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A logarithmic improvement in the Bombieri–Vinogradov theorem


<http://jtnb.centre-mersenne.org/item?id=JTNB_2019__31_3_635_0>
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1. Introduction

For integer number $a$ and $q \geq 1$, let

$$
\psi(x; q, a) = \sum_{\substack{n \leq x \\ n \equiv a (\text{mod} \ q)}} \Lambda(n),
$$

where $\Lambda(n)$ is the von Mangoldt function. The Bombieri–Vinogradov theorem is an estimate for the error terms in the prime number theorem for arithmetic progressions averaged over all $q$ up to $x^{1/2}$, or, rather almost up to $x^{1/2}$. 


2010 Mathematics Subject Classification. 11N13, 11N37, 11N60.

Mots-clefs. primes in arithmetic progressions, large sieve.
Theorem 1.1 (Bombieri–Vinogradov). Let $B > 0$ be given and $A = A(B)$. Then for $Q \leq x^{1/2}/(\log x)^A$ we have

$$\sum_{q \leq Q} \max_{2 \leq y \leq x} \max_{a \equiv 1} |\psi(y; q, a) - \frac{y}{\varphi(q)}| \ll B \frac{x}{(\log x)^B}.$$ 

The implied constant in this theorem is not effective, since we have to take care of characters associated with those $q$ that have small prime factors. At the same time, effective versions — in which the effect of an exceptional character is avoided in one way or another — have been known since Lenstra–Pomerance [9], Timofeev [14], and, very recently, Liu [10].

In the rest of the introduction we state the results of the present paper.

Theorem 1.2 (Bombieri–Vinogradov, ineffective). Let $A > 2$ and $Q \leq x^{1/2}/(\log x)^A$. Then we have the following bound

$$\sum_{q \leq Q} \max_{2 \leq y \leq x} \max_{a \equiv 1} |\psi(y; q, a) - \frac{y}{\varphi(q)}| \ll A \frac{x}{(\log x)^{A-2}}.$$ 

The implied constant in Theorem 1.2 is ineffective.

Previously, the best result of the type of Theorem 1.2 in the literature followed from [4]; it had $A - 5/2$ instead of $A - 2$. While [4] does not state the result in full — focusing on estimating the crucial sum — a complete form can be found in [13] (together with a fully explicit version). It is

$$\sum_{q \leq Q} \max_{l(q) > Q_1} \max_{2 \leq y \leq x} \left|\psi(y; q, a) - \frac{\psi(y)}{\varphi(q)}\right| \ll C \left(\frac{x}{Q_1} + x^{1/2}Q + x^{3/2}Q^{1/2} + x^{5/6} \log \frac{Q}{Q_1}\right) (\log x)^{5/2},$$

where $C$ is an explicit absolute constant (a similar fully explicit result was proven in Akbary–Hambrook [1] with $(\log x)^{9/2}$ instead of $(\log x)^{7/2}$) and $l(q)$ denotes the biggest prime factor of $q$. Another effective variant without explicit constants is given by Lenstra–Pomerance [9, Lemma 11.2] (with a larger power of log) in their work on Gaussian periods.

The key tool for the proof of Theorem 1.4 is Vaughan’s identity, which we have to get in an explicit version for our goal. Let

$$\psi(y, \chi) = \sum_{n \leq y} \Lambda(n) \chi(n),$$

be the twisted summatory function of the von Mangoldt function $\Lambda$ and a Dirichlet character $\chi$ modulo $q$. The key tool in getting Theorem 1.2 is the following estimate.
Proposition 1.3 (Vaughan’s inequality, improved). For \( x \geq 4 \) and any \( \varepsilon > 0 \) we have

\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |\psi(y, \chi)| \ll \varepsilon \left( x + Q^2 x^{1/2} + Q x^{13/14 + \varepsilon} \right)(\log x)^2,
\]

where \( Q \) is any positive real number and \( \sum_{\chi(q)}^* \) stands for a sum over all primitive characters \( \chi \) (mod \( q \)).

