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Sajad SALAMI

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# Twists of the Albanese varieties of cyclic multiple planes with large ranks over higher dimension function fields

par SAJAD SALAMI

RÉSUMÉ. Dans [17], nous avons prouvé un théorème de structure pour les groupes de Mordell–Weil de variétés abéliennes définies sur des corps de fonctions, obtenues comme tordues de variétés abéliennes par des revêtements cycliques de variétés projectives, et ce en terme des variétés de Prym associées à ces revêtements. Dans ce nouvel article, nous donnons une méthode explicite pour construire des variétés abéliennes de grands rangs sur les corps de fonctions. Pour ce faire, nous appliquons le théorème mentionné ci-dessus aux twists des variétés d’Albanese des plans multiples cycliques.

ABSTRACT. In [17], we proved a structure theorem on the Mordell–Weil group of abelian varieties over function fields that arise as the twists of abelian varieties by the cyclic covers of projective varieties in terms of the Prym varieties associated with covers. In this paper, we provide an explicit way to construct the abelian varieties with large ranks over the higher dimension function fields. To do so, we apply the above-mentioned theorem to the twists of Albanese varieties of the cyclic multiple planes.

## 1. Introduction

Let  $\mathcal{A}$  be an abelian variety defined over a given field  $k$  of characteristic 0 or a prime  $p > 0$ . Denote by  $\mathcal{A}(k)$  the set of  $k$ -rational points on  $\mathcal{A}$ . It is well-known [7, 11] that  $\mathcal{A}(k)$  is a finitely generated abelian group for the number fields as well as the function fields under mild conditions. Thus, we have  $\mathcal{A}(k) \cong \mathcal{A}(k)_{tors} \oplus \mathbb{Z}^r$  where  $\mathcal{A}(k)_{tors}$  is a finite group called the *torsion subgroup* of  $\mathcal{A}(k)$ , and  $r$  is a positive number called the *Mordell–Weil rank* or simply the *rank* of  $\mathcal{A}(k)$  and denoted by  $\text{rk}(\mathcal{A}(k))$ . Let  $\mathcal{A}[n](k)$  be the group of  $k$ -rational  $n$ -division points on  $\mathcal{A}$  for any integer  $n \geq 2$ . It is remarkable that studying on the rank of abelian varieties is more difficult than that of the torsion subgroups.

Finding the abelian varieties of a given dimension with large rank is one of the challenging problems in the modern number theory. There are

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some interesting works in the literature depending on the ground field and the dimension of abelian varieties. For example, Shafarevitch and Tate [28] produced isotrivial elliptic curves with arbitrary large rank over the field  $\mathbb{F}_p(t)$ , and a similar result was proved by Ulmer in [30] for non-isotrivial case. In the case of higher dimension abelian varieties over  $\mathbb{F}_p(t)$ , Ulmer showed [31] that for a given prime number  $p \geq 2$  and the integers  $g \geq 2$  and  $r \geq 1$  there exist absolutely simple, non-isotrivial Jacobian varieties of dimension  $g$  and rank  $\geq r$  over  $\mathbb{F}_p(t)$  for which the Birch and Swinnerton–Dyer’s conjecture holds. In [32], Ulmer related the Mordell–Weil group of certain Jacobian varieties over  $k_0(t)$ , where  $k_0$  is an arbitrary field, with the group of homomorphisms of other Jacobian varieties. We note that a similar result has been proved by several authors in the literature with different methods [6, 15, 21, 34]. He also showed the unboundedness of the rank of elliptic curves over  $\bar{\mathbb{F}}_p(t)$  by providing an concrete example of elliptic curve of large rank with the explicit independent points.

In [13, 14], Lapin stated the unboundedness of the rank of elliptic curves over  $\mathbb{C}(t)$ , but his proof had a gap! The geometric methods of Ulmer in [32] led to the construction of elliptic curves of moderate rank over  $\mathbb{C}(t)$ . Moreover, his method suggested a potential way to show the unboundedness of rank of the elliptic curves over  $\mathbb{C}(t)$ , but he mentioned that it seems likely to himself that this does not happen.

In his recent work [33], Ulmer proved that for a very general elliptic curve  $E$  over  $\mathbb{C}(t)$  with height  $d \geq 3$  and for any finite rational extension  $\mathbb{C}(u)$  of  $\mathbb{C}(t)$  the group  $E(\mathbb{C}(u))$  is trivial. It is remarkable that the largest known rank of elliptic curves over  $\mathbb{C}(t)$  is 68 due to T. Shioda [24]. Using the theory of the Mordell–Weil Lattices and symmetry [23, 25, 26], he also showed the existence of high rank Jacobians of curves with genus greater than one defined over function fields with base  $\mathbb{Q}$ . Over the field  $\mathbb{C}(x, y)$ , in [16], Libgober proved that there are certain simple Jacobians of rank  $p - 1$  for any prime number  $p$ .

The aim of this paper is to provide an explicit method of the construction of abelian varieties with the large ranks over function field of certainly defined quotient varieties of high dimensions. The main tools in this work are the theory of twists as well as the Albanese and Prym varieties.

Given a fixed integer  $n \geq 2$ , we assume that  $k$  is a field of characteristic 0 or a prime  $p > 0$  not dividing  $n$ , that contains an  $n$ -th root of unity  $\zeta_n$ . We denote by  $\mathbb{P}^2$  the projective plane over  $\bar{k}$ , an algebraically closed field containing  $k$ . Given any polynomial  $f(x, y) \in k[x, y]$  of degree  $r \geq 2$ , let  $\mathcal{X}_n$  be the non-singular projective model of the hypersurface  $X_n$  defined by the affine equation  $w^n = f(x, y)$ . For any integer  $m \geq 2$ , we define  $\mathcal{U}_m$  to be the fibered product of  $m$  copies of  $\mathcal{X}_n$  over  $k$ . Then, we let  $\mathcal{V}_m$  be the quotient of  $\mathcal{U}_m$  by a certain cyclic subgroup of  $\text{Aut}(\mathcal{U}_m)$ , see Section 5 for

more details. We also denote by  $\tilde{\mathcal{X}}_n$  the twist of  $\mathcal{X}_n$  by the cyclic extension  $\mathcal{L}|\mathcal{K}$ , where  $\mathcal{K} = k(\mathcal{V}_m)$  and  $\mathcal{L} = k(\mathcal{U}_m)$  are the function fields of  $\mathcal{U}_m$  and  $\mathcal{V}_m$  respectively. We let  $\widetilde{\text{Alb}(\mathcal{X}_n)}$  be the twist of Albanese variety  $\text{Alb}(\mathcal{X}_n)$  of  $\mathcal{X}_n$  by the extension  $\mathcal{L}|\mathcal{K}$ . We refer the reader to see Sections 2 and 3 for the definitions and basic properties of the Albanese and Prym varieties, respectively.

The main results of the paper are given as follows.

**Theorem 1.1.** *Notation being as above, we assume that there exists at least one  $k$ -rational point on  $X_n$  and hence on  $\mathcal{X}_n$ . Then, as an isomorphism of abelian groups, we have:*

$$\widetilde{\text{Alb}(\mathcal{X}_n)}(\mathcal{K}) \cong \text{Alb}(\tilde{\mathcal{X}}_n)(\mathcal{K}) \cong (\text{End}_k(\text{Alb}(\mathcal{X}_n)))^m \oplus \text{Alb}(\mathcal{X}_n)[m](k),$$

and hence,  $\text{rk}(\widetilde{\text{Alb}(\mathcal{X}_n)}(\mathcal{K})) = m \cdot \text{rk}(\text{End}_k(\text{Alb}(\mathcal{X}_n)))$ , where  $\text{End}_k(*)$  is the ring of endomorphisms over  $k$  of its origin  $(*)$  and  $\text{rk}(\text{End}_k(*))$  denotes its rank as a  $\mathbb{Z}$ -module.

