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# An exponential sum estimate for systems with linear polynomials

par Shuntaro YAMAGISHI

Résumé. Dans son article [5], W. M. Schmidt a obtenu une estimation de somme exponentielle pour les systèmes de polynômes sans polynômes linéaires, qui a ensuite été utilisée pour appliquer la méthode du cercle de Hardy–Littlewood. Nous démontrons une estimation analogue pour les systèmes qui incluent des polynômes linéaires.

Abstract. In his paper [5], W. M. Schmidt obtained an exponential sum estimate for systems of polynomials without linear polynomials, which was then used to apply the Hardy–Littlewood circle method. We prove an analogous estimate for systems which include linear polynomials.

## 1. Introduction

Let $\mathbf{u} = (\mathbf{u}_d, \dots, \mathbf{u}_1)$ be a system of polynomials in $\mathbb{Q}[x_1, \dots, x_n]$, where $\mathbf{u}_\ell = (u_{\ell,1}, \dots, u_{\ell,r_\ell})$ is the subsystem of all degree $\ell$ polynomials of $\mathbf{u}$ $(1 \leq \ell \leq d)$. We let $\mathbf{U} = (\mathbf{U}_d, \dots, \mathbf{U}_1)$ be the system of forms, where for each $1 \leq \ell \leq d$, $\mathbf{U}_\ell = (U_{\ell,1}, \dots, U_{\ell,r_\ell})$ and $U_{\ell,r}$ is the homogeneous degree $\ell$ portion of $u_{\ell,r}$ $(1 \leq r \leq r_\ell)$. Let $R = r_1 + \dots + r_d$. Let us denote $\mathfrak{B}_0 = [0,1]^n$. We define the following exponential sum associated to $\mathbf{u}$,

$$S(\boldsymbol{\alpha}) = S(\mathbf{u}, \mathfrak{B}_0; \boldsymbol{\alpha}) := \sum_{\mathbf{x} \in P\mathfrak{B}_0 \cap \mathbb{Z}^n} e\left( \sum_{1 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \alpha_{\ell,r} \cdot u_{\ell,r}(\mathbf{x}) \right). \tag{1.1}$$

In his paper [5], W. M. Schmidt obtained an exponential sum estimate for $S(\boldsymbol{\alpha})$ when $\mathbf{u}$ has integer coefficients, does not include linear polynomials, and satisfies certain properties. The estimate was then used in applying the Hardy–Littlewood circle method to obtain the asymptotic formula for the number of integer points of bounded height on the affine variety defined by $\mathbf{u} = \mathbf{0}$. We refer the reader to [5] for more details on this important work. The work of Schmidt was found useful in the breakthrough of B. Cook and Á. Magyar [4], where they count the number of solutions whose coordinates are all primes to diophantine equations, and also in [6]. It makes sense

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for Schmidt in [5] to only consider systems without linear polynomials, because he is concerned with integer points and linear polynomials can be eliminated via substitution in this case. However, if one wants to apply the result of Schmidt to a coordinate dependent problem (where one can not eliminate linear polynomials by substitution), then it may be useful to have an analogous exponential sum estimate for systems with linear polynomials, and this is what we achieve in this paper. In fact, the estimates obtained in this paper were found useful in [7]. It is also worth mentioning the recent work by T. D. Browning and D. R. Heath-Brown [2] where they generalized the seminal work of B. J. Birch [1] to handle systems of polynomials in which the degrees need not all be the same. Similarly as in [5], they consider systems of polynomials which do not include linear polynomials (see [2, l. 7, p. 359]). Thus the approach of this paper may be used to generalize their work to include linear polynomials as well.

We need to introduce some notations before we can state our result. Let $1 < \ell \leq d$ and $r_\ell > 0$. We let $\mathbb{M}_\ell = \mathbb{M}_\ell(\mathbf{U}_\ell)$ be the affine variety in $(\mathbb{C}^n)^{\ell-1}$ associated to $\mathbf{U}_\ell$, for which the definition we provide in (2.1) of Section 2. For $R_0 > 0$, we denote $z_{R_0}(\mathbb{M}_\ell)$ to be the number of integer points $(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1})$ on $\mathbb{M}_\ell$ such that

$$\max_{1 \leq i \leq \ell-1} \max_{1 \leq j \leq n} |x_{ij}| \leq R_0,$$

where $\mathbf{x}_i = (x_{i1}, \dots, x_{in}) (1 \leq i \leq \ell - 1)$. We define $g_\ell(\mathbf{U}_\ell)$ to be the largest real number such that

$$z_P(\mathbb{M}_\ell) \ll P^{n(\ell-1)-g_\ell(\mathbf{U}_\ell)+\varepsilon} \tag{1.2}$$

holds for each $\varepsilon > 0$.

Let

$$\gamma_\ell = \frac{2^{\ell-1}(\ell-1)r_\ell}{g_\ell(\mathbf{U}_\ell)}$$

when $r_\ell > 0$ and $g_\ell(\mathbf{U}_\ell) > 0$. We let $\gamma_\ell = 0$ if $r_\ell = 0$, and let $\gamma_\ell = +\infty$ if $r_\ell > 0$ and $g_\ell(\mathbf{U}_\ell) = 0$.

These quantities are not defined for linear polynomials. When $\ell = 1$, following [4] we instead consider $\mathcal{B}_1(\mathbf{u}_1)$ which is defined to be the minimum number of non-zero coefficients in a non-trivial linear combination

$$\lambda_1 U_1 + \cdots + \lambda_{r_1} U_{r_1},$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_{r_1}) \in \mathbb{Q}^{r_1} \backslash \{\mathbf{0}\}$. Clearly $\mathcal{B}_1(\mathbf{u}_1) > 0$ if and only if the linear forms $U_1, \dots, U_{r_1}$ are linearly independent over $\mathbb{Q}$. If $r_1 = 0$ then we let $\mathcal{B}_1(\mathbf{u}_1) = +\infty$.

For $\alpha \in \mathbb{R}$, let $\|\alpha\|$ denote the distance from $\alpha$ to the closest integer. Let $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_d, \dots, \boldsymbol{\alpha}_1) \in \mathbb{R}^R$, where $R = r_1 + \cdots + r_d$ and $\boldsymbol{\alpha}_\ell = (\alpha_{\ell,1}, \dots, \alpha_{\ell,r_\ell}) \in$

$\mathbb{R}^{r_\ell}$ $(1 \le \ell \le d)$. We define

$$\|\boldsymbol{\alpha}\| = \max_{\substack{1\le \ell\le d\\ 1\le r\le r_\ell}} \|\alpha_{\ell,r}\| \quad \text{and} \quad |\boldsymbol{\alpha}| = \max_{\substack{1\le \ell\le d\\ 1\le r\le r_\ell}} |\alpha_{\ell,r}|.$$

The following theorem is the main result of this paper.

