sequence associated to $X_\infty \hookrightarrow \overline{X}$. The diagram (4.6) gives $\mathrm{Ker}(\rho_{\overline{X}}^\vee) \simeq \mathrm{Ker}(\rho_X^\vee)$. By Theorem 4.1 (1), we obtain

$$\widehat{\mathbb{Z}}^{r(\overline{X})} \simeq \mathrm{Coker}_{\mathrm{top}}(\rho_{\overline{X}}) \simeq \mathrm{Ker}(\rho_{\overline{X}}^\vee)^\vee \simeq \mathrm{Ker}(\rho_X^\vee)^\vee \simeq \mathrm{Coker}_{\mathrm{top}}(\rho_X).$$

The assertion follows from this. $\square$

For any effective Weil divisor $D$ on $\overline{X}$ whose support $|D| \subset X_\infty$, we have canonical surjective homomorphisms

$$\pi_1^{\mathrm{ab}}(X) \twoheadrightarrow \pi_1^{\mathrm{ab}}(X, D) \twoheadrightarrow \pi_1^{\mathrm{ab}}(\overline{X})$$

from the very definition of $\pi_1^{\mathrm{ab}}(X, D)$ (Definition 3.4). The above Theorem 4.2 and Theorem 4.1 (1) imply also

$$\mathrm{Coker}_{\mathrm{top}}(\rho_{X,D}) := \pi_1^{\mathrm{ab}}(X, D)/\overline{\mathrm{Im}(\rho_{X,D})} \simeq \widehat{\mathbb{Z}}^{r(\overline{X})}. \tag{4.7}$$

**Lemma 4.3.** *For the map $\rho_X^{\mathrm{t},0} : C^{\mathrm{t}}(X)^0 \to \pi_1^{\mathrm{t,ab}}(X)^0$ (defined in (3.5)), we have $\#\,\mathrm{Im}(\rho_X^{\mathrm{t},0}) < \infty$.*

*Proof.* For each $x \in X_\infty$, let $I_{k(X)_x} = I^0_{k(X)_x}$ be the inertia subgroup of $G^{\mathrm{ab}}_{k(X)_x}$, that is, the kernel of the restriction $G^{\mathrm{ab}}_{k(X)_x} \to G^{\mathrm{ab}}_{k(x)}$. Theorem 2.3 and Proposition 2.5 imply that $\rho_{k(X)_x}$ induces

$$U^0 K_2(k(X)_x)/U^1 K_2(k(X)_x) \to I^0_{k(X)_x}/I^1_{k(X)_x}.$$

This gives the following commutative diagram with exact rows:

$$\begin{CD}
\bigoplus_{x\in X_\infty} U^0K_2(k(X)_x)/U^1K_2(k(X)_x) @>>> C^{\mathrm{t}}(X)^0 @>>> C(\overline{X})^0 @>>> 0\\
@VVV @VV{\rho_X^{\mathrm{t},0}}V @VV{\rho^0_{\overline{X}}}V\\
\bigoplus_{x\in X_\infty} I^0_{k(X)_x}/I^1_{k(X)_x} @>>> \pi_1^{\mathrm{t,ab}}(X)^0 @>{\varphi}>> \pi_1^{\mathrm{ab}}(\overline{X})^0 @>>> 0,
\end{CD}$$

where $\varphi$ is the induced homomorphism from the open immersion $X \hookrightarrow \overline{X}$. For each $x \in X_\infty$, we have

- $\partial_x : K_2(k(X)_x)/U^0K_2(k(X)_x) \xrightarrow{\simeq} k(x)^\times$ (by $U^0K_2(k(X)_x) = \mathrm{Ker}(\partial_x)$), and
- $K_2(k(X)_x)/U^1K_2(k(X)_x) \simeq K_2(k(x)) \oplus k(x)^\times$ (cf. [7, Chap. IX, Prop. 2.2]).

These isomorphisms give $U^0K_2(k(X)_x)/U^1K_2(k(X)_x) \simeq K_2(k(x))$. It is known that $K_2(k(x))$ is the sum of a finite group and a divisible subgroup ([7, Chap. IX, Thm. 4.3]). By Theorem 2.3, $\rho^\vee_{k(X)_x}$ induces an injective homomorphism $(I^0_{k(X)_x}/I^1_{k(X)_x})^\vee \hookrightarrow (U^0K_2(k(X)_x)/U^1K_2(k(X)_x))^\vee$. Therefore, the quotient $I^0_{k(X)_x}/I^1_{k(X)_x}$ is finite and so is $\mathrm{Ker}(\varphi)$. The assertion $\#\,\mathrm{Im}(\rho_X^{\mathrm{t},0}) < \infty$ follows from $\#\,\mathrm{Im}(\rho^0_{\overline{X}}) < \infty$ (Theorem 4.1 (3)). ☐

From Lemma 4.3 and (4.7), we have a short exact sequence

$$0 \longrightarrow \pi_1^{\mathrm{t,ab}}(X)^0_{\mathrm{tor}} = \mathrm{Im}(\rho_X^{\mathrm{t},0}) \longrightarrow \pi_1^{\mathrm{t,ab}}(X)^0 \longrightarrow \widehat{\mathbb{Z}}^{r(\overline{X})} \longrightarrow 0. \tag{4.8}$$

**Open curves.** The rest of this section is devoted to show Theorem 1.1 (2) (=Theorem 4.6 below). Recall $\mathbb{N}' = \{m \in \mathbb{Z}_{\geq 1} \,|\, m \text{ is prime to } p\}$ and the reciprocity map $\rho_X$ induces $\rho_{X,m} : C(X)/m \to \pi_1^{\mathrm{ab}}(X)/m$ for each $m \in \mathbb{Z}_{\geq 1}$.