This theorem generalizes the main results of [6, 21, 34] for the higher dimensional abelian varieties risen as the twist of Albanese variety of the cyclic  $n$ -covers of a projective plane. For a given cyclic  $n$ -cover  $\mathcal{X}_n$ , by enlarging the integer  $m$ , one may obtain the abelian varieties of arbitrary large rank over the function field of  $\mathcal{W}_m$  with the base field  $k$  of arbitrary characteristic.

On the other hand, for a fixed integer  $m \geq 2$ , one may be interested to compute the rank of  $\widetilde{\text{Alb}(\mathcal{X}_n)}(\mathcal{K})$  in terms of  $m$  and the other geometric quantities of  $\text{Alb}(\mathcal{X}_n)$ , for example its dimension  $d_n = \dim(\text{Alb}(\mathcal{X}_n))$ . In general,  $d_n \leq \dim H^0(\mathcal{X}_n, \Omega^1_{\mathcal{X}_n})$ , where  $\Omega^1_{\mathcal{X}_n}$  is the vector space of differential 1-forms on  $\mathcal{X}_n$ . We note that “=” holds when the field  $k$  is of characteristic zero. In the following we are going to calculate  $\text{rk}(\text{End}_k(\text{Alb}(\mathcal{X}_n)))$  in terms of  $d_n$  when  $k$  is a number field.

We assume that  $k \subset \mathbb{C}$  is a number field which contains  $\mathbb{Q}(\zeta_n)$  and let  $\pi_n : \mathcal{X}_n \rightarrow \mathbb{P}^2$  be a cyclic  $n$ -cover. Its Galois group is a cyclic group generated by the order  $n$  automorphism  $\bar{\tau}$  of  $\mathcal{X}_n$  induced by

$$\tau : (x, y, w) \mapsto (x, y, \zeta_n \cdot w),$$

an automorphism of  $X_n$  which acts on  $\mathcal{X}_n$  and induces an action on  $\text{Alb}(\mathcal{X}_n)$ . Let  $\text{Alb}(\mathcal{X}_n) \cong W_n/\Lambda_n$ , where  $W_n$  is a  $\mathbb{C}$ -vector space of dimension  $d_n$  and  $\Lambda_n$  is a lattice. Then  $\Lambda_n \otimes \mathbb{C}$  is of dimension  $2d_n$  as a  $\mathbb{C}$ -vector space and we have  $\Lambda_n \otimes \mathbb{C} = \bigoplus_{\lambda} H_{\lambda}$ , where  $H_{\lambda}$  is the eigenspace of  $\Lambda_n \otimes \mathbb{C}$  corresponding to each character  $\lambda$  of  $\mu_n$  the cyclic group of primitive  $n$ -th roots. Therefore, we can write  $W_n = \bigoplus_{\lambda} W_{n,\lambda}$ , where  $W_{n,\lambda} = W_n \cap H_{\lambda}$ . To state the next results, we shall consider the following assumption.

**Assumption 1.2.** *Keeping the above notations, we further assume that  $\Lambda_n$  is a  $\mathbb{Z}[\zeta_n]$ -module and no conjugate character appears in  $W_n$ .*

**Theorem 1.3.** *Keeping the hypothesis of Theorem 1.1 and the above notation, we suppose further that  $k \subset \mathbb{C}$  is a number field containing  $\mathbb{Q}(\zeta_n)$  and Assumption 1.2 holds on  $\text{Alb}(\mathcal{X}_n)$  for the positive integer  $n$  satisfying  $3 \leq n \leq 12$  but  $n \neq 7, 9, 11$ . Then,*

$$\text{rk}(\widetilde{\text{Alb}}(\mathcal{X}_n)(\mathcal{K})) \geq m \cdot c_n,$$

where  $c_n = 2d_n^2$  for  $n = 3, 4, 5, 6, 10$  and  $c_n = d_n^2 - 8n_1n_2$  with  $n_1 + n_2 = d_n/2$  for  $n = 8, 12$ . Moreover, the equality holds for all cases except  $n = 8$ .

The next theorem gives a sufficient condition that provides a way to find explicit examples for the case  $n = 6$  in the above theorem.

**Theorem 1.4.** *Keeping the hypothesis of Theorem 1.1, we assume that  $k \subset \mathbb{C}$  is a number field containing  $\mathbb{Q}(\zeta_3)$  and Assumption 1.2 holds on  $\text{Alb}(\mathcal{X}_6)$ . Let  $F(u_0, u_1, u_2)$  be a homogeneous polynomial that can be written in two forms  $F = G_1^2 + H_1^3 = G_2^2 + H_2^3$ , where  $G_1, G_2, H_1, H_2 \in k[u_0, u_1, u_2]$  are homogeneous polynomials such that*

$$\{\lambda_0 G_1^2 + \lambda_1 H_1^3 : [\lambda_0 : \lambda_1] \in \mathbb{P}^1\}, \quad \{\lambda_0 G_2^2 + \lambda_1 H_2^3 : [\lambda_0 : \lambda_1] \in \mathbb{P}^1\},$$

are two different pencils. Then,  $\text{rk}(\widetilde{\text{Alb}}(\mathcal{X}_6)(\mathcal{K})) = 2md_6^2$ . One may consider the following two cases as explicit examples:

- (i)  $\mathcal{X}_6$  is associated to  $\mathcal{C} = \mathcal{E}^\vee$  the dual of a smooth plane cubic  $\mathcal{E}$ ;
- (ii)  $\mathcal{X}_6$  is associated to the affine curve

$$C : f(x, y) = x^3 - 3xy(y^3 - 8) + 2(y^6 - 20y^3 - 8) = 0.$$

**Remark 1.5.**

- (1) The Theorem 1.1 can be generalized for the twists of Albanese variety associated with  $n$ -covers of the projective space  $\mathbb{P}^\ell$  of dimension  $\ell \geq 3$ . Indeed, given any polynomial  $f(u_1, \dots, u_\ell)$  with coefficients in  $k$ , if  $\mathcal{X}_n$  denotes a non-singular projective model of the hypersurface defined by the affine equation  $w^n = f(u_1, \dots, u_\ell)$ , and  $\tilde{\mathcal{X}}_n, \mathcal{U}_m, \mathcal{V}_m, \mathcal{K}$  and  $\mathcal{L}$  are defined in a similar way, then the assertion of Theorem 1.1 hold for  $\tilde{\mathcal{X}}_n$  by adapting its proof. Furthermore, it is remarkable that the cyclic  $n$ -covers of the projective line  $\mathbb{P}^1$  has been treated by the author in [21].
- (2) One can use the Theorems 1.3 and 1.4 to obtain abelian varieties of arbitrary large ranks over the number fields by considering the Silverman’s specialization theorem [27] and study on the  $k$ -rational points on the variety  $\mathcal{V}_m$ .

The present paper is organized as follows. The Section 2 is devoted to recalling the definitions and fundamental properties of the Albanese and Prym varieties. In Section 3, we briefly review the basics of twisting theory of algebraic varieties. Then, we provide some results on the multiple planes  $\mathcal{X}_n$  in Section 4. We give the proofs of Theorems 1.1, 1.3 and 1.4 in Sections 5 and 6.

### 2. Albanese and Prym varieties

In this section, we let  $\mathcal{X}$  be a non-singular projective variety over  $\bar{k}$  and  $\text{Pic}(\mathcal{X})$  denotes its reduced Picard variety. By an *abelian variety over  $\bar{k}$*  we mean an algebraic group  $\mathcal{A}$  over  $\bar{k}$  which is non-singular, proper, and connected as a variety. In below, we are going to give the definition and basic properties of an abelian variety associated to  $\mathcal{X}$ . For more details, we refer the reader to [10, 22].