**Theorem 1.1.** *Suppose* $\mathbf{u}$ *has coefficients in* $\mathbb{Z}$, *and that*

$$\mathcal{B}_1(\mathbf{u}_1) > 2r_1 \left( \max\left\{ 4(r_1+1)\left(\sum_{j=2}^{d} 4^{j-2}\gamma_j\right), \frac{1}{4(R+1)} \right\}\right)^{-1}.$$

*Let*

$$0 < \Omega < \min\left\{ \frac{1}{8r_1+9}\left(\sum_{j=2}^{d} 4^{j-2}\gamma_j\right)^{-1}, \left(\frac{1}{2(R+1)} + \sum_{j=2}^{d} 4^{j-2}\gamma_j\right)^{-1} \right\}.$$

*Let* $0 < \Delta \le 1$, *and let* $P$ *be sufficiently large with respect to* $n$, $d$, $r_d, \dots, r_1$, $\Delta$, $\Omega$, *and* $\mathbf{u}$. *Then one of the following two alternatives must hold:*

(1) $|S(\boldsymbol{\alpha})| \le P^{n-\Delta\Omega}$.

(2) *There exists* $q \in \mathbb{N}$ *such that*

$$q \le P^{\Delta} \quad \textit{and} \quad \|q\boldsymbol{\alpha}_\ell\| \le P^{-\ell+\Delta} (1 \le \ell \le d).$$

In Section 2, we also prove a lemma on estimating the quantity known as the singular integral, which comes up in the Hardy–Littlewood circle method. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation. We also use $e(x)$ to denote $e^{2\pi i x}$ when $x \in \mathbb{R}$. For $q \in \mathbb{N}$, we use the numbers from $\{0, 1, \dots, q-1\}$ to represent the residue classes of $\mathbb{Z}/q\mathbb{Z}$.

## 2. Proof of Theorem 1.1

First we present the following lemma from [4].

**Lemma 2.1.** [4, Lemma 3] *Let* $\mathbf{G} = (G_1, \dots, G_{r'})$ *be a system of linear forms in* $\mathbb{Q}[x_1, \dots, x_n]$. *Given any* $1 \le j \le n$, *we have*

$$\mathcal{B}_1(\mathbf{G}|_{x_j=0}) \ge \mathcal{B}_1(\mathbf{G}) - 1.$$

Let $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,n})$ for $j \ge 1$. Given a function $G(\mathbf{x})$, we define

$$\Gamma_{\ell,G}(\mathbf{x}_1, \dots, \mathbf{x}_\ell) = \sum_{t_1=0}^{1} \cdots \sum_{t_\ell=0}^{1} (-1)^{t_1+\cdots+t_\ell} G(t_1\mathbf{x}_1 + \cdots + t_\ell\mathbf{x}_\ell).$$

Then it follows that $\Gamma_{\ell,G}$ is symmetric in its $\ell$ arguments, and that $\Gamma_{\ell,G}(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}, \mathbf{0}) = 0$ [5, Section 11]. We also have that if $G$ is a form of degree $d$ and $\ell > d > 0$, then $\Gamma_{\ell,G} = 0$ [5, Lemma 11.2].

Let $\mathbf{u} = (\mathbf{u}_d, \dots, \mathbf{u}_1)$ and $\mathbf{U} = (\mathbf{U}_d, \dots, \mathbf{U}_1)$ be as in Section 1. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis vectors of $\mathbb{C}^n$. Let $1 < \ell \le d$. We define

$\mathbb{M}_\ell = \mathbb{M}_\ell(\mathbf{U}_\ell)$ to be the set of $(\ell-1)$-tuples $(\mathbf{x}_1, \dots, \mathbf{x}_{\ell-1}) \in (\mathbb{C}^n)^{\ell-1}$ for which the matrix

$$(2.1) \qquad [m_{ir}] = [\Gamma_{\ell, U_{\ell,r}}(\mathbf{x}_1, \dots, \mathbf{x}_{d-1}, \mathbf{e}_i)] \quad (1 \leq r \leq r_\ell, 1 \leq i \leq n)$$

has rank strictly less than $r_\ell$.

Lemma 2.2 below is the inhomogeneous polynomials version of [5, Lemma 15.1], and it is obtained by essentially the same proof. We refer the reader to [5, Section 9] and "Remark on inhomogeneous polynomials" in [5, p. 262] for further explanation. We remark that the implicit constants may depend on $\mathbf{u}$ here, and not only on $\mathbf{U}$. We also note that [5, Lemma 15.1] is for systems without linear polynomials in contrast to Lemma 2.2 below. However, it is clear from the proof of [5, Lemma 15.1] that the lemma is not affected with the presence of linear polynomials.

**Lemma 2.2.** [5, Lemma 15.1] *Suppose* $\mathbf{u}$ *has coefficients in* $\mathbb{Z}$. *Let* $Q > 0$ *and* $\varepsilon > 0$. *Let* $2 \leq \ell \leq d$ *with* $r_\ell > 0$. *Let* $P$ *be sufficiently large with respect to* $d$ *and* $r_d, \dots, r_1$. *If* $\ell = d$, *then let* $\theta = 0$ *and* $q = 1$. *On the other hand, if* $2 \leq \ell < d$, *then suppose* $0 \leq \theta < 1/4$ *and that there is* $q \in \mathbb{N}$ *with*

$$q \leq P^\theta \quad \textit{and} \quad \|q\boldsymbol{\alpha}_j\| \leq P^{\theta - j} (\ell < j \leq d).$$

*Let* $S(\boldsymbol{\alpha})$ *be the sum associated to* $\mathbf{u}$ *as in* (1.1). *Given* $\eta > 0$ *with* $\eta + 4\theta \leq 1$, *one of the following three alternatives must hold:*

(1) $|S(\boldsymbol{\alpha})| \leq P^{n-Q}$.