**Lemma 4.4.** *For any $m \in \mathbb{N}'$, $\rho_{X,m} : C(X)/m \to \pi_1^{\mathrm{ab}}(X)/m$ is injective.*

*Proof.* For any $m \in \mathbb{N}'$, we have $H^3_c(X, \mathbb{Z}/m(2)) = H^3(\overline{X}, j_!\mathbb{Z}/m(2))$ ([9, §7.4]), where $\mathbb{Z}/m(n) = \mu_m^{\otimes n}$ and $j : X \hookrightarrow \overline{X}$ is the open immersion. We define a commutative diagram:

$$\begin{CD}
K_2(k(X))/m @>>> \bigoplus_{x\in X_0} k(x)^\times/m \oplus \bigoplus_{x\in X_\infty} K_2(k(X)_x)/m @>>> C(X)/m @>>> 0\\
@V{h}VV @VVV @VVV\\
H^2_{\mathrm{Gal}}(k(X), \mathbb{Z}/m(2)) @>>> \bigoplus_{x\in \overline{X}_0} H^3_x(\overline{X}, j_!\mathbb{Z}/m(2)) @>>> H^3(\overline{X}, j_!\mathbb{Z}/m(2))\,.
\end{CD}$$

Here, the horizontal sequences are exact, and the left vertical map $h$ is bijective by the Merkurjev–Suslin theorem [21]. The middle vertical map is also bijective from the following facts:

- For $x \in X_0$, the Kummer theory gives

$$K_1(k(x))/m \xrightarrow{\simeq} H^1_{\mathrm{Gal}}(k(x), \mathbb{Z}/m(1)) \simeq H^3_x(\overline{X}, j_!\mathbb{Z}/m(2)),$$

where the latter isomorphism follows from the excision theorem ([9, Prop. 5.6.12]): $H^3_x(\overline{X}, j_!\mathbb{Z}/m(2)) \simeq H^3_x(X, \mathbb{Z}/m(2))$, the purity theorem ([9, Cor. 8.5.6]) for the closed immersion $i : x \hookrightarrow X$:

$$R^t i^! \mathbb{Z}/m(2) = \begin{cases} 0 & \text{if } t \neq 2, \\ i^*\mathbb{Z}/m(1) & \text{if } t = 2, \end{cases}$$

and the Leray spectral sequence ([9, Prop. 5.6.11]):

$$E_2^{s,t} = H^s(x, R^t i^! \mathbb{Z}/m(2)) \Rightarrow H^{s+t}_x(X, \mathbb{Z}/m(2)).$$

- For $x \in X_\infty$, the Merkurjev–Suslin theorem again gives

$$K_2(k(X)_x)/m \xrightarrow{\simeq} H^2_{\mathrm{Gal}}(k(X)_x, \mathbb{Z}/m(2)) \simeq H^3_x(\overline{X}, j_!\mathbb{Z}/m(2)).$$

Here, the latter isomorphism is given by the excision theorem (see [22, Chap. III, Cor. 1.28]):

$$H^3_x(\overline{X}, j_!\mathbb{Z}/m(2)) \simeq H^3_x(\mathrm{Spec}(\mathscr{O}^h_{\overline{X},x}), j_!\mathbb{Z}/m(2)),$$

and [23, Chap. II, Prop. 1.1]:

$$\begin{aligned} H^3_x(\mathrm{Spec}(\mathscr{O}^h_{\overline{X},x}), j_!\mathbb{Z}/m(2)) &\simeq H^2_{\mathrm{Gal}}(k(X)^h_x, \mathbb{Z}/m(2)) \\ &\simeq H^2_{\mathrm{Gal}}(k(X)_x, \mathbb{Z}/m(2)), \end{aligned}$$

where $\mathscr{O}^h_{\overline{X},x}$ is the henselization of the local ring $\mathscr{O}_{\overline{X},x}$, and $k(X)^h_x$ is its fraction field. Thus, the induced homomorphism $C(X)/m \to H^3_c(X, \mathbb{Z}/m(2))$ is injective from the above diagram. By the duality theorem ([30]) through $(\pi_1^{\mathrm{ab}}(X)/m)^\vee \simeq H^1(X, \mathbb{Z}/m)$ (cf. (4.4)), we have $\pi_1^{\mathrm{ab}}(X)/m \simeq H^3_c(X, \mathbb{Z}/m(2))$ so that $\rho_{X,m} : C(X)/m \to \pi_1^{\mathrm{ab}}(X)/m$ is injective. □

Before proving Theorem 1.1 (2) (=Theorem 4.6 below), we prepare some notation (following [8, §3]) and quote a lemma from [14]. For a set of primes $\mathbb{L}$ with $p \notin \mathbb{L}$, define

- $\mathbb{N}(\mathbb{L}) := \{m \in \mathbb{N}' \mid \text{the prime divisors of } m \text{ are in } \mathbb{L}\}$ as a sub monoid of $\mathbb{N}'$.