**Definition 2.1.** The *Albanese variety* of  $\mathcal{X}$  is an abelian variety  $\text{Alb}(\mathcal{X})$  together with the morphism of varieties  $\alpha : \mathcal{X} \rightarrow \text{Alb}(\mathcal{X})$  satisfying the following universal property:

For any morphism  $\alpha' : \mathcal{X} \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is an abelian variety, there exists a unique homomorphism  $\alpha'' : \text{Alb}(\mathcal{X}) \rightarrow \mathcal{A}$ , up to a translation, such that the following diagram commutes:

$$(2.1) \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\alpha} & \text{Alb}(\mathcal{X}) \\ \downarrow \text{id} & & \downarrow \exists! \alpha'' \\ \mathcal{X} & \xrightarrow{\alpha'} & \mathcal{A} \end{array}$$

The morphism  $\alpha$  is called the *Albanese morphism* and  $\alpha(\mathcal{X})$  is known as the *Albanese image* of  $\mathcal{X}$ . Moreover, the dimension of  $\alpha(\mathcal{X})$  is called the *Albanese dimension* of  $\mathcal{X}$ .

The notion of Albanese variety is a classic one in the case  $\bar{k} \cong \mathbb{C}$ . Indeed, for a smooth complex variety  $\mathcal{X}$  one has  $\text{Alb}(\mathcal{X}) = H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1)^\vee / H^1(\mathcal{X}, \mathbb{Z})$ , and the Albanese map is given by  $x \mapsto \int_{x_0}^x \omega$  where  $x_0$  is a base point and the integral is viewed as a linear function on  $H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1)$  well-defined up to the periods.

It can be shown that if  $\text{Alb}(\mathcal{X})$  exists, then it is unique up to isomorphism. Moreover, it is equal to  $\text{Pic}^\vee(\mathcal{X})$ , the dual of reduced Picard variety of  $\mathcal{X}$ . For example, when  $\mathcal{X}$  is a non-singular projective curve, then the Albanese variety is nothing but the Jacobian variety of  $\mathcal{X}$ .

We note that the Albanese variety  $\text{Alb}(\mathcal{X})$  is generated by  $\alpha(\mathcal{X})$ , i.e., there is no abelian subvariety of  $\text{Alb}(\mathcal{X})$  containing  $\alpha(\mathcal{X})$ . In particular,  $\alpha(\mathcal{X})$  is not reduced to a point if  $\text{Alb}(\mathcal{X})$  is not a singleton set. In general, one has  $\dim \text{Alb}(\mathcal{X}) \leq \dim H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1)$ , where  $\Omega_{\mathcal{X}}^1$  is the vector space of

regular 1-forms on  $\mathcal{X}$  and the equality holds if  $\bar{k}$  is of characteristic zero. We note that the quantity  $\dim H^0(\mathcal{X}, \Omega_{\mathcal{X}}^1)$  is well-known as the *irregularity* of  $\mathcal{X}$ .

The assignment  $\mathcal{X} \mapsto \text{Alb}(\mathcal{X})$  is a covariant functor from the category of non-singular projective varieties to the category of abelian varieties. In other words, for any morphism  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  of the non-singular projective varieties  $\mathcal{X}$  and  $\mathcal{X}'$ , there exists a unique morphism  $\tilde{\pi} : \text{Alb}(\mathcal{X}') \rightarrow \text{Alb}(\mathcal{X})$  such that the digram (2.2) in below commutes. In particular, if the morphism  $\pi$  is surjective, then the morphism  $\tilde{\pi}$  is also surjective.

$$(2.2) \quad \begin{array}{ccc} \mathcal{X}' & \xrightarrow{\pi} & \mathcal{X} \\ \downarrow \alpha' & & \downarrow \alpha \\ \text{Alb}(\mathcal{X}') & \xrightarrow{\tilde{\pi}} & \text{Alb}(\mathcal{X}) \end{array}$$

In the rest of this section, we recall the definition of Prym varieties for cyclic  $n$ -cover  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  of non-singular projective varieties. Classically, the notion of Prym variety has been introduced by Mumford [19] for the double covers of curves over the complexes and it is extensively studied by Beauville in [1]. Then, it has been considered for the double covers of smooth surfaces by Khashin in [8], and for the double covers of projective varieties by Hazama in [6]. Recently, the Prym variety of cyclic  $n$ -covers of curves are studied in [12]. In [18], it is defined in a more abstract setting, but is used for arbitrary covers of curves over the complexes.

**Definition 2.2.** The *Prym variety* of the cyclic  $n$ -covers  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  of non-singular projective varieties is defined as the quotient abelian variety,

$$\text{Prym}_{\mathcal{X}'/\mathcal{X}} := \frac{\text{Alb}(\mathcal{X}')}{\text{Im}(\text{id} + \tilde{\gamma} + \dots + \tilde{\gamma}^{n-1})},$$

where  $\tilde{\gamma}$  is the automorphism on  $\text{Alb}(\mathcal{X}')$  induced by an order  $n$  automorphism  $\gamma$  of  $\mathcal{X}'$ .

When  $\mathcal{X}, \mathcal{X}'$  are irreducible and both of them as well as the cyclic  $n$ -cover  $\pi$  are defined over  $k$ , there is a  $k$ -isogeny of abelian varieties,

$$(2.3) \quad \text{Prym}_{\mathcal{X}'/\mathcal{X}} \sim_k \ker(\text{id} + \tilde{\gamma} + \dots + \tilde{\gamma}^{n-1} : \text{Alb}(\mathcal{X}') \rightarrow \text{Alb}(\mathcal{X}'))^\circ,$$

where  $(*)^\circ$  means the unique connected component of the origin  $(*)$ .

It is useful to construct the new cyclic  $n$ -covers using the given ones. To this end, we let  $\pi_i : \mathcal{X}'_i \rightarrow \mathcal{X}_i$  ( $i = 1, 2$ ) be two cyclic  $n$ -covers of irreducible non-singular projective varieties,  $\gamma_i \in \text{Aut}(\mathcal{X}'_i)$  be an automorphism of order  $n \geq 2$  for  $i = 1, 2$ , all defined over  $k$ . Moreover, we assume that there exist  $k$ -rational points  $x'_i \in \mathcal{X}'_i(k)$  for  $i = 1, 2$ . Then, we have a  $k$ -rational isogeny of abelian varieties,

$$(2.4) \quad \text{Prym}_{\mathcal{X}'_1 \times \mathcal{X}'_2/\mathcal{Y}} \sim_k \text{Prym}_{\mathcal{X}'_1/\mathcal{X}_1} \times \text{Prym}_{\mathcal{X}'_2/\mathcal{X}_2},$$

where  $\mathcal{Y} = \mathcal{X}'_1 \times \mathcal{X}'_2/G$  is the intermediate cover and  $G$  is the cyclic group generated by  $\gamma = (\gamma_1, \gamma_2) \in \text{Aut}(\mathcal{X}'_1 \times \mathcal{X}'_2)$ .

For the proof of above assertions, we refer the reader to [21]. Note that we have to restrict ourselves to non-singular projective varieties in Section 2 of [21] instead of the quasi-projective ones. Because,  $\text{Alb}(\mathcal{X})$  is a semi-abelian variety for any quasi-projective variety  $\mathcal{X}$ . This means that it is an extension of an algebraic group by a torus. For more details, consult [22].

### 3. $G$ -sets and twists

In this section, we briefly recall the two equivalent definitions of the twist of an algebraic variety and its basic properties. To see more on the subject, consult [2].