(2) *There exists* $n_0 \in \mathbb{N}$ *such that*

$$n_0 \ll P^{r_\ell(\ell-1)\eta} \quad \textit{and} \quad \|qn_0\boldsymbol{\alpha}_\ell\| \ll P^{-\ell + 4\theta + r_\ell(\ell-1)\eta}.$$

(3) $z_{R_0}(\mathbb{M}_\ell) \gg R_0^{(\ell-1)n - 2^{\ell-1}(Q/\eta) - \varepsilon}$ *holds with* $R_0 = P^\eta$.

*The implicit constants depend at most on* $n, d, r_d, \dots, r_1, \eta, \varepsilon$, *and* $\mathbf{u}$.

We are left to deal with the case $\ell = 1$ in Lemma 2.2. Given $\boldsymbol{\epsilon} \in (\mathbb{N} \cup \{0\})^n$ and a sufficiently differentiable function $f : \mathbb{R}^n \to \mathbb{C}$, put

$$\partial^{\boldsymbol{\epsilon}} f = \frac{\partial^{\epsilon_1 + \dots + \epsilon_n} f}{\partial x_1^{\epsilon_1} \dots \partial x_n^{\epsilon_n}}.$$

Let $\mathcal{C}^n(\mathbb{R}^n)$ be the set of $n$-th continuously differentiable functions defined on $\mathbb{R}^n$.

For $\boldsymbol{\epsilon} \in \{0,1\}^n$, we define $\overline{\boldsymbol{\epsilon}} = (1, 1, \dots, 1) - \boldsymbol{\epsilon}$. Given $\mathbf{t} = (t_1, \dots, t_n)$, we let $\mathbf{t}_{\boldsymbol{\epsilon}}$ be the vector whose $i$-th coordinate equals zero if $\epsilon_i = 0$ and equals $t_i$ if $\epsilon_i = 1$. Similarly, given $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{Z}^n$, we let $\mathbf{N}_{\overline{\boldsymbol{\epsilon}}}$ be the vector whose $i$-th coordinate equals $N_i$ if $\epsilon_i = 0$ and equals zero if $\epsilon_i = 1$. The following is a generalization of the partial summation formula obtained by applying induction on the dimension.

**Lemma 2.3.** [3, Lemma 2.1] *Let* $\varrho : \mathbb{Z}^n \to \mathbb{C}$ *be a function, and let*

$$T_\varrho(\mathbf{t}) = \sum_{0 \le x_1 \le t_1} \dots \sum_{0 \le x_n \le t_n} \varrho(\mathbf{x}).$$

*Let* $N_i \in \mathbb{R}_{\ge 0}$ $(1 \le i \le n)$*. Then for any* $f \in \mathcal{C}^n(\mathbb{R}^n)$ *we have*

$$\sum_{\substack{0 \le x_i \le N_i \\ (1 \le i \le n)}} f(\mathbf{x})\varrho(\mathbf{x}) = \sum_{\boldsymbol{\epsilon} \in \{0,1\}^n} \left( \prod_{1 \le i \le n} (-1)^{\epsilon_i} N_i^{\epsilon_i - 1} \right)$$

$$\cdot \int_{[0,N_1]} \dots \int_{[0,N_n]} \partial^{\boldsymbol{\epsilon}} f(\mathbf{N}_{\overline{\boldsymbol{\epsilon}}} + \mathbf{t}_{\boldsymbol{\epsilon}}) T_\varrho(\mathbf{N}_{\overline{\boldsymbol{\epsilon}}} + \mathbf{t}_{\boldsymbol{\epsilon}}) \mathrm{d}t_n \dots \mathrm{d}t_1.$$

Let us use the following notations. For $\mathbf{a} = (a_1, \dots, a_{r_1}) \in (\mathbb{Z}/q\mathbb{Z})^{r_1}$, we let

$$\mathfrak{M}_{\mathbf{a},q}(C) = \left\{ \boldsymbol{\alpha}_1 \in [0,1)^{r_1} : \max_{1 \le r \le r_1} |\alpha_{1,r} - a_r/q| \le P^{C-1} \right\},$$

$$\mathfrak{M}(C) = \bigcup_{\substack{\gcd(\mathbf{a},q)=1 \\ \mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^{r_1} \\ 1 \le q \le P^C}} \mathfrak{M}_{\mathbf{a},q}(C),$$

and

$$\mathfrak{m}(C) = [0,1)^{r_1} \backslash \mathfrak{M}(C).$$

We also let

$$\mathfrak{N}_{a,q}(C) = \left\{ \alpha \in [0,1) : |\alpha - a/q| \le P^{C-1} \right\},$$

$$\mathfrak{N}(C) = \bigcup_{\substack{\gcd(a,q)=1 \\ 0 \le a < q \\ 1 \le q \le P^C}} \mathfrak{N}_{a,q}(C),$$

and

$$\mathfrak{n}(C) = [0,1) \backslash \mathfrak{N}(C).$$

With the use of Lemma 2.3, we obtain the following result when $r_1 > 0$.

**Lemma 2.4.** *Suppose* $\mathbf{u}$ *has coefficients in* $\mathbb{Z}$ *and that* $r_1 > 0$*. Let* $0 < \theta_0 < 1$*, and suppose there exists* $q \in \mathbb{N}$ *with*

$$q \le P^{\theta_0} \quad \textit{and} \quad \|q\boldsymbol{\alpha}_j\| \le P^{\theta_0 - j} (1 < j \le d).$$

*Let $S(\boldsymbol{\alpha})$ be the sum associated to $\mathbf{u}$ as in* (1.1)*. Let $\varepsilon_0 > 0$ be sufficiently small. Let $Q > 0$ and $0 < Q_0 < 1$ be two real numbers such that*

$$\theta_0 < \frac{Q_0/2 - \varepsilon_0}{2r_1}$$

*and*

$$Q < \mathcal{B}_1(\mathbf{u}_1)\left(\frac{Q_0/2 - \varepsilon_0}{r_1} - 2\theta_0\right).$$

*Suppose $P$ is sufficiently large with respect to $d$, $n$, $r_d, \ldots, r_1$, $\varepsilon_0$, $\theta_0$, $Q_0$, $Q$, and $\mathbf{u}$. Then one of the following two alternatives must hold:*

(1) $|S(\boldsymbol{\alpha})| \leq P^{n-Q}$.