The improvement here consists in having a factor of \((\log x)^2\), rather than \((\log x)^{5/2}\). In order to prove Proposition 1.3 we use the weighted version of Vaughan’s identity (see Lemma 1.5) and a corollary of Graham [6], which was also given in a less general form in an earlier work of Barban–Vehov [2]. While Graham uses the Siegel–Walfisz theorem, there is an effective (and explicit) version of it in Helfgott [7, Chapter 5]. We follow methods developed in [7, Chapter 5] in the proof.

Analogously to [13] we formulate an effective version of Proposition 1.3 in the following corollary.

Corollary 1.4 (Bombieri–Vinogradov, with exceptional character taken out). Let \( x \geq 4 \), \( 1 \leq Q_1 \leq Q \leq x^{1/2} \). Denote by \( l(q) \) the smallest prime divisor of \( q \). Then for any positive \( \varepsilon > 0 \) we have

\[
\sum_{q \leq Q} \max_{l(q) > Q_1} \max_{2 \leq y \leq x} a \left| \frac{\psi(y, q, a)}{\varphi(q)} - \frac{\psi(y)}{\varphi(q)} \right| \ll \varepsilon \frac{x^{1/2} Q (\log x)^2}{Q_1} + \frac{x}{Q_1} (\log x)^3 + x^{13/14 + \varepsilon} (\log x)^4.
\]

The implied constant is effective and can be made explicit using [7, Chapter 5] together with the best available constant in Pólya–Vinogradov inequality given by Frolenkov–Soundararajan [5, Theorem 2]. The effectiveness is attained by getting rid of those moduli that have small prime divisors, thus of a possible exceptional character. Analogously to [1] one can state the corollary above for \( \pi(y; q, a) \) instead of \( \psi(y; q, a) \), where \( \pi(y; q, a) \) counts the number of primes up to \( x \) congruent to \( a \) mod \( q \).

Proposition 1.3 allows us to prove the Bombieri–Vinogradov theorem in the form of Theorem 1.2 and, hence, Corollary 1.4. In addition to Theorem 1.4, the proof uses the Siegel–Walfisz theorem, which states that

\[
\psi(x, \chi) - \delta(\chi)x \ll_A x \frac{1}{e^{c \sqrt{\log x}}}
\]

uniformly for \( q \leq (\log x)^A \). Here \( A > 0 \) is a fixed real number, \( c \) is an absolute positive constant, and \( \delta(\chi) = 1 \) if \( \chi \) is principal and is zero otherwise. The implied constant in the Bombieri–Vinogradov theorem is ineffective since the implied constant in the Siegel–Walfisz theorem is ineffective. To prove Corollary 1.2 we use the Siegel–Walfisz theorem to deal with moduli.
$q \leq Q$ having small prime divisors and Theorem 1.4 to deal with the sum over the remaining moduli.

The recent work of Liu [10] gives us a genuinely effective Bombieri–Vinogradov theorem. This is ultimately due to the fact that we can use an effective Landau–Page result (see Page [12], Landau [8] and also Vinogradov [15, Chapter 10]), which is non-trivial up to $(\log x)^2$ instead of making a standard ineffective step on applying Siegel–Walfisz theorem.

In [10] various applications of the statement are considered, such as an asymptotic formula for the representation of a large integer as the sum of two squares and a prime and Titchmarsh divisor problem (both results obviously become effective).

Acknowledgments. The author is grateful to Henryk Iwaniec for his crucial advice. The author also thanks her former supervisor Harald Helfgott for his help.

Auxiliary lemmas

We start with a so-called weighted Vaughan identity. It allows us to get cancellation in type II sums.