Let  $\mathcal{K}$  be a field and  $\mathcal{L}|\mathcal{K}$  a finite extension with Galois group  $G = \text{Gal}(\mathcal{L}|\mathcal{K})$ . A  $G$ -set is a discrete topological space  $\mathcal{X}$  such that the left action of  $G$  on  $\mathcal{X}$  is continuous. For every  $x \in \mathcal{X}$  and  $u \in G$ , we denote by  ${}^u x$  the left action of  $u$  on  $x$ . A  $G$ -group is a  $G$ -set  $\mathcal{A}$  equipped with a group structure invariant under the action of  $G$ , i.e.,  ${}^u(x \cdot y) = {}^u x \cdot {}^u y$  for  $x, y \in \mathcal{A}$  and  $u \in G$ . Any continuous application  $a : u \mapsto a_u$  of  $G$  to a  $G$ -set  $\mathcal{A}$  is called a *cochain* of  $G$  with values in  $\mathcal{A}$ . A cochain  $a = (a_u)$  is called a *1-cocycle* of  $G$  with values in  $\mathcal{A}$  if  $a_{uv} = a_u \cdot {}^u a_v$  for  $u, v \in G$ . For any 1-cocycle  $a = (a_u)$ , one has  $a_{\text{id}} = 1$  and  $a_u \cdot {}^u a_{u^{-1}} = 1$ , where  $u \in G$ , and  $1 \in \mathcal{A}$  is the identity element. The set of 1-cocycles of  $G$  with values in a  $G$ -set  $\mathcal{A}$ , is denoted by  $\mathcal{Z}^1(G, \mathcal{A})$ . We say that a  $G$ -group  $\mathcal{A}$  *acts on the  $G$ -set  $\mathcal{X}$  from left*, in a compatible way with the action of  $G$ , if there is an application  $(a, x) \rightarrow a \cdot x$  of  $\mathcal{A} \times \mathcal{X}$  to  $\mathcal{X}$  satisfying the following conditions:

- (i)  ${}^u(a \cdot x) = {}^u a \cdot {}^u x \quad (a \in \mathcal{A}, x \in \mathcal{X}, u \in G)$
- (ii)  $a \cdot (b \cdot x) = (a \cdot b) \cdot x$ , and  $1 \cdot x = x, \quad (a, b \in \mathcal{A}, x \in \mathcal{X})$ .

Let  $\mathcal{A}$  be a  $G$ -group,  $a = (a_u) \in \mathcal{Z}^1(G, \mathcal{A})$  a 1-cocycle of  $\mathcal{A}$ , and  $\mathcal{X}$  a  $G$ -set that is compatible with the group action of  $G$ . For any  $u \in G$  and  $x \in \mathcal{X}$ , define  ${}^{u'} x := a_u \cdot {}^u x$ . The  $G$ -set with this action of  $G$  is denoted by  $\mathcal{X}'_a$  and is called the *twist of  $\mathcal{X}$*  obtained by the 1-cocycle  $a = (a_u)$ .

In what follows, we will give another definition of the twist in terms of the schemes and its relation with the previous definition. Let  $\mathcal{X}$  be a projective scheme defined over  $k$  and denote its function field by  $\mathcal{K}$ . Let  $\text{Aut}(\mathcal{X})$  be the automorphism scheme of  $\mathcal{X}$  and  $a = (a_u) \in \mathcal{Z}^1(G, \text{Aut}(\mathcal{X}))$  be a 1-cocycle. Then,  $\text{Alb}(a) := (\text{Alb}(a_u))$  satisfies the 1-cocycle condition, i.e.,  $\text{Alb}(a) \in \mathcal{Z}^1(G, \text{Aut}(\text{Alb}(\mathcal{X})))$ . Indeed, the equality  $a_{uv} = a_u \circ {}^u a_v$  implies  $\text{Alb}(a_{uv}) = \text{Alb}(a_u) \circ {}^u \text{Alb}(a_v)$ , since the construction of the Albanese variety is compatible with base change. Here, we have a proposition that provides the second definition for the twist of  $\mathcal{X}$ . For its proof, one can see the Propositions 2.6 and 2.7 in [2].



**Proposition 3.1.** *Keeping the above notations, there exist a unique quasi-projective  $\mathcal{K}$ -scheme  $\tilde{\mathcal{X}}$  and a unique  $\mathcal{L}$ -isomorphism*

$$g : \mathcal{X} \otimes_{\mathcal{K}} \mathcal{L} \rightarrow \tilde{\mathcal{X}} \otimes_{\mathcal{K}} \mathcal{L}$$

such that  ${}^u g = g \circ a_u$  holds for any  $u \in G$ . The map  $g$  induces an isomorphism of the twisted  $G$ -set  $\mathcal{X}_a(\mathcal{L})$  onto the  $G$ -set  $\tilde{\mathcal{X}}(\mathcal{L})$ .

The scheme  $\tilde{\mathcal{X}}$  in the above theorem is called the *twist of  $\mathcal{X}$*  by the extension  $\mathcal{L}|\mathcal{K}$ , or equivalently by the 1-cocycle  $a = (a_u)$ .

The following proposition shows the relation between Albanese variety and the twists.

**Proposition 3.2.** *Keeping the above notations, the twist  $\text{Alb}(\mathcal{X})_{\text{Alb}(a)}$  of  $\text{Alb}(\mathcal{X})$  by the 1-cocycle  $\text{Alb}(a)$  is  $\mathcal{K}$ -isomorphic to  $\text{Alb}(\mathcal{X}_a)$ . Equivalently,  $\widetilde{\text{Alb}(\mathcal{X})}$  the twist of  $\text{Alb}(\mathcal{X})$  is  $\mathcal{K}$ -isomorphic to  $\text{Alb}(\tilde{\mathcal{X}})$ .*

*Proof.* If  $g : \mathcal{X} \otimes_{\mathcal{K}} \mathcal{L} \rightarrow \mathcal{X}_a \otimes_{\mathcal{K}} \mathcal{L}$  denotes the isomorphism such that  ${}^u g = g \circ a_u$ , then the induced isomorphism of Albanese varieties

$$\text{Alb}(g) : \text{Alb}(\mathcal{X}) \otimes_{\mathcal{K}} \mathcal{L} \rightarrow \text{Alb}(\mathcal{X}_a) \otimes_{\mathcal{K}} \mathcal{L}$$

satisfies  ${}^u \text{Alb}(g) = \text{Alb}(g) \circ {}^u \text{Alb}(a_u)$  for any  $u \in G$ , by the functoriality of twists. Therefore, the uniqueness of the twist implies that  $\text{Alb}(\mathcal{X})_{\text{Alb}(a)}$  is  $\mathcal{K}$ -isomorphic to  $\text{Alb}(\mathcal{X}_a)$ . Equivalently, we have  $\widetilde{\text{Alb}(\mathcal{X})} \cong_{\mathcal{K}} \text{Alb}(\tilde{\mathcal{X}})$ .  $\square$

In [21], we proved a structure theorem on the set of rational points of the twists of abelian varieties by cyclic covers over function fields of (quasi)-projective varieties. In order to make this work be self-contained and for convenience of the reader, we recall here the main result of [21] which plays an essential rule in the proof of Theorem 1.1.

Given an integer  $n \geq 2$ , let  $\pi : \mathcal{X}' \rightarrow \mathcal{X}$  be a cyclic  $n$ -cover of irreducible projective varieties, both as well as  $\pi$  defined over a field  $k$ . Denote by  $\mathcal{K}$  and  $\mathcal{L}$  the function fields of  $\mathcal{X}$  and  $\mathcal{X}'$  respectively. Assume that  $\mathcal{A}$  is an abelian variety with an automorphism  $\sigma$  of order  $n$ , and let  $\mathcal{A}[n](k)$  be the group of  $k$ -rational  $n$ -division points on  $\mathcal{A}$ . Define  $\tilde{\mathcal{A}}$  to be the twist of  $\mathcal{A}$  by the cyclic extension  $\mathcal{L}|\mathcal{K}$ , or equivalently, by the 1-cocycle  $a = (a_u) \in \mathcal{Z}^1(G, \text{Aut}(\mathcal{A}))$ , where  $a_{\text{id}} = \text{id}$ ,  $a_{\gamma^j} = \sigma^j$  for  $j = 0, \dots, n - 1$ , and  $G$  is the Galois group of the extension  $\mathcal{L}|\mathcal{K}$ . The following theorem describes the structure of Mordell–Weil group of  $\mathcal{K}$ -rational points on the twist  $\tilde{\mathcal{A}}$  which generalizes the main result of [6, 34].