(2) *There exists $n_0 \in \mathbb{N}$ such that*

$$n_0 \leq P^{Q_0} \quad \text{and} \quad \|n_0 \boldsymbol{\alpha}_1\| \leq P^{Q_0 - 1}.$$

*Proof.* If the alternative (2) holds then we are done. Thus let us assume it is not the case. Suppose $\boldsymbol{\alpha}_1 \in \mathfrak{M}(Q_0/2)$. Then for some $1 \leq q' \leq P^{Q_0/2}$ and $a_1, \ldots, a_{r_1} \in \mathbb{Z}$, we have

$$\max_{1 \leq r \leq r_1} |\alpha_{1,r} - a_r/q'| \leq P^{(Q_0/2)-1}$$

from which it follows that

$$\|q' \boldsymbol{\alpha}_1\| \leq P^{Q_0 - 1},$$

and this is a contradiction. Therefore, we have $\boldsymbol{\alpha}_1 \in \mathfrak{m}(Q_0/2)$.

For simplicity we denote $B = \mathcal{B}_1(\mathbf{u}_1)$ and $Q_0' = Q_0/2$. Let us also denote

$$\sum_{r=1}^{r_1} \alpha_{1,r} \cdot U_{1,r}(\mathbf{x}) = \xi_1 x_1 + \cdots + \xi_n x_n.$$

We let $\widetilde{M_1}$ be the $n \times r_1$ matrix, where its $(j,r)$-th entry is the $x_j$ coefficient of $U_{1,r}(\mathbf{x})$. Since this matrix has full rank (because $B > 0$), let us take an invertible $r_1 \times r_1$ submatrix, which we assume without loss of generality to be the first $r_1$ rows of $\widetilde{M_1}$, and denote it $M_1$.

Suppose $\xi_1, \ldots, \xi_{r_1} \in \mathfrak{N}(C')$ for some $C' > 0$. Then there exist integers $a_1, \ldots, a_{r_1}$ and $q_1, \ldots, q_{r_1}$ such that $\gcd(a_r, q_r) = 1$, $0 < q_r \leq P^{C'}$, and $|\xi_r - a_r/q_r| \leq P^{C'}/P$ $(1 \leq r \leq r_1)$. Let us define

$$\begin{bmatrix} a_1'/q' \\ \vdots \\ a_{r_1}'/q' \end{bmatrix} = M_1^{-1} \cdot \begin{bmatrix} a_1/q_1 \\ \vdots \\ a_{r_1}/q_{r_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \beta_1' \\ \vdots \\ \beta_{r_1}' \end{bmatrix} = M_1^{-1} \cdot \begin{bmatrix} \xi_1 - a_1/q_1 \\ \vdots \\ \xi_{r_1} - a_{r_1}/q_{r_1} \end{bmatrix}.$$

It is easy to deduce that we have

$$q' \leq P^{r_1 C' + \varepsilon_0} \quad \text{and} \quad |\beta_r'| \leq \frac{P^{r_1 C' + \varepsilon_0}}{P} (1 \leq r \leq r_1)$$

when $P$ is sufficiently large with respect to the coefficients of $\mathbf{U}_1$. Since $\alpha_{1,r} = \frac{a'_r}{q'} + \beta'_r$ $(1 \le r \le r_1)$, we see that $\boldsymbol{\alpha}_1 \in \mathfrak{M}(r_1 C' + \varepsilon_0)$. However, since $\boldsymbol{\alpha}_1 \in \mathfrak{m}(Q'_0)$, it follows from this argument that at least one of $\xi_1, \dots, \xi_{r_1}$ is in $\mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$. Without loss of generality, we suppose that $\xi_1 \in \mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$.

Let $\widetilde{M}_2$ be the matrix obtained by removing the first row of $\widetilde{M}_1$. Then $\widetilde{M}_2$ is the coefficient matrix of linear forms $\mathbf{U}_1|_{x_1=0}$, and we know from Lemma 2.1 that $\mathcal{B}_1(\mathbf{U}_1|_{x_1=0}) \ge B - 1$. Thus if $B - 1 > 0$ then it follows that $\widetilde{M}_2$ has full rank. Let us take an invertible $r_1 \times r_1$ submatrix, which we assume without loss of generality to be the first $r_1$ rows of $\widetilde{M}_2$, and denote it $M_2$. By the same argument as above, we obtain without loss of generality that $\xi_2 \in \mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$. In fact we can repeat the argument $B$ times, and obtain that $\xi_1, \xi_2, \dots, \xi_B \in \mathfrak{n}((Q'_0 - \varepsilon_0)/r_1)$.

Since $q \le P^{\theta_0} \le P^{(Q'_0 - \varepsilon_0)/r_1}$, it then follows that

$$\frac{P^{(Q'_0 - \varepsilon_0)/r_1}}{P} < \|q\xi_i\| (1 \le i \le B). \tag{2.2}$$

For each $2 \le \ell \le d, 1 \le r \le r_\ell$, let $a_{\ell,r} \in \mathbb{Z}$ and $\beta_{\ell,r} \in \mathbb{R}$ be such that

$$\alpha_{\ell,r} - a_{\ell,r}/q = \beta_{\ell,r} \quad \text{and} \quad |\beta_{\ell,r}| \le P^{\theta_0 - \ell}. \tag{2.3}$$

We then consider

$$\begin{aligned} |S(\boldsymbol{\alpha})| &= \Bigg| \sum_{\substack{0 \le k_i < q \\ (1 \le i \le n)}} \sum_{\substack{\mathbf{x} \in [0,P]^n \\ x_i \equiv k_i (\mathrm{mod}\ q) \\ (1 \le i \le n)}} e\Bigg( \sum_{1 \le \ell \le d} \sum_{1 \le r \le r_\ell} \alpha_{\ell,r} \cdot u_{\ell,r}(\mathbf{x}) \Bigg) \Bigg| \\ &\le q^n \max_{\substack{0 \le k_i < q \\ (1 \le i \le n)}} \Bigg| \sum_{\substack{0 \le y_i \le (P - k_i)/q \\ (1 \le i \le n)}} e\Bigg( \sum_{1 \le \ell \le d} \sum_{1 \le r \le r_\ell} \alpha_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) \Bigg) \Bigg|. \end{aligned} \tag{2.4}$$

Let us denote

$$f(\mathbf{y}) = e\Bigg( \sum_{2 \le \ell \le d} \sum_{1 \le r \le r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y} + \mathbf{k}) \Bigg).$$