For an abelian group $G$, the natural surjective homomorphisms $G \to G/m$ for $m \in \mathbb{N}(\mathbb{L})$ induces a homomorphism

$$\phi_{G,\mathbb{L}} : G \longrightarrow G_{\mathbb{L}} := \varprojlim_{m \in \mathbb{N}(\mathbb{L})} G/m. \tag{4.9}$$

**Lemma 4.5** ([14, Lem. 7.7])**.** *Let $A$ be an abelian group, $\{B_m\}_{m\in\mathbb{N}(\mathbb{L})}$ a projective system of abelian groups, and a morphism $\{\varphi_m : A/m \to B_m\}_{m\in\mathbb{N}(\mathbb{L})}$ of the projective systems. Put $B_{\mathbb{L}} := \varprojlim_{m\in\mathbb{N}(\mathbb{L})} B_m$. If we assume that*

(1) *$\varphi_m$ is injective for all $m \in \mathbb{N}(\mathbb{L})$, and*
(2) *there exists $N \in \mathbb{N}(\mathbb{L})$ such that $N \cdot (B_{\mathbb{L}})_{\mathrm{tor}} = 0$,*

*then $\mathrm{Ker}(\phi_{A,\mathbb{L}} : A \to A_{\mathbb{L}})$ is $l$-divisible for any prime $l \in \mathbb{L}$.*

**Theorem 4.6.** *Let $k$ be a local field of $\mathrm{char}(k) = p$, and $X \subset \overline{X}$ an open curve over $k$. Then $\mathrm{Ker}(\rho_X)$ is the maximal $l$-divisible subgroup of $C(X)$ for all prime number $l \neq p$.*

*Proof.* Since any profinite group does not contain non-trivial divisible elements, it is enough to show that, for any set of primes $\mathbb{L}$ with $p \notin \mathbb{L}$, $\mathrm{Ker}(\rho_X)$ is $l$-divisible for all $l \in \mathbb{L}$. From Lemma 4.4, we have an injective homomorphism $\rho_{X,\mathbb{L}} := \varprojlim_{m\in\mathbb{N}(\mathbb{L})} \rho_{X,m} : C(X)_{\mathbb{L}} \hookrightarrow \pi_1^{\mathrm{ab}}(X)_{\mathbb{L}}$ which commutes with $\rho_X$ as in the following commutative diagram:

$$\begin{tikzcd} C(X) \arrow[r, "\rho_X"] \arrow[d, "\psi"'] & \pi_1^{\mathrm{ab}}(X) \arrow[d, "\phi"] \\ C(X)_{\mathbb{L}} \arrow[r, hook, "\rho_{X,\mathbb{L}}"] & \pi_1^{\mathrm{ab}}(X)_{\mathbb{L}}, \end{tikzcd}$$

where the vertical maps are the natural one $\psi = \phi_{C(X),\mathbb{L}}$ and $\phi = \phi_{\pi_1^{\mathrm{ab}}(X),\mathbb{L}}$ (defined in (4.9)). This diagram gives an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\rho_X) \longrightarrow \mathrm{Ker}(\psi) \xrightarrow{\rho_X} \mathrm{Ker}(\phi). \tag{4.10}$$

**Claim.** For any prime number $l \in \mathbb{L}$, we have

(1) $\mathrm{Ker}(\psi)$ is $l$-divisible, and
(2) $\mathrm{Ker}(\phi)$ is $l$-torsion free, that is, if we have $lx = 0$ with $x \in \mathrm{Ker}(\phi)$ then $x = 0$.

*Proof.* (1) Put $A := C(X), B_m := \pi_1^{\mathrm{ab}}(X)/m$ and $\varphi_m := \rho_{X,m} : A/m \to B_m$. Using Lemma 4.5, we show that $\mathrm{Ker}(\psi) = \mathrm{Ker}(\phi_{A,\mathbb{L}})$ is $l$-divisible for any $l \in \mathbb{L}$. By Lemma 4.4, $\varphi_m = \rho_{X,m}$ is injective for all $m \in \mathbb{N}(\mathbb{L})$: the condition (1) in Lemma 4.5 holds.

The tame fundamental group $\pi_1^{\mathrm{t,ab}}(X)$ is defined by the wild inertia subgroups $I^1_{k(X)_x}$ for $x \in X_\infty$ in Definition 3.4 and (3.4). This group $I^1_{k(X)_x}$ is pro-$p$ so that we have $B_m = \pi_1^{\mathrm{ab}}(X)/m \xrightarrow{\simeq} \pi_1^{\mathrm{t,ab}}(X)/m$ for each $m \in \mathbb{N}(\mathbb{L})$. Taking the inverse limit,

$$B_{\mathbb{L}} = \pi_1^{\mathrm{ab}}(X)_{\mathbb{L}} \xrightarrow{\simeq} \pi_1^{\mathrm{t,ab}}(X)_{\mathbb{L}}. \tag{4.11}$$

By local class field theory (of $k$) and the structure of the base field $k$ (e.g. [26, Prop. 5.7(ii)]), $(G_k^{\mathrm{ab}})_{\mathbb{L}}$ is (topologically) finitely generated. For $(\pi_1^{\mathrm{t,ab}}(X)^0)_{\mathbb{L}}$ is finitely generated (4.8), so is $B_{\mathbb{L}}$ by (4.11). Using the finiteness of the torsion part $(B_{\mathbb{L}})_{\mathrm{tor}}$, there exists $N \in \mathbb{N}(\mathbb{L})$ such that $N \cdot$

$(B_{\mathbb{L}})_{\mathrm{tor}} = 0$: the condition (2) in Lemma 4.5 holds. The claim (1) follows from Lemma 4.5.