**Theorem 3.3.** *Notation being as above, we assume that there exists at least one  $k$ -rational point on  $\mathcal{X}'$ . Then,*

$$\tilde{\mathcal{A}}(\mathcal{K}) \cong \text{Hom}_k(\text{Prym}_{\mathcal{X}'/\mathcal{X}}, \mathcal{A}) \oplus \mathcal{A}[n](k).$$

as an isomorphism of abelian groups. Moreover, if  $\text{Prym}_{\mathcal{X}'/\mathcal{X}}$  is  $k$ -isogenous with  $\mathcal{A}^m \times \mathcal{B}$  for some integer  $m > 0$  and  $\mathcal{B}$  is an abelian variety defined over  $k$  so that none of its irreducible components is  $k$ -isogenous to  $\mathcal{A}$ , then

$$\text{rk}(\tilde{\mathcal{A}}(\mathcal{K})) = m \cdot \text{rk}(\text{End}_k(\mathcal{A})).$$

We used this opportunity to correct Theorem 1.1 in [21] as the above form by removing the irrelevant conditions on the dimension of  $\mathcal{B}$  as well as changing the symbol “ $\geq$ ” with “ $=$ ” in the assertion of the theorem. Indeed, by rechecking and specially considering the series of isogenies in the proof of above theorem, one can conclude that the dimension of  $\mathcal{B}$  does not have any role in the proof and we have “ $=$ ” instead of “ $\geq$ ”.

### 4. Cyclic multiple planes

Given an integer  $n \geq 2$  and a polynomial  $f(x, y) \in k[x, y]$  of degree  $r \geq 2$ , we suppose that  $F(u_0, u_1, u_2)$  is a homogeneous polynomial such that  $f(x, y) = F(1, x, y)$ . Denote by  $C$  and  $\mathcal{C}$  the affine and projective plane curves defined by  $f = 0$  and  $F = 0$ , respectively. Let  $L_\infty$  be the line at infinity, say  $u_0 = 0$ . Assume that  $e$  is the smallest integer satisfying  $e \geq n/r$  and set  $n_0 = ne - r$ . Define  $X_n$  to be the projective surface given by the affine equation  $w^n = f(x, y)$ , which can be expressed by the equation  $u_3^n = u_0^{n_0} F(u_0, u_1, u_2)$  in the weighted projective space  $\mathbb{P}^3_{(1,1,1,n_0)}$ . Let  $B$  be the branch locus of the map  $p : X_n \rightarrow \mathbb{P}^2$  which drops the last coordinate. Then  $B$  is  $\mathcal{C}$  if  $n_0 = 0$  and  $\mathcal{C} \cup L_\infty$  otherwise. Doing a series of blow-ups gives us a map  $\psi : Y \rightarrow \mathbb{P}^2$  so that  $B' := \psi^{-1}(B)$  has normal crossings. Hence, the projection on the second factor  $p' : X'_n = X_n \times Y \rightarrow Y$  will be a cyclic  $n$ -cover of  $Y$ . Let  $\nu : X''_n \rightarrow X'_n$  be the normalization map and  $p'' : X''_n \rightarrow Y$  the composition of  $\nu$  and  $p'$ , which is again a cyclic  $n$ -cover of  $Y$ . Denote by  $\mathcal{X}_n$  the desingularization of  $X''_n$  and let  $\beta : \mathcal{X}_n \rightarrow X''_n$  be a morphism such that the map  $\tilde{p} = p'' \circ \beta : \mathcal{X}_n \rightarrow Y$  is a cyclic  $n$ -cover of  $Y$  over an open subset of  $Y$  and the Albanese map  $\alpha : \mathcal{X}_n \rightarrow \text{Alb}(\mathcal{X}_n)$  factor through a map  $\alpha' : X''_n \rightarrow \text{Alb}(\mathcal{X}_n)$ . Thus, we obtain a cyclic  $n$ -cover  $\pi_n : \mathcal{X}_n \rightarrow \mathbb{P}^2$  which is the composition  $\psi \circ \tilde{p}$  and fits in the following commutative diagram.

$$(4.1) \quad \begin{array}{ccccc} & & X''_n & \xrightarrow{\alpha'} & \text{Alb}(\mathcal{X}_n) \\ & \swarrow \nu & \downarrow p'' & \swarrow \beta & \uparrow \alpha \\ X'_n & \xrightarrow{p'} & Y & \xleftarrow{\tilde{p}} & \mathcal{X}_n \\ \downarrow & & \downarrow \psi & \swarrow \pi_n & \\ X_n & \xrightarrow{p} & \mathbb{P}^2 & & \end{array}$$

Recall that  $d_n$  is the dimension of Albanese variety  $\text{Alb}(\mathcal{X}_n)$ . It is a well known fact that  $d_n \leq \dim H^0(\mathcal{X}_n, \Omega_{\mathcal{X}_n})$  and “=” holds when the base field  $k$  is of characteristic zero. If we assume that the polynomial  $f(x, y)$  has a decomposition  $f = f_1^{m_1} \cdots f_d^{m_d}$  into the irreducible factors over  $\bar{k} \cong \mathbb{C}$  such that  $\gcd(n_0, m_1, \dots, m_d) = 1$ , which implies the irreducibility of  $\mathcal{X}_n$ , then letting  $r_i = \deg(f_i)$  for  $1 \leq i \leq d$ , we have

$$(4.2) \quad 0 \leq d_n \leq \begin{cases} \frac{1}{2}(n-1)(\sum_{i=1}^d r_i - 2), & \text{if } n|r \\ \frac{1}{2}(n-1)(\sum_{i=1}^d r_i - 1), & \text{otherwise,} \end{cases}$$

One can see Proposition 1 and its corollary in [20] for the proof of above inequality.

Here, there exists an example for which  $d_n = \dim H^0(\mathcal{X}_n, \Omega_{\mathcal{X}_n}) > 0$ .

**Example 4.1.** Denote by  $\mathcal{D}_\ell$  an  $\ell$ -cyclic covering of the projective line  $\mathbb{P}^1$  defined by the equation  $v_2^\ell = \prod_{i=1}^t (b_i v_0 - a_i v_1)^{s_i}$  with  $\ell | (s_1 + \dots + s_t)$ . Let  $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^1$  be a rational map given by

$$\varphi((u_0 : u_1 : u_2)) = (F_1(u_0, u_1, u_2) : F_2(u_0, u_1, u_2)),$$

where both of  $F_1$  and  $F_2$  are homogeneous polynomials of degree  $s$ . If  $n$  divides  $s \cdot (s_1 + \dots + s_t)$  and  $\ell$  is a divisor of  $n$ , then the cyclic multiple plane  $\mathcal{X}_n$  associated with  $X_n$  defined by the affine equation,

$$w^n = f(x, y) = \prod_{i=1}^t (b_i F_1(1, x, y) - a_i F_2(1, x, y))^{s_i},$$

factors through  $\mathcal{D}_\ell$ . In this case, it is said that  $\mathcal{X}_n$  factors through a pencil. One can see that if  $\mathcal{D}_\ell$  is a curve of genus  $\geq 1$ , then  $\dim H^0(\mathcal{X}_n, \Omega_{\mathcal{X}_n}) > 0$ .