Using the fact that $e(m) = 1$ for $m \in \mathbb{Z}$, we can simplify the above inequality (2.4) further,

$$
\begin{aligned}
|S(\boldsymbol{\alpha})| \leq q^n \max_{\substack{0 \leq k_i < q \\ (1 \leq i \leq n)}} \Bigg| \sum_{\substack{0 \leq y_i \leq (P-k_i)/q \\ (1 \leq i \leq n)}} e\Bigg( \sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y}+\mathbf{k}) \\
+ \sum_{1 \leq r \leq r_1} \alpha_{1,r} \cdot U_{1,r}(q\mathbf{y}) \Bigg) \Bigg| \\
\leq q^n \max_{\substack{0 \leq k_i < q \\ (1 \leq i \leq n)}} \sum_{\substack{0 \leq y_i \leq (P-k_i)/q \\ (B < i \leq n)}} \Bigg| \sum_{\substack{0 \leq y_i \leq (P-k_i)/q \\ (1 \leq i \leq B)}} f(\mathbf{y}) e\Bigg( \sum_{1 \leq i \leq B} q\xi_i y_i \Bigg) \Bigg|.
\end{aligned}
$$

Let $0 \leq y_i \leq (P-k_i)/q$ $(B < i \leq n)$. Given $\boldsymbol{\epsilon} \in \{0,1\}^B$, let $\left(\frac{(P-\mathbf{k})}{q}\right)_{\overline{\boldsymbol{\epsilon}}}$ be the vector whose $i$-th coordinate, for $1 \leq i \leq B$, equals $(P-k_i)/q$ if $\epsilon_i = 0$ and equals zero if $\epsilon_i = 1$, and for $B < i \leq n$, equals $y_i$. We also let $\mathbf{t}_{\boldsymbol{\epsilon}}$ be the vector whose $i$-th coordinate, for $1 \leq i \leq B$, equals $0$ if $\epsilon_i = 0$ and equals $t_i$ if $\epsilon_i = 1$, and for $B < i \leq n$, equals zero.

We prove that given $\boldsymbol{\epsilon} \in \{0,1\}^B$ and $0 \leq t_i \leq (P-k_i)/q$ $(1 \leq i \leq B)$, we have

$$
\left. \frac{\partial^{\epsilon_1+\cdots+\epsilon_B} f}{\partial y_1^{\epsilon_1} \ldots \partial y_B^{\epsilon_B}} \right|_{\mathbf{y}=\left(\frac{(P-\mathbf{k})}{q}\right)_{\overline{\boldsymbol{\epsilon}}}+\mathbf{t}_{\boldsymbol{\epsilon}}} \ll q^{\epsilon_1+\cdots+\epsilon_B} P^{(\theta_0-1)(\epsilon_1+\cdots+\epsilon_B)}, \tag{2.5}
$$

where the implicit constant is independent of $k_1, \ldots, k_n$, $y_{B+1}, \ldots, y_n$, and $\mathbf{t}$. In order to prove this statement, without loss of generality suppose $\epsilon_i = 1$ for $1 \leq i \leq E$ and $\epsilon_i = 0$ for $E < i \leq B$. The statement is trivial if $\epsilon_i = 0$ for all $1 \leq i \leq B$. Let $i_1 < \cdots < i_m \leq E$. First note when $m \leq d$, we have from (2.3) that

$$
\begin{aligned}
\frac{\partial^m}{\partial y_{i_1} \ldots \partial y_{i_m}} \Bigg( \sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y}+\mathbf{k}) \Bigg) \Bigg|_{\mathbf{y}=\left(\frac{(P-\mathbf{k})}{q}\right)_{\overline{\boldsymbol{\epsilon}}}+\mathbf{t}_{\boldsymbol{\epsilon}}} \\
\ll q^m \sum_{\max\{2,m\} \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} P^{\ell-m} \\
\ll q^m P^{\theta_0 - m},
\end{aligned} \tag{2.6}
$$

and when $m > d$,

$$
\frac{\partial^m}{\partial y_{i_1} \ldots \partial y_{i_m}} \Bigg( \sum_{2 \leq \ell \leq d} \sum_{1 \leq r \leq r_\ell} \beta_{\ell,r} \cdot u_{\ell,r}(q\mathbf{y}+\mathbf{k}) \Bigg) = 0. \tag{2.7}
$$

Thus we have

$$\left.\frac{\partial^E f}{\partial y_1 \dots \partial y_E}\right|_{\mathbf{y}=\left(\frac{(P-\mathbf{k})}{q}\right)_{\overline{\boldsymbol{\epsilon}}}+\mathbf{t}_{\boldsymbol{\epsilon}}} \ll \max_{\substack{m_1+\cdots+m_j=E\\ 1\le m_i\le d\\ (1\le i\le j)}} q^E P^{j\theta_0-E} \ll q^E P^{(\theta_0-1)E}, \tag{2.8}$$

from which we can deduce (2.5). We now prepare to apply Lemma 2.3. Let $0 \le t_i \le (P-k_i)/q$ $(1 \le i \le B)$. It follows from (2.2) that

$$\left|\sum_{0\le y_i\le t_i} e(q\xi_i y_i)\right| \ll \min\left\{t_i+1, \|q\xi_i\|^{-1}\right\} \le P^{1-(Q_0'-\varepsilon_0)/r_1} \qquad (1 \le i \le B). \tag{2.9}$$

Then for $\boldsymbol{\epsilon} \in \{0,1\}^B$, we have by (2.5) and (2.9) that

$$\int_{[0,(P-k_1)/q]} \cdots \int_{[0,(P-k_B)/q]} \partial^{\boldsymbol{\epsilon}} f\left(\left(\frac{(P-\mathbf{k})}{q}\right)_{\overline{\boldsymbol{\epsilon}}}+\mathbf{t}_{\boldsymbol{\epsilon}}\right) \cdot \sum_{\substack{0\le y_i\le (P-k_i)/q\\ \epsilon_i=0}} \sum_{\substack{0\le y_i\le t_i\\ \epsilon_i=1}} e\left(\sum_{1\le i\le B} q\xi_i y_i\right) \mathrm{d}t_B \dots \mathrm{d}t_1 \ll q^{\epsilon_1+\cdots+\epsilon_B} P^{(\theta_0-1)(\epsilon_1+\cdots+\epsilon_B)} \left(\prod_{1\le i\le B} \frac{P-k_i}{q}\right) \cdot P^{B-B(Q_0'-\varepsilon_0)/r_1}. \tag{2.10}$$