(2) Putting $\phi^{\mathrm{t}} = \phi_{\pi_1^{\mathrm{t,ab}}(X),\mathbb{L}}$ (cf. (4.9)), the commutative diagram

$$\begin{tikzcd} \pi_1^{\mathrm{ab}}(X) \arrow[r, two heads] \arrow[d, "\phi"'] & \pi_1^{\mathrm{t,ab}}(X) \arrow[d, "\phi^{\mathrm{t}}"] \\ \pi_1^{\mathrm{ab}}(X)_{\mathbb{L}} \arrow[r, "\simeq"] & \pi_1^{\mathrm{t,ab}}(X)_{\mathbb{L}} \end{tikzcd}$$

induces a short exact sequence

$$\bigoplus_{x \in X_\infty} I^1_{k(X)_x} \longrightarrow \mathrm{Ker}(\phi) \longrightarrow \mathrm{Ker}(\phi^{\mathrm{t}}) \longrightarrow 0.$$

Recall that the wild inertia subgroup $I^1_{k(X)_x}$ is pro-$p$, in particular, it has no $l$-torsion quotient. It is enough to show that $\mathrm{Ker}(\phi^{\mathrm{t}})$ is $l$-torsion free. We further consider the commutative diagram below with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \pi_1^{\mathrm{t,ab}}(X)^0 \arrow[r] \arrow[d, "\phi^{\mathrm{t},0}"] & \pi_1^{\mathrm{t,ab}}(X) \arrow[r] \arrow[d, "\phi^{\mathrm{t}}"] & G_k^{\mathrm{ab}} \arrow[r] \arrow[d, "\phi_k"] & 0 \\ 0 \arrow[r] & (\pi_1^{\mathrm{t,ab}}(X)^0)_{\mathbb{L}} \arrow[r] & \pi_1^{\mathrm{t,ab}}(X)_{\mathbb{L}} \arrow[r] & (G_k^{\mathrm{ab}})_{\mathbb{L}}, & \end{tikzcd}$$

where $\phi^{\mathrm{t},0} := \phi_{\pi_1^{\mathrm{t,ab}}(X)^0,\mathbb{L}}$ and $\phi_k := \phi_{G_k^{\mathrm{ab}},\mathbb{L}}$ (cf. (4.9)). Since $\pi_1^{\mathrm{t,ab}}(X)^0$ is finitely generated (4.8), $\mathrm{Ker}(\phi^{\mathrm{t},0})$ is $l$-torsion free. By local class field theory, $\mathrm{Ker}(\phi_k)$ is also $l$-torsion free. Therefore, the same holds on $\mathrm{Ker}(\phi^{\mathrm{t}})$. □

By the exact sequence (4.10) and the claim above, $\mathrm{Ker}(\rho_X)$ is $l$-divisible for any prime $l \in \mathbb{L}$ as required. □

**Restricted Ramification.** In closing this section, we derive the class field theory with *modulus* from Theorem 4.6 above.

**Theorem 4.7.** *Let $D \geq 0$ be an effective Weil divisor on $\overline{X}$ with support $|D| \subset X_\infty$. For $\rho_{X,D} : C(X,D) \to \pi_1^{\mathrm{ab}}(X,D)$, we have:*

(1) $\mathrm{Ker}(\rho_{X,D})$ *is the maximal $l$-divisible subgroup of $C(X,D)$ for any prime $l \neq p$, and*

(2) $\mathrm{Coker}_{\mathrm{top}}(\rho_{X,D}) \simeq \widehat{\mathbb{Z}}^{r(\overline{X})}$.

*Proof.* The assertion (2) is already given in (4.7). Furthermore, the surjective homomorphism $C(X) \twoheadrightarrow C(X,D)$ gives a homomorphism $\mathrm{Ker}(\rho_X) \twoheadrightarrow \mathrm{Ker}(\rho_{X,D})$ which is also surjective. From Theorem 4.6, $\mathrm{Ker}(\rho_{X,D})$ is $l$-divisible for a prime $l \neq p$. Since profinite groups contain no non-trivial divisible elements, the assertion (1) follows. □

## 5. Proof of Theorem 1.2

We keep the notation of Section 4.

### Unramified class field theory.

**Corollary 5.1.** *The induced homomorphism* $\rho_{\overline{X}}^{\vee} : H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \to C(\overline{X})^{\vee}$ *from the reciprocity map* $\rho_{\overline{X}}$ *satisfies the following:*

(1) $\mathrm{Ker}(\rho_{\overline{X}}^{\vee}) \simeq (\mathbb{Q}/\mathbb{Z})^{r(\overline{X})}$, *and*
(2) $\rho_{\overline{X}}^{\vee}$ *is surjective.*