From now on, we assume that  $k \subset \mathbb{C}$  is a number field that contains an  $n$ -th root of unity denoted by  $\zeta_n$ . In order to give a description of the structure of  $\text{Alb}(\mathcal{X}_n)$  over  $\mathbb{C}$ , we also suppose that the assumption (1.2) holds for  $\text{Alb}(\mathcal{X}_n)$  for  $2 \leq n \leq 12$  but  $n \neq 7, 9, 11$ . Let  $\mathcal{E}_i$  and  $\mathcal{E}_\rho$  be the elliptic curves associated to the lattices  $\Lambda_\rho = \mathbb{Z} \oplus \rho\mathbb{Z}$  and  $\Lambda_i = \mathbb{Z} \oplus i\mathbb{Z}$ , where  $\rho = \zeta_3$  and  $i = \zeta_4$ . It is easy to see that the affine Weierstrass forms of the elliptic curves  $\mathcal{E}_i$  and  $\mathcal{E}_\rho$  are  $y^2 = x^3 + x$  and  $y^2 = x^3 + 1$ , respectively. We denote by  $C_1$  and  $C_2$  the genus 2 curves which are the normalization of the projective closure of the affine curves  $y^2 = x^5 + 1$  and  $y^2 = x^5 + x$  and let  $J(C_1)$  and  $J(C_2)$  be their Jacobians varieties, respectively.

**Proposition 4.2.** *Keeping the above notations and assumptions, for a given  $n \in \{2, 3, 4, 5, 6, 8, 10, 12\}$ , we have*

- (i)  $\mathcal{X}_2$  factors through a pencil;
  - (ii) For  $n = 3, 6$ , if  $\alpha(\mathcal{X}_n)$  is a surface, then  $\text{Alb}(\mathcal{X}_n) \cong \mathcal{E}_\rho^{d_n}$ ;
  - (iii) For  $n = 4$ , either  $\mathcal{X}_4$  factors through a pencil, or  $\text{Alb}(\mathcal{X}_4) \cong \mathcal{E}_i^{d_4}$ .
- For  $n = 5, 8, 10$ , and  $12$ , the dimension of  $\text{Alb}(\mathcal{X})$  is an even integer, and

- (iv)  $\text{Alb}(\mathcal{X}_n) \cong J(C_1)^{d_n/2}$  for  $n = 5, 10$ ;
- (v)  $\text{Alb}(\mathcal{X}_8) \cong J(C_2)^{n_1} \times \mathcal{E}_i^{2n_2}$  with  $n_1 + n_2 = d_8/2$ ;
- (vi)  $\text{Alb}(\mathcal{X}_{12}) \cong \mathcal{E}_i^{2n_1} \times \mathcal{E}_\rho^{2n_2}$  with  $n_1 + n_2 = d_{12}/2$ .

*Proof.* The part (i) is a result of de Franchis [5]. When  $\alpha(\mathcal{X}_n)$  is a surface, the part (ii) for  $n = 3$  is due to Comessati in [4], which is proved in Theorem 4.10 of [3] with a different method; Otherwise, it is a consequence of Theorem 5.7 and the remarks (5.5) and (5.6) in [3]. The part (iii) is Theorem 4 of [20] or one can conclude it from Theorem 5.7 of [3] again. All of the assertions in (iv-vi) are consequences of Theorem 5.8 of [3], when  $\alpha(\mathcal{X}_n)$  is a surface; and they can be concluded in general by Theorem 5.9 and remark (5.11) of [3].  $\square$

We recall that the *Albanese dimension* of the affine curve

$$C : f(x, y) = F(1, x, y) = 0$$

is defined as  $\text{Albdim}(C) = \max_{n \in \mathbb{N}} \dim \alpha(\mathcal{X}_n)$  where  $\alpha_n : \mathcal{X}_n \rightarrow \text{Alb}(\mathcal{X}_n)$  is the Albanese map. A priori,  $\text{Albdim}(C)$  can take the values 0, 1, or 2. In Theorem 1(ii) of [9], Kulikov provides a sufficient condition to have  $\text{Albdim}(C) = 2$  without giving any concrete example.

**Theorem 4.3.** *Assume that  $F(u_0, u_1, u_2)$  can be written in two different forms  $F = G_1^a + H_1^b = G_2^a + H_2^b$ , where  $a, b$  are co-prime integers  $\geq 2$  and  $G_1, G_2, H_1, H_2 \in k[u_0, u_1, u_2]$ , such that the two pencils*

$$\{\lambda_0 G_1^a + \lambda_1 H_1^b : [\lambda_0 : \lambda_1] \in \mathbb{P}^1\}, \quad \{\lambda_0 G_2^a + \lambda_1 H_2^b : [\lambda_0 : \lambda_1] \in \mathbb{P}^1\},$$

*are different over  $\mathbb{C}$ . Then  $\alpha(\mathcal{X}_{ab})$  is a surface and hence  $\text{Albdim}(C) = 2$  for  $C : f(x, y) = F(1, x, y) = 0$ .*

In Theorem 0.2 of [29], Tokunaga demonstrated an explicit irreducible affine plane curve satisfying the Kulikov’s condition as follows.

**Proposition 4.4.** *The Albanese image  $\alpha(\mathcal{X}_6)$  is a surface in the following two cases:*

- (i)  $\mathcal{X}_6$  is associated to  $C = \mathcal{E}^\vee$  the dual of a smooth plane cubic  $\mathcal{E}$ ;
- (ii)  $\mathcal{X}_6$  is associated to the projective plane sextic curve  $C$  defined by

$$F(u_0, u_1, u_2) = u_0^3 u_1^3 - 3u_0 u_1 u_2 (u_2^3 - 8) + 2(u_2^6 + 20u_0^3 u_2^3 - 8u_0^6) = 0.$$

*Moreover, we have  $\text{Albdim}(C) = 2$  for  $C$  the affine model of  $C$  in both cases.*

### 5. Proof of Theorem 1.1

It is clear that the hypersurface  $X_n$  defined by  $w^n = f(x, y)$  admits an order  $n$  automorphism  $\tau : (x, y, w) \mapsto (x, y, \zeta_n \cdot w)$ , which induces an automorphism on its non-singular projective model  $\mathcal{X}_n$  denoted by  $\bar{\tau}$ . For

any integer  $m \geq 1$ , we define  $U_m := X_n^{(1)} \times_k \cdots \times_k X_n^{(m)}$  where  $X_n^{(i)}$  is a copy of  $X_n$  given by the affine equation  $w_i^n = f(x_i, y_i)$  for  $1 \leq i \leq m$ . Denote by  $\tau_i$  the corresponding automorphism. Then  $\gamma = (\tau_1, \dots, \tau_m)$  is an order  $n$  automorphism of  $U_m$  which naturally induces an order  $n$  automorphism  $\bar{\gamma} = (\bar{\tau}_1, \dots, \bar{\tau}_m)$  of the fibered product  $\mathcal{U}_m = \mathcal{X}_n^{(1)} \times_k \cdots \times_k \mathcal{X}_n^{(m)}$ , where  $\mathcal{X}_n^{(i)}$  is a non-singular projective model of  $X_n^{(i)}$  for each  $1 \leq i \leq m$ . Note that  $\mathcal{U}_m$  can be viewed as a non-singular model of  $U_m$ . Denote by  $L$  and  $\mathcal{L}$  the function fields of  $U_m$  and  $\mathcal{U}_m$ , respectively. Let  $\text{Aut}(\ast)$  be the automorphism group of its origin  $(\ast)$  and  $G = \langle \gamma \rangle$  and  $\bar{G} = \langle \bar{\gamma} \rangle$  be the cyclic subgroup of  $\text{Aut}(U_m)$  and  $\text{Aut}(\mathcal{U}_m)$  generated by  $\gamma$  and  $\bar{\gamma}$ , respectively. Let  $V_m$  and  $\mathcal{V}_m$  be the quotient of  $U_m$  and  $\mathcal{U}_m$  by  $G$  and  $\bar{G}$ , and denote by  $K$  and  $\mathcal{K}$  the function fields of  $V_m$  and  $\mathcal{V}_m$ , respectively. Then, both of the extensions  $L|K$  and  $\mathcal{L}|\mathcal{K}$  are finite cyclic extension of order  $n$ . Indeed, we have  $L \subset k(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m, w_1, \dots, w_m)$ , where  $x_i$ 's and  $y_i$ 's are independent transcendental variables and each  $w_i$  satisfies in the following equations,

$$(5.1) \quad w_i^n - f(x_i, y_i) = 0 \quad (i = 1, \dots, m).$$

Then, the field  $K$  is the  $G$ -invariant elements of  $L$ , i.e.,

$$K = L^G \subseteq k(x_1, \dots, x_m, y_1, \dots, y_m, w_1^{n-1}w_2, \dots, w_1^{n-1}w_{m-1}).$$