Therefore, by Lemma 2.3 and (2.10) we obtain for any $0 \le y_i \le (P-k_i)/q$ $(B < i \le n)$,

$$\left|\sum_{\substack{0\le y_i\le (P-k_i)/q\\ (1\le i\le B)}} f(\mathbf{y}) e\left(\sum_{1\le i\le B} q\xi_i y_i\right)\right| \ll \sum_{\boldsymbol{\epsilon}\in\{0,1\}^B} \left(\prod_{1\le i\le B}\left(\frac{P-k_i}{q}\right)^{\epsilon_i-1}\right) q^{\epsilon_1+\cdots+\epsilon_B} \cdot P^{(\theta_0-1)(\epsilon_1+\cdots+\epsilon_B)} \left(\prod_{1\le i\le B} \frac{P-k_i}{q}\right) P^{B-B(Q_0'-\varepsilon_0)/r_1} \ll P^{B\theta_0} P^{B-B(Q_0'-\varepsilon_0)/r_1}. \tag{2.11}$$

Thus we obtain that (2.4) is bounded by

$$\begin{aligned}|S(\boldsymbol{\alpha})| &\ll q^n \left(\frac{P}{q}\right)^{n-B} P^{B\theta_0} P^{B-B(Q_0'-\varepsilon_0)/r_1} \\ &\leq q^B P^{n+B\theta_0-B(Q_0'-\varepsilon_0)/r_1} \\ &\leq P^{n+2B\theta_0-B(Q_0'-\varepsilon_0)/r_1}.\end{aligned}$$

Since we chose $Q$ to satisfy

$$Q < B\left(\frac{Q_0/2-\varepsilon_0}{r_1} - 2\theta_0\right),$$

it follows that we are in alternative $(i)$ as long as $P$ is sufficiently large with respect to $\mathbf{u}$, $d$, $n$, $r_d, \dots, r_1$, and $Q$. □

Recall the definition of $g_\ell(\mathbf{U}_\ell)$ and $\gamma_\ell$ from above. For $\ell$ with $r_\ell > 0$, we also define

$$\gamma_\ell' = \frac{2^{\ell-1}}{g_\ell(\mathbf{U}_\ell)} = \frac{\gamma_\ell}{(\ell-1)r_\ell}. \tag{2.12}$$

From Lemma 2.2, we obtain the following corollary which is the inhomogeneous polynomials version of [5, p. 276, Corollary], and it is also obtained by essentially the same proof.

**Corollary 2.5** ([5, p. 276, Corollary])**.** *Suppose* $\mathbf{u}$ *has coefficients in* $\mathbb{Z}$*. Let* $Q>0$ *and* $\varepsilon>0$*. Let* $2 \leq \ell \leq d$ *with* $r_\ell > 0$*. Let* $P$ *be sufficiently large with respect to* $d$ *and* $r_d, \dots, r_1$*. If* $\ell = d$*, then let* $\theta = 0$ *and* $q = 1$*. On the other hand, if* $2 \leq \ell < d$*, then suppose* $0 \leq \theta < 1/4$ *and that there is* $q \in \mathbb{N}$ *with*

$$q \leq P^\theta \quad \textit{and} \quad \|q\boldsymbol{\alpha}_j\| \leq P^{\theta-j} (\ell < j \leq d).$$

*Let* $S(\boldsymbol{\alpha})$ *be the sum associated to* $\mathbf{u}$ *as in* (1.1)*. Suppose*

$$4\theta + Q\gamma_\ell' < 1.$$

*Then one of the following two alternatives must hold:*

(1) $|S(\boldsymbol{\alpha})| \leq P^{n-Q}$*.*
(2) *There exists* $n_0 \in \mathbb{N}$ *such that*

$$n_0 \ll P^{Q\gamma_\ell + \varepsilon} \quad \textit{and} \quad \|n_0 q \boldsymbol{\alpha}_\ell\| \ll P^{-\ell+4\theta+Q\gamma_\ell+\varepsilon}.$$

*The implicit constants depend at most on* $n, d, r_d, \dots, r_1, \varepsilon$*, and* $\mathbf{u}$*.*

The above corollary does not deal with the case $\ell = 1$, and we take care of this in the following lemma.

**Lemma 2.6.** [5, Lemma 15.2] *Suppose* $\mathbf{u}$ *has coefficients in* $\mathbb{Z}$*, and that*

$$\mathcal{B}_1(\mathbf{u}_1) > 2r_1 \left( \max\left\{ 4(r_1+1)\left(\sum_{j=2}^d 4^{j-2}\gamma_j\right), \frac{1}{4(R+1)} \right\}\right)^{-1}.$$

*Let $\varepsilon > 0$ be sufficiently small. Let $Q > 0$ satisfy*

$$Q(8r_1+8)\left(\sum_{j=2}^{d} 4^{j-2}\gamma_j\right) < 1 \quad \textit{and} \quad \frac{Q}{2(R+1)} < 1.$$

*Let $S(\boldsymbol{\alpha})$ be the sum associated to $\mathbf{u}$ as in* (1.1)*. Suppose $P$ is sufficiently large with respect to $d$, $n$, $r_d, \dots, r_1$, $\varepsilon$, $Q$, and $\mathbf{u}$. Then one of the following two alternatives must hold:*

(1) $|S(\boldsymbol{\alpha})| \leq P^{n-Q}$.

(2) *There exist $n_1, n_2, \dots, n_d \in \mathbb{N}$ such that*

$$n_\ell \ll P^{Q\gamma_\ell+\varepsilon} \quad \textit{and} \quad \|n_d \dots n_\ell \boldsymbol{\alpha}_\ell\| \ll P^{-\ell+Q\left(\sum_{j=\ell}^{d} 4^{j-\ell}\gamma_j\right)+\varepsilon} (2 \leq \ell \leq d),$$
$$n_1 \leq P^{M_0 Q} \quad \textit{and} \quad \|n_1 \boldsymbol{\alpha}_1\| \leq P^{-1+M_0 Q},$$

*where*

$$M_0 = \max\left\{8(r_1+1)\left(\sum_{j=2}^{d} 4^{j-2}\gamma_j\right), \frac{1}{2(R+1)}\right\}.$$

*The implicit constants depend at most on $n, d, r_d, \dots, r_1, \varepsilon$, and $\mathbf{u}$.*