*Proof.* The assertion (1) follows from Theorem 4.1 (1). By Theorem 4.1 (2), $\rho_{\overline{X}}^0$ defined in (3.2) induces an injection $\rho_{\overline{X},m}^0 : C(\overline{X})^0/m \hookrightarrow \pi_1^{\mathrm{ab}}(\overline{X})^0/m$. Since the quotient $C(\overline{X})^0/m$ is finite (Theorem 4.1 (3)), we obtain the surjective homomorphism

$$(\rho_{\overline{X},m}^0)^{\vee} : (\pi_1^{\mathrm{ab}}(\overline{X})^0/m)^{\vee} \twoheadrightarrow (C(\overline{X})^0/m)^{\vee} \tag{5.1}$$

on the dual groups for any $m \in \mathbb{Z}_{\geq 1}$. Now, we show that

$$(\rho_{\overline{X}}^0)^{\vee} : (\pi_1^{\mathrm{ab}}(\overline{X})^0)^{\vee} \longrightarrow (C(\overline{X})^0)^{\vee}$$

is surjective. Take a character $\varphi \in (C(\overline{X})^0)^{\vee}$. By the very definition of $(C(\overline{X})^0)^{\vee}$, the character $\varphi$ has finite order (cf. Notation). Hence, there exists $m \in \mathbb{Z}_{\geq 1}$ and $\varphi_m \in (C(\overline{X})^0/m)^{\vee}$ such that $\varphi$ is the image of $\varphi_m$ by the natural map $(C(\overline{X})^0/m)^{\vee} \to (C(\overline{X})^0)^{\vee}$. Since $(\rho_{\overline{X},m}^0)^{\vee}$ is surjective (5.1), there exists $\chi_m \in (\pi_1^{\mathrm{ab}}(\overline{X})^0/m)^{\vee}$ such that $(\rho_{\overline{X},m}^0)^{\vee}(\chi_m) = \varphi_m$. From the commutative diagram

$$\begin{tikzcd}
(\pi_1^{\mathrm{ab}}(\overline{X})^0/m)^{\vee} \arrow[r] \arrow[d, "(\rho_{\overline{X},m}^0)^{\vee}"'] & (\pi_1^{\mathrm{ab}}(\overline{X})^0)^{\vee} \arrow[d, "(\rho_{\overline{X}}^0)^{\vee}"] \\
(C(\overline{X})^0/m)^{\vee} \arrow[r] & (C(\overline{X})^0)^{\vee},
\end{tikzcd}$$

the image $\chi$ of $\chi_m$ by $(\pi_1^{\mathrm{ab}}(\overline{X})^0/m)^{\vee} \to (\pi_1^{\mathrm{ab}}(\overline{X})^0)^{\vee}$ gives $\varphi = (\rho_{\overline{X}}^0)^{\vee}(\chi)$. Hence, $(\rho_{\overline{X}}^0)^{\vee}$ is surjective.

On the other hand, the commutative diagram (3.2) and the Hochschild–Serre spectral sequence $H^s_{\mathrm{Gal}}(k, H^t(\overline{X}_{\overline{k}}, \mathbb{Q}/\mathbb{Z})) \Rightarrow H^{s+t}(\overline{X}, \mathbb{Q}/\mathbb{Z})$ associated with the projection $\overline{X}_{\overline{k}} \to \overline{X}$ (cf. [3, Exp. VIII, Cor. 8.5]) give the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^1_{\mathrm{Gal}}(k, \mathbb{Q}/\mathbb{Z}) \arrow[r, hook] \arrow[d, "\simeq"', "\rho_k^{\vee}"] & H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\rho_{\overline{X}}^{\vee}"] & H^1(\overline{X}_{\overline{k}}, \mathbb{Q}/\mathbb{Z})^{G_k} \arrow[r] \arrow[d] & H^2_{\mathrm{Gal}}(k, \mathbb{Q}/\mathbb{Z}) \\
(k^{\times})^{\vee} \arrow[r, "N_{\overline{X}}^{\vee}"] & C(\overline{X})^{\vee} \arrow[r] & (C(\overline{X})^0)^{\vee} &
\end{tikzcd} \tag{5.2}$$

.

Here, for a $G_k$-module $M$, we denote by $M^{G_k}$ the $G_k$-invariant submodule of $M$ and $H^2_{\mathrm{Gal}}(k, \mathbb{Q}/\mathbb{Z}) = 0$ as in (4.2). By local class field theory, the left vertical map $\rho_k^\vee$ in (5.2) is bijective. Because of $H^1_{\mathrm{Gal}}(k, \mathbb{Q}/\mathbb{Z}) \simeq (G_k^{\mathrm{ab}})^\vee$, and $H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \simeq \pi_1^{\mathrm{ab}}(\overline{X})^\vee$, we obtain $H^1(\overline{X}_{\overline{k}}, \mathbb{Q}/\mathbb{Z})^{G_k} \simeq (\pi_1^{\mathrm{ab}}(\overline{X})^0)^\vee$. The right vertical map in the diagram (5.2) coincides with $(\rho^0_{\overline{X}})^\vee$ and is surjective by (5.1). Therefore, $\rho^\vee_{\overline{X}}$ is surjective. □

**Corollary 5.2.** *We assume that we have $r(\overline{X}) = 0$. Then $\rho^\vee_{\overline{X},m} : H^1(\overline{X}, \mathbb{Z}/m) \to (C(\overline{X})/m)^\vee$ is bijective for any $m \in \mathbb{Z}_{\geq 1}$.*