Since  $(w_1^{n-1}w_{i+1})^n = f(x_1, y_1)^{n-1}f(x_{i+1}, y_{i+1})$  for  $1 \leq i \leq m - 1$ , so by defining  $z_i := w_1^{n-1}w_{i+1}$  the variety  $V_m$  can be expressed by the equations

$$(5.2) \quad z_i^n = f(x_1, y_1)^{n-1}f(x_{i+1}, y_{i+1}) \quad (i = 1, \dots, m - 1).$$

Thus,  $L|K$  is a cyclic extension of degree  $n$  determined by the equation  $w_1^n = f(x_1, y_1)$ , i.e.,

$$L = K(w_1) \subseteq k(x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_{m-1})(w_1).$$

Note that the cyclic field extension  $\mathcal{L}|\mathcal{K}$  can be determined by considering the homogenization of the equations (5.1). Furthermore, the variety  $\mathcal{V}_m$  can be expressed by the homogenization of equations (5.2).

Let  $\tilde{X}_n$  and  $\tilde{\mathcal{X}}_n$  be the twists of  $X_n$  and  $\mathcal{X}_n$  by the extensions  $L|K$  and  $\mathcal{L}|\mathcal{K}$ , respectively. Then, one can check that the twist  $\tilde{X}_n$  is given by the affine equation,

$$(5.3) \quad f(x_1, y_1)z^n = f(x, y),$$

and  $\tilde{\mathcal{X}}_n$  can be determined by its homogenization. Moreover, it is easy to check that  $\tilde{X}_n$  contains the  $m$   $K$ -rational points:

$$(5.4) \quad P_1 := (x_1, y_1, 1) \text{ and } P_{i+1} := (x_{i+1}, y_{i+1}, w_{i+1}/w_1) \text{ for } 1 \leq i \leq m - 1.$$

Let us denote by  $\tilde{P}_i$  the point corresponding to  $P_i$  on  $\tilde{\mathcal{X}}_n$  for  $i = 1, \dots, m$ . Now, by applying (2.3) to the  $n$ -cover  $\pi : \mathcal{X}_n \rightarrow \mathbb{P}^2$ , we obtain

$$\text{Prym}_{\mathcal{X}_n^{(i)}/\mathbb{P}^2} = \frac{\text{Alb}(\mathcal{X}_n^{(i)})}{\text{Im}(\text{id} + \tilde{\gamma} + \dots + \tilde{\gamma}^{n-1})} \sim_k \ker(\text{id} + \tilde{\gamma} + \dots + \tilde{\gamma}^{n-1})^\circ.$$

Since  $0 = \text{id} - \tilde{\gamma}^n = (\text{id} - \tilde{\gamma})(\text{id} + \tilde{\gamma} + \dots + \tilde{\gamma}^{n-1})$  and  $\text{id} \neq \tilde{\gamma}$ , we have

$$0 = \text{id} + \tilde{\gamma} + \dots + \tilde{\gamma}^{n-1} \in \text{End}(\text{Alb}(\mathcal{X}_n^{(i)})) = \text{End}(\text{Alb}(\mathcal{X}_n)),$$

which implies that  $\text{Prym}_{\mathcal{X}_n^{(i)}/\mathbb{P}^2} = \text{Alb}(\mathcal{X}_n^{(i)}) = \text{Alb}(\mathcal{X}_n)$  for  $i = 1, \dots, m$ .

Then, using (2.4), we get a  $k$ -isogeny of abelian varieties,

$$(5.5) \quad \text{Prym}_{\mathcal{U}_m/\mathcal{V}_m} \sim_k \prod_{i=1}^m \text{Prym}_{\mathcal{X}_n^{(i)}/\mathbb{P}^2} = \text{Alb}(\mathcal{X}_n)^m.$$

Now, let us consider the 1-cocycle  $a = (a_u) \in \mathcal{Z}^1(\bar{G}, \text{Aut}(\text{Alb}(\mathcal{X}_n)))$  defined by  $a_{\text{id}} = \text{id}$  and  $a_{\tilde{\tau}^j} = \tilde{\tau}_*^j$  where  $\tilde{\gamma}^j \in \bar{G}$  and  $\tilde{\tau}_* : \text{Alb}(\mathcal{X}_n) \rightarrow \text{Alb}(\mathcal{X}_n)$  is the automorphism induced by  $\tilde{\tau} : \mathcal{X}_n \rightarrow \mathcal{X}_n$ . Then, using Proposition 3.2, we conclude that  $\widetilde{\text{Alb}(\mathcal{X}_n)} \sim_{\mathcal{K}} \text{Alb}(\tilde{\mathcal{X}}_n)$ . Thus, by Theorem 1.1 in [21], for  $\mathcal{X}' = \mathcal{U}_m$ ,  $\mathcal{X} = \mathcal{V}_m$ , and  $\mathcal{A} = \text{Alb}(\mathcal{X}_n)$ , we get:

$$\begin{aligned} \widetilde{\text{Alb}(\mathcal{X}_n)}(\mathcal{K}) &\cong \text{Hom}_k(\text{Prym}_{\mathcal{U}_m/\mathcal{V}_m}, \text{Alb}(\mathcal{X}_n)) \oplus \text{Alb}(\mathcal{X}_n)[m](k) \\ &\cong \text{Hom}_k(\text{Alb}(\mathcal{X}_n)^m, \text{Alb}(\mathcal{X}_n)) \oplus \text{Alb}(\mathcal{X}_n)[m](k) \\ &\cong (\text{End}_k(\text{Alb}(\mathcal{X}_n)))^m \oplus \text{Alb}(\mathcal{X}_n)[m](k). \end{aligned}$$

We denote by  $\tilde{Q}_i$  the image of  $\tilde{P}_i$  by the Albanese map  $\tilde{\alpha}_n : \tilde{\mathcal{X}}_n \rightarrow \text{Alb}(\tilde{\mathcal{X}}_n)$  for  $i = 1, \dots, m$ . Then, by tracing back the above isomorphisms, one can see that the points  $\tilde{Q}_1, \dots, \tilde{Q}_m$  form a subset of independent generators for the Mordell–Weil group  $\text{Alb}(\tilde{\mathcal{X}}_n)(\mathcal{K})$ . Hence, as  $\mathbb{Z}$ -modules, we have

$$\text{rk}(\widetilde{\text{Alb}(\mathcal{X}_n)}(\mathcal{K})) = \text{rk}(\text{Alb}(\tilde{\mathcal{X}}_n)(\mathcal{K})) = m \cdot \text{rk}(\text{End}_k(\text{Alb}(\mathcal{X}_n))).$$

Therefore, we have completed the proof of Theorem 1.1.

### 6. Proofs of Theorems 1.3 and 1.4

In order to prove Theorem 1.3, we examine the ring of endomorphisms of  $\text{Alb}(\mathcal{X}_n)$  for  $3 \leq n \leq 12$  and  $n \neq 7, 9, 11$  case by case. We suppose that  $k$  contains  $\mathbb{Q}(\zeta_n)$  and there exists a  $k$ -rational point on  $X_n$  and hence on  $\mathcal{X}_n$ . Furthermore, we assume that (1.2) holds for  $\text{Alb}(\mathcal{X}_n)$ . By Theorem 1.1,

$$\widetilde{\text{Alb}(\mathcal{X}_n)}(\mathcal{K}) \cong (\text{End}_k(\text{Alb}(\mathcal{X}_n)))^m \oplus \text{Alb}(\mathcal{X}_n)[m](k).$$

Since  $\text{Alb}(\mathcal{X}_n)[m](k)$  is a trivial or a finite group, so it does not distribute to the rank of  $\text{Alb}(\mathcal{X}_n)(\mathcal{K})$  as a  $\mathbb{Z}$ -module, but the rank of  $\text{End}_k(\text{Alb}(\mathcal{X}_n))$  does. We recall that  $d_n$  is the dimension of Albanese variety  $\text{Alb}(\mathcal{X}_n)$ .