**Lemma 1.5** (Weighted Vaughan identity). Let $U, V \geq 1$. Let $\eta(t) : \mathbb{Z}^+ \to \mathbb{R}$ be any function such that $\eta(t) = 1$ for $t \leq V$. We have

$$\Lambda(n) = \lambda_0(n) + \lambda_1(n) + \lambda_2(n) + \lambda_3(n),$$

where

$$\lambda_0(n) = \begin{cases} \Lambda(n), & n \leq U, \\ 0, & n > U, \end{cases} \quad \lambda_1(n) = \sum_{d|n} \mu(d) \eta(d) \log \frac{n}{d},$$

$$\lambda_2(n) = -\sum_{c \leq U} \sum_{d|n} \mu(d) \Lambda(c) \eta(d), \quad \lambda_3(n) = \sum_{c > U} \sum_{d|n} \mu(d) \Lambda(c)(1 - \eta(d)).$$

**Proof.** Let $n > U$, since otherwise the statement is trivial. Define the following quantities

$$\Lambda_1(n) = \sum_{d|n, d \leq V} \mu(d) \log \frac{n}{d}$$

$$= \lambda_1(n) - \sum_{d|n, d > V} \mu(d) \eta(d) \log \frac{n}{d} = \lambda_1(n) + \lambda_1'(n),$$
\( \Lambda_2(n) = - \sum_{c \leq U} \sum_{d | n} \mu(d) \Lambda(c) \)
\[= \lambda_2(n) + \sum_{c \leq U} \sum_{d | n} \mu(d) \Lambda(c) \eta(d) = \lambda_2(n) + \lambda'_2(n), \]
\( \Lambda_3(n) = \sum_{c > U} \sum_{d | n} \mu(d) \Lambda(c) \)
\[= \lambda_3(n) + \sum_{c > U} \sum_{d | n} \mu(d) \Lambda(c) \eta(d) = \lambda_3(n) + \lambda'_3(n). \]

Vaughan’s identity in its classical form is
\( \Lambda(n) = \Lambda_1(n) + \Lambda_2(n) + \Lambda_3(n), \)
so it remains to show that \( \lambda'_1(n) + \lambda'_2(n) + \lambda'_3(n) = 0 \) for every \( n \). Let us rewrite this sum
\[\sum_{i=1}^3 \lambda'_i(n) = \sum_{d | n} \left( -\mu(d) \eta(d) \log \frac{n}{d} + \sum_{c | n/d} \mu(d) \Lambda(c) \eta(d) + \sum_{c | n/d} \mu(d) \Lambda(c) \eta(d) \right)\]
\[= \sum_{d | n} \left( -\mu(d) \eta(d) \log \frac{n}{d} + \mu(d) \eta(d) \sum_{c | n/d} \Lambda(c) \right) = 0, \]
where in the last equality we used the fact that \( \sum_{a | b} \Lambda(a) = \log b. \) \( \square \)

**Lemma 1.6** (Graham [6]). Let \( 1 \leq N_1 \leq N_2 \leq N \) and define
\[f_i(d) = \begin{cases} \mu(d) \log(N_i/d), & d \leq N_i, \\ 0, & d > N_i. \end{cases} \]

We have
\[\sum_{n=1}^N \left( \sum_{d_1 | n} f_1(d_1) \right) \left( \sum_{d_2 | n} f_2(d_2) \right) = N \log N_1 + O(N). \]

From the lemma above one can deduce the following.

**Corollary 1.7.** Define a function \( \eta(t) \), that is equal to 1 for \( t \leq V \), to 0 for \( t > V_0 \) and
\[\eta(t) = \frac{\log(V_0/t)}{\log(V_0/V)}, \quad V < t \leq V_0. \]

Then
\[\sum_{k \leq Y} \left| \sum_{d | k} \mu(d) \eta(d) \right|^2 \ll \frac{Y}{\log(V_0/V)}. \]
The constant here can be made explicit using [7, Chapter 5].
We also need the large sieve inequality as stated in a classical form in
(see for example, Montgomery [11, p. 561]),

\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \left| \sum_{m=m_{0}+1}^{m_{0}+M} a_{m} \chi(m) \right|^{2} \leq (M + Q^{2}) \sum_{m=m_{0}+1}^{m_{0}+M} |a_{m}|^{2},
\]

from which we deduce the following.