*Proof.* For each $m \in \mathbb{Z}_{\geq 1}$, we have the following commutative diagram:

$$\begin{tikzcd} H^1(\overline{X}, \mathbb{Z}/m) \arrow[r, hook, "q^\vee"] \arrow[d, "\rho^\vee_{\overline{X},m}"] & H^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\rho^\vee_{\overline{X}}", "\simeq"'] \\ (C(\overline{X})/m)^\vee \arrow[r, hook, "q^\vee"] & C(\overline{X})^\vee, \end{tikzcd}$$

where the horizontal maps $q^\vee$ are induced from the quotient maps $q : \pi_1^{\mathrm{ab}}(\overline{X}) \twoheadrightarrow \pi_1^{\mathrm{ab}}(\overline{X})/m$ and $q : C(\overline{X}) \twoheadrightarrow C(\overline{X})/m$. From Corollary 5.1, $\rho^\vee_{\overline{X},m}$ is injective. To show that $\rho^\vee_{\overline{X},m}$ is surjective, take any $\varphi \in (C(\overline{X})/m)^\vee$. We denote by $\widetilde{\varphi} := q^*(\varphi)$ its image in $C(\overline{X})^\vee$. As $\rho^\vee_{\overline{X}}$ is surjective, there exists $\widetilde{\chi} \in H^1(\overline{X}, \mathbb{Q}/\mathbb{Z})$ such that $\rho^\vee_{\overline{X}}(\widetilde{\chi}) = \widetilde{\varphi}$. Since $\rho^\vee_{\overline{X}}$ is injective, $\rho^\vee_{\overline{X}}(m\widetilde{\chi}) = m\widetilde{\varphi} = 0$ implies $m\widetilde{\chi} = 0$. The character $\widetilde{\chi} : \pi_1^{\mathrm{ab}}(\overline{X}) \to \mathbb{Q}/\mathbb{Z}$ factors through $q : \pi_1^{\mathrm{ab}}(X) \twoheadrightarrow \pi_1^{\mathrm{ab}}(\overline{X})/m$ and this induces $\chi \in H^1(\overline{X}, \mathbb{Z}/m)$. By diagram chasing, we obtain $\rho^\vee_{\overline{X},m}(\chi) = \varphi$ and the assertion follows. □

**Open curves.** Recall that $X \subset \overline{X}$ is a non-empty open subscheme and $\rho_X$ induces $\rho_X^\vee : H^1(X, \mathbb{Q}/\mathbb{Z}) \to C(X)^\vee$ and $\rho^\vee_{X,m} : H^1(X, \mathbb{Z}/m) \to (C(X)/m)^\vee$ for each $m \in \mathbb{Z}_{\geq 1}$.

**Proposition 5.3.** *Assume $r(\overline{X}) = 0$. Then*

$$\rho^\vee_{X,m} : H^1(X, \mathbb{Z}/m) \to (C(X)/m)^\vee$$

*is bijective for any $m \in \mathbb{Z}_{\geq 1}$.*

*Proof.* From the assumption $r(\overline{X}) = 0$ and Theorem 4.2, $\rho_X$ and hence $\rho_{X,m}$ has dense image. On the dual groups, $\rho_X^\vee$ and $\rho^\vee_{X,m}$ for any $m \in \mathbb{Z}_{\geq 1}$ are injective. In the following, we show that $\rho^\vee_{X,m}$ is surjective.

*Prime to p-part.* For $m \in \mathbb{N}'$, we have an isomorphism $\pi_1^{\mathrm{ab}}(X)/m \simeq \pi_1^{\mathrm{t,ab}}(X)/m$ of *finite groups* as noted in the proof of Theorem 4.6 (cf. (4.11)). Since $\rho_{X,m}$ is an injective homomorphism of finite groups (Lemma 4.4), the dual $\rho^\vee_{X,m}$ becomes surjective.

*p-part.* Instead of using $\mathbb{Z}/p^n$ with $\mathbb{Q}/\mathbb{Z}$ in (4.3), for each $x \in X_\infty$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\mathrm{Gal}}(k(x), \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\rho^\vee_{k(x),p^n}", "\simeq"'] & H^1_{\mathrm{Gal}}(k(X)_x, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\rho^\vee_{k(X)_x,p^n}", "\simeq"'] & H^2_x(\overline{X}, \mathbb{Z}/p^n) \arrow[r] \arrow[d, dashed, "\phi_{x,p^n}"] & 0 \\
0 \arrow[r] & (k(x)^\times/p^n)^\vee \arrow[r] & (K_2(k(X)_x)/p^n)^\vee \arrow[r] & (U^0K_2(k(X)_x)/p^n)^\vee, &
\end{tikzcd}$$

where the middle vertical map is bijective by (2.3). As in (4.6), the localization sequence and (2.3) give the following diagram with exact rows:

$$\begin{tikzcd}
H^1(\overline{X}, \mathbb{Z}/p^n) \arrow[r, hook] \arrow[d, "\rho^\vee_{\overline{X},p^n}", "\simeq"'] & H^1(X, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\rho^\vee_{X,p^n}"] & \bigoplus_{x \in X_\infty} H^2_x(\overline{X}, \mathbb{Z}/p^n) \arrow[r, "j"] \arrow[d, hook, "\phi"] & H^2(\overline{X}, \mathbb{Z}/p^n) \\
(C(\overline{X})/p^n)^\vee \arrow[r, hook] & (C(X)/p^n)^\vee \arrow[r, "i"] & \bigoplus_{x \in X_\infty} (U^0K_2(k(X)_x)/p^n)^\vee, &
\end{tikzcd}$$

where $\phi := \oplus \phi_{x,p^n}$. From Corollary 5.2, $\rho^\vee_{\overline{X},p^n}$ is bijective.