*Proof.* We begin by proceeding as in the proof of [5, Lemma 15.2]. Suppose we have

$$|S(\boldsymbol{\alpha})| > P^{n-Q}.$$

Let $\varepsilon_d > 0$ be sufficiently small. Since $Q\gamma'_d < 1$, by Corollary 2.5 there exists $n_d \in \mathbb{N}$ with

$$n_d \ll P^{Q\gamma_d+\varepsilon_d} \quad \text{and} \quad \|n_d \boldsymbol{\alpha}_d\| \ll P^{-d+Q\gamma_d+\varepsilon_d}.$$

Suppose now that $r_{d-1} > 0$. Since $4Q\gamma_d + Q\gamma'_{d-1} < 1$, we can apply Corollary 2.5 again with $\ell = d-1$, $\theta = Q\gamma_d + 2\varepsilon_d$, and $q = n_d$. Note we have by our assumption on $Q$ that $\theta < 1/4$. Let $\varepsilon_{d-1} > 0$ be sufficiently small. Thus there exists $n_{d-1} \in \mathbb{N}$ with

$$(2.13) \quad n_{d-1} \ll P^{Q\gamma_{d-1}+\varepsilon_{d-1}} \quad \text{and} \quad \|n_d n_{d-1} \boldsymbol{\alpha}_{d-1}\| \ll P^{-(d-1)+4Q\gamma_d+8\varepsilon_d+Q\gamma_{d-1}+\varepsilon_{d-1}}.$$

In the case $r_{d-1} = 0$, we have $\gamma_{d-1} = 0$ and obtain (2.13) trivially with $n_{d-1} = 1$. It is clear we can continue in this manner. By repeating the argument, we ultimately obtain that there exist $n_2, \dots, n_d \in \mathbb{N}$ such that

$$n_\ell \ll P^{Q\gamma_\ell+\varepsilon} \quad \text{and} \quad \|n_d \dots n_\ell \boldsymbol{\alpha}_\ell\| \ll P^{-\ell+Q\left(\sum_{j=\ell}^{d} 4^{j-\ell}\gamma_j\right)+\varepsilon} (2 \leq \ell \leq d).$$

If $r_1 = 0$, then we are done trivially with $n_1 = 1$. Let $r_1 > 0$. We now apply Lemma 2.4 with

$$\theta_0 = \left( \sum_{j=2}^{d} 4^{j-2} \gamma_j \right) Q + d\varepsilon < 1,$$

where $\varepsilon > 0$ is sufficiently small,

$$Q_0/2 = \max \left\{ 4(r_1+1) \left( \sum_{j=2}^{d} 4^{j-2} \gamma_j \right) Q, \frac{Q}{4(R+1)} \right\} < \frac{1}{2},$$

and

$$q = (n_d \dots n_2) \leq P^{\theta_0},$$

where the last inequality holds for $P$ sufficiently large. Let $\varepsilon_0 > 0$ be sufficiently small. With these choices of $\theta_0$ and $Q_0$, we have

$$2\theta_0 < (Q_0/2 - \varepsilon_0)/(2r_1) < (Q_0/2 - \varepsilon_0)/r_1.$$

With our assumption on $\mathcal{B}_1(\mathbf{u}_1)$, it is clear that we have

$$Q < \mathcal{B}_1(\mathbf{u}_1) \left( \frac{Q_0/2 - \varepsilon_0}{2r_1} \right) < \mathcal{B}_1(\mathbf{u}_1) \left( \frac{Q_0/2 - \varepsilon_0}{r_1} - 2\theta_0 \right).$$

Therefore, it follows by Lemma 2.4 that there exists $n_1 \in \mathbb{N}$ such that

$$n_1 \leq P^{Q_0} \quad \text{and} \quad \|n_1 \boldsymbol{\alpha}_1\| \leq P^{Q_0 - 1}. \qquad \square$$

We are now in position to prove our main result.

*Proof of Theorem 1.1.* By the hypotheses, we know that

$$(8r_1+8)\Delta\Omega \left( \gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \cdots + 4^{d-2}\gamma_d \right) < 1,$$
$$\frac{\Delta\Omega}{2(R+1)} < 1,$$

and

$$\Omega \left( \gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \cdots + 4^{d-2}\gamma_d \right) + \Omega M_0 < 1, \tag{2.14}$$

where

$$M_0 = \max \left\{ 8(r_1+1) \left( \sum_{j=2}^{d} 4^{j-2} \gamma_j \right), \frac{1}{2(R+1)} \right\}$$

as in the statement of Lemma 2.6.

Let

$$\varepsilon_0' = \frac{1}{2\Omega} \left( 1 - \Omega \left( \gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \cdots + 4^{d-2}\gamma_d \right) - \Omega M_0 \right). \tag{2.15}$$

We apply Lemma 2.6 with $Q = \Delta\Omega$. If the alternative (1) of Lemma 2.6 holds then we are done. Let us suppose we have the alternative (2) of Lemma 2.6. Then for $P$ sufficiently large, we have

$$q := n_d \dots n_2 n_1 \leq P^{\Delta\Omega\left(\sum_{j=2}^{d} 4^{j-2}\gamma_j\right) + \Delta\Omega M_0 + \Delta\Omega\varepsilon_0'},$$

and

$$\|q\boldsymbol{\alpha}_\ell\| \leq P^{-\ell + \Delta\Omega\left(\sum_{j=2}^{d} 4^{j-2}\gamma_j\right) + \Delta\Omega M_0 + \Delta\Omega\varepsilon_0'} \quad (1 \leq \ell \leq d).$$

Since

$$\Omega\left(\gamma_2 + 4\gamma_3 + 4^2\gamma_4 + \dots + 4^{d-2}\gamma_d\right) + \Omega M_0 + \Omega\varepsilon_0' < 1,$$

we obtain our result. □

We also prove the following lemma which becomes useful in some applications of the Hardy–Littlewood circle method. The proof is based on that of [5, Lemma 8.1]. Let

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) = \int_{\mathbf{v} \in \mathfrak{B}_0} e\left(\sum_{\ell=1}^{d}\sum_{r=1}^{r_\ell} \tau_{\ell,r} \cdot U_{\ell,r}(\mathbf{v})\right) \mathbf{dv}.$$

**Lemma 2.7** ([5, Lemma 8.1]). *Suppose* $\mathbf{u}$ *has coefficients in* $\mathbb{Z}$*, and that* $\mathcal{B}_1(\mathbf{u}_1)$ *is sufficiently large with respect to* $r_d, \dots, r_1$*, and* $d$*. Furthermore, suppose* $\gamma_2, \dots, \gamma_d$ *are sufficiently small with respect to* $r_d, \dots, r_1$*, and* $d$*. Then we have*