First, we let  $n = 3$  and assume that  $\alpha(\mathcal{X}_3)$  is a surface. Then, by the part (ii) of Proposition 4.2, we have  $\text{Alb}(\mathcal{X}_3) \cong \mathcal{E}_\rho^{d_3}$  which implies that

$$\text{End}_k(\text{Alb}(\mathcal{X}_3)) \otimes \mathbb{Q} = M_{d_3}(\text{End}_k(\mathcal{E}_\rho) \otimes \mathbb{Q}),$$

where  $\mathcal{E}_\rho : y^2 = x^3 + 1$  and  $M_*(\cdot)$  denotes the ring of  $(*, *)$ -matrices with entries in its origin  $(\cdot)$ . Since we have assumed that  $\rho = \zeta_3 \in k$ , we have  $\text{End}_k(\mathcal{E}_\rho) \cong \mathbb{Z}[\rho]$  which is of rank 2 as a  $\mathbb{Z}$ -module. Thus,  $M_{d_3}(\text{End}_k(\mathcal{E}_\rho) \otimes \mathbb{Q})$  has rank  $d_3^2$  as a  $\mathbb{Z}[\rho]$ -module and hence it is of rank  $2d_3^2$  as a  $\mathbb{Z}$ -module. Therefore, we conclude that

$$\text{rk}(\widetilde{\text{Alb}(\mathcal{X}_3)}(\mathcal{K})) = m \cdot \text{rk}(\text{End}_k(\text{Alb}(\mathcal{X}_3))) = 2md_3^2.$$

Similar arguments work for the case  $n = 4$  and 6, where in the former case we have to use the fact that  $\text{End}_k(\mathcal{E}_i) \cong \mathbb{Z}[i]$ .

Second, we consider the case  $n = 5$ . By Proposition 4.2(iv),  $d_5$  is an even number and we have  $\text{Alb}(\mathcal{X}_5) = J(C_1)^{d_5/2}$ , where  $C_1 : y^2 = x^5 + 1$ . This implies that  $\text{End}_k(\text{Alb}(\mathcal{X}_5)) \otimes \mathbb{Q} = M_{d_5/2}(\text{End}_k(J(C_1)) \otimes \mathbb{Q})$ . We note that the Jacobian variety  $J(C_1)$  is a simple abelian variety and its endomorphism ring  $\text{End}_k(J(C_1))$  contains  $\mathbb{Z}[\zeta_5]$  as a  $\mathbb{Z}$ -submodule of rank 4. Thus,  $M_{d_5/2}(\text{End}_k(J(C_1)) \otimes \mathbb{Q})$  contains  $M_{d_5/2}(\mathbb{Q}(\zeta_5))$ . This gives us that  $\text{End}_k(\text{Alb}(\mathcal{X}_5))$  has rank at least  $d_5^2$  as a  $\mathbb{Z}$ -module. Therefore,

$$\text{rk}(\widetilde{\text{Alb}(\mathcal{X}_5)}(\mathcal{K})) = m \cdot \text{rk}(\text{End}_k(\text{Alb}(\mathcal{X}_5))) \geq md_5^2.$$

A similar arguments leads to the proof in the case  $n = 10$ .

Finally, we consider the case  $n = 8$  and leave  $n = 12$  for the reader. The part (v) of Proposition 4.2 implies that  $d_8$  is an even number and one has  $\text{Alb}(\mathcal{X}_8) = J(C_2)^{n_1} \times \mathcal{E}_i^{2n_2}$ , where  $C_2$  is the normalization of the projective closure of the affine curve  $y^2 = x^5 + x$  and  $n_1$  and  $n_2$  are positive integers such that  $n_1 + n_2 = d_8/2$ . Thus,

$$\text{End}_k(\text{Alb}(\mathcal{X}_5)) \otimes \mathbb{Q} = M_{n_1}(\text{End}_k(J(C_2))) \oplus M_{2n_2}(\text{End}_k(\mathcal{E}_i)).$$

Since the Jacobian variety  $J(C_2)$  splits as the product of the elliptic curves,

$$\mathcal{E}_1 : y^2 = x^3 + x^2 - 3x + 1, \text{ and } \mathcal{E}_2 : y^2 = x^3 - x^2 - 3x + 1,$$

we have  $\text{End}_k(J(C_2)) = \text{End}_k(\mathcal{E}_1) \oplus \text{End}_k(\mathcal{E}_2)$  and hence

$$\text{End}_k(\text{Alb}(\mathcal{X}_5)) \otimes \mathbb{Q} = M_{n_1}(\text{End}_k(\mathcal{E}_1)) \oplus M_{n_1}(\text{End}_k(\mathcal{E}_2)) \oplus M_{2n_2}(\text{End}_k(\mathcal{E}_i)).$$

One can check that the endomorphism ring of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  contains the rings  $\mathbb{Z}[\sqrt{-2}]$  and  $\mathbb{Z}[\sqrt{-3}]$ , respectively, which are of rank 2 as  $\mathbb{Z}$ -modules. This means that the rank of  $\text{End}_k(J(C_2))$  is at least 4 as a  $\mathbb{Z}$ -module. Thus, considering the fact  $\text{End}_k(\mathcal{E}_i) = \mathbb{Z}[i]$  and  $n_1 + n_2 = d_8/2$ , one can conclude that  $\text{End}_k(\text{Alb}(\mathcal{X}_5))$  is a  $\mathbb{Z}$ -module of rank at least  $4(n_1^2 + n_2^2) = d_8^2 - 8n_1n_2$ . Hence,

$$\text{rk}(\widetilde{\text{Alb}(\mathcal{X}_8)}(\mathcal{K})) \geq m \cdot (d_8^2 - 8n_1n_2).$$

We refer the reader to [17] for more details on the above assertions on the elliptic curves  $\mathcal{E}_i$ 's, the Jacobians  $J(C_i)$ 's, for  $i = 1, 2$ , and their endomorphism rings. Therefore, we have completed the proof of Theorem 1.3 as desired.

To prove Theorem 1.4, we note that the Albanese image  $\alpha(\mathcal{X}_6)$  is a surface by Theorem 4.3. Thus  $\text{Alb}(\mathcal{X}_6) \cong \mathcal{E}_\rho^{d_6}$  by Proposition 4.2 (ii). Then, applying Theorem 1.3 leads to the equality in the first part of 1.4,

$$\text{rk}(\widetilde{\text{Alb}(\mathcal{X}_6)}(\mathcal{K})) = \text{rk}(\widetilde{\text{Alb}(\tilde{\mathcal{X}}_6)}(\mathcal{K})) = 2md_6.$$

If we assume that  $\mathcal{X}_6$  is associated to  $\mathcal{C}$ , the dual of a smooth cubic  $\mathcal{E}$  or the projective model of the affine curve

$$C : f(x, y) = x^3 - 3xy(y^3 - 8) + 2(y^6 - 20y^3 - 8) = 0,$$

then the above equality and Proposition 4.4 proves the last assertion of Theorem 1.4.

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Sajad SALAMI

Instituto de Matemática e Estatística

Universidade Estadual do Rio do Janeiro, Brazil

E-mail: [sajad.salami@ime.uerj.br](mailto:sajad.salami@ime.uerj.br)

URL: <https://sites.google.com/a/ime.uerj.br/sajadsalami/home>