**Claim.** $\mathrm{Im}(i) \subset \mathrm{Im}(\phi)$.

*Proof.* The map $i$ can be written as the composition

$$(C(X)/p^n)^\vee \longrightarrow \bigoplus_{x \in X_\infty} (K_2(k(X)_x)/p^n)^\vee \xrightarrow{\oplus i_x} \bigoplus_{x \in X_\infty} (U^0K_2(k(X)_x)/p^n)^\vee,$$

where the first map is given by the natural map $K_2(k(X)_x) \to C(X)$ for each $x \in X_\infty$ and the latter which is denoted by $\oplus i_x$ is induced from the inclusion $U^0K_2(k(X)_x) \hookrightarrow K_2(k(X)_x)$ for each $x \in X_\infty$. For each $x \in X_\infty$, as in (4.3), there exists a commutative diagram

$$\begin{tikzcd}
H^1_{\mathrm{Gal}}(k(X)_x, \mathbb{Z}/p^n) \arrow[r] \arrow[d, "\simeq", "\rho^\vee_{k(X)_x,p^n}"'] & H^2_x(\overline{X}, \mathbb{Z}/p^n) \arrow[d, "\phi_{x,p^n}"] \\
(K_2(k(X)_x)/p^n)^\vee \arrow[r, "i_x"] & (U^0K_2(k(X)_x)/p^n)^\vee.
\end{tikzcd}$$

Here, the left vertical map is bijective from (2.3) and the claim follows. $\square$

To show that $\rho^\vee_{X,p^n}$ is surjective, take $\varphi \in (C(X)/p^n)^\vee$. From the above Claim, there exists $\gamma \in \bigoplus_x H^2_x(\overline{X}, \mathbb{Z}/p^n)$ such that $i(\varphi) = \phi(\gamma)$. From $H^2(\overline{X} \otimes_k \overline{k}, \mathbb{Z}/p^n) = 0$ ([3, Exp. X, Cor. 5.2]), there exists a finite Galois extension $k'$ of $k$ such that the image of $j(\gamma)$ by the homomorphism $\overline{f}^* : H^2(\overline{X}, \mathbb{Z}/p^n) \to H^2(\overline{X}', \mathbb{Z}/p^n)$ becomes zero, where $\overline{f} : \overline{X}' := \overline{X} \otimes_k k' \to \overline{X}$ is the projection. Put also $X' := X \otimes_k k'$ and let $f : X' \to X$ be the induced morphism. In this setting, we have the norm homomorphism $N := N_{X'/X}$ :

$C(X') \to C(X)$ defined in Definition 3.7. By (3.7), this makes the following diagram commutative:

$$\begin{tikzcd} H^1(X,\mathbb{Z}/p^n) \arrow[r, "f^*"] \arrow[d, "\rho^\vee_{X,p^n}"'] & H^1(X',\mathbb{Z}/p^n) \arrow[d, "\rho^\vee_{X',p^n}"] \\ (C(X)/p^n)^\vee \arrow[r, "N^\vee_{p^n}"] & (C(X')/p^n)^\vee, \end{tikzcd}$$

where $N^\vee_{p^n}$ is the induced homomorphism by $N = N_{X'/X}$. Thus, there exists $\chi' \in H^1(X',\mathbb{Z}/p^n)$ such that $\varphi' := N^\vee_{p^n}(\varphi) = \rho^\vee_{X',p^n}(\chi')$ in $(C(\overline{X}')/p^n)^\vee$ by the diagram chase. It is left to show that $\varphi$ comes from $H^1(X,\mathbb{Z}/p^n)$.

Let $H$ be the $p$-Sylow subgroup of $G := \mathrm{Gal}(k'/k)$ and $k_H$ the fixed field of $H$ in $k'$. Putting $X_H := X \otimes_k k_H$, the diagram

$$\begin{tikzcd} & H^1(X_H,\mathbb{Z}/p^n) \arrow[r] \arrow[d, "\rho^\vee_{X_H,p^n}"'] & H^1(X,\mathbb{Z}/p^n) \arrow[d, "\rho^\vee_{X,p^n}"] \\ (C(X)/p^n)^\vee \arrow[r, "N^\vee_{X_H/X,p^n}"] \arrow[rr, bend right, "{[k_H:k]}"'] & (C(X_H)/p^n)^\vee \arrow[r, "i^\vee_{X_H/X,p^n}"] & (C(X)/p^n)^\vee \end{tikzcd}$$

is commutative by (3.7). From Lemma 3.8, we have $N_{X_H/X} \circ i_{X_H/X} = [k_H : k]\,\mathrm{id}_{C(X)}$. Since the order of $\varphi$ is a power of $p$, using the above diagram, we may assume $k_H = k$ and $G = \mathrm{Gal}(k'/k)$ is a $p$-group. Take a field extensions $k = k_0 \subset k_1 \subset \cdots \subset k_s = k'$ such that $k_{i+1}/k_i$ is a cyclic extension of degree $p$. By induction on $i$, we may assume that the Galois group $G$ is a cyclic group of the order $p$. We denote by $\widetilde{\varphi} \in C(X)^\vee$ and $\widetilde{\varphi}' \in C(X')^\vee$ the characters induced from $\varphi$ and $\varphi'$ respectively. We also denote by $\widetilde{\chi}' \in H^1(X',\mathbb{Q}/\mathbb{Z})$ the lift of $\chi'$. From the equality $\varphi' = N^\vee_{p^n}(\varphi) = \rho^\vee_{X',p^n}(\chi')$, we have $\widetilde{\varphi}' = N^\vee(\widetilde{\varphi}) = \rho^\vee_{X'}(\widetilde{\chi}')$ in $C(X')^\vee$. We consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & G^\vee \arrow[r] \arrow[d, "\psi"] & H^1(X,\mathbb{Q}/\mathbb{Z}) \arrow[r, "f^*"] \arrow[d, hook, "\rho^\vee_X"] & H^1(X',\mathbb{Q}/\mathbb{Z})^G \arrow[r] \arrow[d, hook, "\rho^\vee_{X'}"] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(N^\vee) \arrow[r] & C(X)^\vee \arrow[r, "N^\vee"] & (C(X')^\vee)^G. \end{tikzcd} \tag{5.3}$$