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) \ll \min(1, |\boldsymbol{\tau}|^{-R-1}), \tag{2.16}$$

*where the implicit constant depends at most on* $n$*,* $d$*,* $r_d, \dots, r_1$*, and* $\mathbf{U}$*.*

*Proof.* Given $\mathbf{a} = (\mathbf{a}_d, \dots, \mathbf{a}_1) \in (\mathbb{Z}/q\mathbb{Z})^R$, where $\mathbf{a}_\ell = (a_{\ell,1}, \dots, a_{\ell,r_\ell}) \in (\mathbb{Z}/q\mathbb{Z})^{r_\ell}$ $(1 \leq \ell \leq d)$ and $\gcd(\mathbf{a}, q) = 1$, let us define

$$\begin{aligned}&\widetilde{\mathfrak{M}}_{\mathbf{a},q}((R+2)^{-1}) \\ &\quad = \left\{\boldsymbol{\alpha} \in [0,1)^R : \max_{1 \leq r \leq r_\ell} |q\alpha_{\ell,r} - a_{\ell,r}| \leq P^{(R+2)^{-1}}/P^{\ell} (1 \leq \ell \leq d)\right\},\end{aligned}$$

and let

$$\widetilde{\mathfrak{M}} = \bigcup_{q \leq P^{(R+2)^{-1}}} \bigcup_{\substack{\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^R \\ \gcd(\mathbf{a},q)=1}} \widetilde{\mathfrak{M}}_{\mathbf{a},q}((R+2)^{-1}).$$

Note the boxes $\widetilde{\mathfrak{M}}_{\mathbf{a},q}((R+2)^{-1})$ with $q \leq P^{(R+2)^{-1}}$, $\mathbf{a} \in (\mathbb{Z}/q\mathbb{Z})^R$, and $\gcd(\mathbf{a}, q) = 1$ are disjoint when $P$ is sufficiently large.

Suppose $|\boldsymbol{\tau}| > 2$. Let $P\mathbf{v} = \mathbf{v}'$ so that we have

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) = \frac{1}{P^n} \int_{P\mathfrak{B}_0} e\left(\sum_{\ell=1}^{d}\sum_{r=1}^{r_\ell} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}')\right) \mathbf{dv}',$$

where

$$\beta_{\ell,r} = \frac{\tau_{\ell,r}}{P^{\ell}} (1 \leq \ell \leq d, 1 \leq r \leq r_{\ell}). \tag{2.17}$$

Let $P = |\boldsymbol{\tau}|^{R+2}$, and consider the exponential sum

$$S(\boldsymbol{\beta}) = \sum_{\mathbf{x} \in P\mathfrak{B}_0 \cap \mathbb{Z}^n} e\left( \sum_{\ell=1}^{d} \sum_{r=1}^{r_{\ell}} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{x}) \right).$$

Then $\boldsymbol{\beta}$ lies on the boundary of the box $\widetilde{\mathfrak{M}}_{\mathbf{0},1}((R+2)^{-1})$. Thus for $|\boldsymbol{\tau}|$ sufficiently large, $\boldsymbol{\beta}$ lies on the boundary of the set $\widetilde{\mathfrak{M}}$, which is precisely the set considered in the alternative $(ii)$ of Proposition 1.1 with $\Delta = (R+2)^{-1}$. Consequently, $\boldsymbol{\beta}$ also lies on the boundary of $[0,1)^R \backslash \widetilde{\mathfrak{M}}$. Since $|S(\boldsymbol{\alpha})|$ is a continuous function, we obtain via Theorem 1.1 (with $\Omega = R+1$) that

$$|S(\boldsymbol{\beta})| \leq P^{n-(R+2)^{-1}\Omega} = P^n |\boldsymbol{\tau}|^{-\Omega} = P^n |\boldsymbol{\tau}|^{-R-1}. \tag{2.18}$$

Note with the hypothesis of this lemma, we have

$$\begin{aligned} \min \left\{ \frac{1}{8r_1+9} \left( \sum_{j=2}^{d} 4^{j-2}\gamma_j \right)^{-1}, \left( \frac{1}{2(R+1)} + \sum_{j=2}^{d} 4^{j-2}\gamma_j \right)^{-1} \right\} & \\ = \left( \frac{1}{2(R+1)} + \sum_{j=2}^{d} 4^{j-2}\gamma_j \right)^{-1} & \\ > R+1, & \end{aligned}$$

which justifies our application of Theorem 1.1 with $\Omega = R+1$. We also have

$$\begin{aligned} S(\boldsymbol{\beta}) - \int_{P\mathfrak{B}_0} & e\left( \sum_{\ell=1}^{d} \sum_{r=1}^{r_{\ell}} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}') \right) \mathbf{dv}' \\ &= \sum_{\mathbf{x} \in [0,P)^n} \int_{x_1}^{x_1+1} \cdots \int_{x_n}^{x_n+1} e\left( \sum_{\ell=1}^{d} \sum_{r=1}^{r_{\ell}} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{x}) \right) \\ &\qquad - e\left( \sum_{\ell=1}^{d} \sum_{r=1}^{r_{\ell}} \beta_{\ell,r} \cdot U_{\ell,r}(\mathbf{v}') \right) \mathbf{dv}' + O(P^{n-1}) \\ &\ll P^n \frac{|\boldsymbol{\tau}|}{P} + O(P^{n-1}) \\ &\ll P^{n-1} |\boldsymbol{\tau}|, \end{aligned}$$

where we applied the mean value theorem and (2.17) to obtain the second last inequality. Therefore, it follows that

$$S(\boldsymbol{\beta}) = P^n \mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) + O(P^{n-1}|\boldsymbol{\tau}|).$$

It is then easy to deduce from (2.18) that

$$\mathcal{I}(\mathfrak{B}_0, \boldsymbol{\tau}) \ll \min\{1, |\boldsymbol{\tau}|^{-R-1}\}. \qquad \square$$

Finally, let us remark that from the above proof it is clear that the only assumptions on $\mathcal{B}_1(\mathbf{u}_1), \gamma_2, \dots, \gamma_d$ we needed to prove Lemma 2.7 were that they satisfy the hypotheses of Theorem 1.1 with $\Omega = R + 1$.

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