The upper horizontal sequence is exact which comes from the Hochschild–Serre spectral sequence $H^s_{\mathrm{Gal}}(G, H^t(X',\mathbb{Q}/\mathbb{Z})) \Rightarrow H^{s+t}(X,\mathbb{Q}/\mathbb{Z})$ associated with $f : X' \to X$ (cf. [3, Exp. VIII, Cor. 8.5]) and $H^2_{\mathrm{Gal}}(G,\mathbb{Q}/\mathbb{Z}) = 0$ ([31, Chap. VIII, §5]). Since $\widetilde{\varphi}' = N^\vee(\widetilde{\varphi})$ is fixed by $G$ and $\rho^\vee_{X'}$ is *injective*, $\widetilde{\chi}'$ is also fixed by $G$. The left vertical map $\psi$ in (5.3) is injective. Note that from the assumption $r(\overline{X}) = 0$, there are no completely split coverings of $X$ (cf.

Section 1). The lemma below (Lemma 5.4) implies that $\psi$ is bijective. From the diagram chase, one can find $\widetilde{\chi} \in H^1(X, \mathbb{Z}/p^n)$ such that $\rho_X^\vee(\widetilde{\chi}) = \widetilde{\varphi}$. For $\rho_X^\vee(p^n\widetilde{\chi}) = p^n\widetilde{\varphi} = 0$ and $\rho_X^\vee$ is injective, we obtain $p^n\widetilde{\chi} = 0$. Therefore, $\widetilde{\chi}$ induces $\chi \in H^1(X, \mathbb{Z}/p^n)$. This satisfies $\rho_{X,p^n}^\vee(\chi) = \varphi$ and thus $\rho_{X,p^n}^\vee$ is surjective. $\square$

**Lemma 5.4.** *Let $k'/k$ be a Galois extension with $[k' : k] = p$. We assume that the base change $X' := X \otimes_k k' \to X$ is not a completely split covering. Then, the following sequence is exact:*

$$0 \longrightarrow G^\vee \longrightarrow C(X)^\vee \xrightarrow{N^\vee} C(X')^\vee,$$

*where $G = \mathrm{Gal}(k'/k)$ and $N = N_{X'/X}$.*

*Proof.* A character $\varphi \in \mathrm{Ker}(N^\vee)$ induces an element $\varphi_x$ of $K_2(k(X)_x)^\vee$ for each $x \in X_\infty$. Since $\varphi_x$ is in the kernel of $N_{k'k(X)_x/k(X)_x}^\vee : K_2(k(X)_x)^\vee \to K_2(k'k(X)_x)^\vee$, the corresponding character $\chi_x := (\rho_{k(X)_x}^\vee)^{-1}(\varphi_x) \in H^1(k(X)_x)$ (Theorem 2.3 (1)) is annihilated by the unramified extension $k'k(X)_x/k(X)_x$. In particular, $\chi_x$ is unramified so that $\varphi_x$ annihilates $U^0K_2(k(X)_x)$ (Theorem 2.3 (2)). Thus, the assertion is reduced to the case of $X = \overline{X}$, that is, the exactness of

$$0 \longrightarrow G^\vee \longrightarrow C(\overline{X})^\vee \xrightarrow{N_{\overline{X}'/\overline{X}}^\vee} C(\overline{X}')^\vee,$$

where $\overline{X}' = \overline{X} \otimes_k k'$. This follows from Corollary 5.1. $\square$

**Theorem 5.5.** *Suppose that we have $r(\overline{X}) = 0$. Then, the dual of the reciprocity homomorphism $\rho_X^\vee : \pi_1^{\mathrm{ab}}(X)^\vee \to C(X)^\vee$ is bijective.*

*Proof.* We use $\pi_1^{\mathrm{ab}}(X)^\vee \simeq H^1(X, \mathbb{Q}/\mathbb{Z})$ given in (4.5). The injectivity of $\rho_X^\vee$ follows from Theorem 4.2. To show that $\rho_X^\vee$ is surjective, take any $\varphi \in C(X)^\vee$. Since $\varphi$ has finite order (cf. Notation), $\varphi$ defines $\varphi_m \in (C(X)/m)^\vee$ for some $m \in \mathbb{Z}_{\geq 1}$. Consider the commutative diagram

$$\begin{tikzcd}
H^1(X, \mathbb{Z}/m) \arrow[r, hook] \arrow[d, "\rho_{X,m}^\vee"', "\simeq"] & H^1(X, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\rho_X^\vee"] \\
(C(X)/m)^\vee \arrow[r, hook] & C(X)^\vee,
\end{tikzcd}$$

where the horizontal maps are injective (as in the proof of Corollary 5.2) and the left vertical map is bijective (Proposition 5.3). There exists $\chi \in H^1(X, \mathbb{Q}/\mathbb{Z})$ such that $\rho_X^\vee(\chi) = \varphi$ and hence $\rho_X^\vee$ is surjective. $\square$

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