**Lemma 1.8 (Large sieve inequality).** Let \(a_{m}, b_{n}\) be arbitrary complex numbers with \(m_{0} \leq m \leq M, n_{0} \leq n \leq N\) and \(m_{0}, n_{0}, M, N \in \mathbb{N}\). Then

\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y} \left| \sum_{m=m_{0}}^{M} \sum_{n=n_{0}}^{N} a_{m} b_{n} \chi(mn) \right|
\]

\[
\leq c_{3}(M' + Q^{2})^{\frac{1}{2}}(N' + Q^{2})^{\frac{1}{2}} \left( \sum_{m=m_{0}}^{M} |a_{m}|^{2} \right)^{\frac{1}{2}} \left( \sum_{n=n_{0}}^{N} |b_{n}|^{2} \right)^{\frac{1}{2}} \log(2MN),
\]

where \(c_{3} = 2.64\ldots\) and \(M' = M - m_{0} + 1, N' = N - n_{0} + 1\) are the number
of terms in the sums over \(m\) and \(n\) respectively.

For the proof see [1, Lemma 6.1] or [3, Theorem 8.3.3].

**2. Proof of Proposition 1.3**

We proceed now with the proof of Proposition 1.3. Fix arbitrary real
numbers \(Q > 0\) and \(x \geq 4\). Without loss of generality we can assume that
\(2 \leq Q \leq x^{1/2}\) and decompose the von Mangoldt function using a weighted
form of Vaughan’s identity, namely Lemma 1.5.

\[
\Lambda(n) = \lambda_{0}(n) + \lambda_{1}(n) + \lambda_{2}(n) + \lambda_{3}(n),
\]

where \(\lambda_{i}(n), i = 0, 1, 2, 3\) are as in the statement of the lemma and \(U, V, V_{0}\)
are parameters. Notice also that we are free to choose \(\eta(t)\) as we wish, we
only need to fulfill the conditions stated in Lemma 1.5.

Assume \(y \leq x, q \leq Q\), and \(\chi\) is a primitive character mod \(q\). We use the
above decomposition to write

\[
\psi(y, \chi) = s_{0} + s_{1} + s_{2} + s_{3},
\]

where

\[
s_{i} = \sum_{n \leq y} \lambda_{i}(n) \chi(n).
\]
Denote the contributions to our main sum by

\[(2.1)\quad S_i = \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} |s_i|.
\]

Easily we obtain

\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{*} \max_{y \leq x} |\psi(y, \chi)| \leq S_0 + S_1 + S_2 + S_3,
\]

where

\[s_1 = \sum_{d \leq y} \mu(d)\chi(d)e(d) \sum_{h \leq \frac{y}{d}} \chi(h)\log h,
\]

\[s_2 = \sum_{dcr \leq y} \chi(dc)\mu(d)\Lambda(c)\eta(d),
\]

\[s_3 = \sum_{n \leq y} \chi(n) \sum_{c > U} \sum_{d|n} \mu(d)\Lambda(c)(1 - \eta(d)).
\]

and \(S_0 \ll UQ^2\) by the Chebyshev’s estimate

\[|s_0| \leq \sum_{n \leq U} \Lambda(n) \ll U.
\]

In what follows we choose \(\eta(\cdot)\) from the paper by Graham [6], i. e.

\[(2.2)\quad \eta(d) = \frac{\log(V_0/d)}{\log(V_0/V)}, \quad V \leq d \leq V_0.
\]

We remind that \(\eta(d) = 1\) for \(d \leq V\) and \(\eta(d) = 0\) for \(d > V_0\). This choice immediately allows us to win \((\log x)^{\frac{1}{2}}\) in the last sum, that is of type II.

**Type I sums.** We start with linear sums among \(s_i\) and work with \(s_1\) first. Write

\[
\sum_{h \leq \frac{y}{d}} \chi(h)\log h = \sum_{h \leq \frac{y}{d}} \chi(h) \int_1^{h} \frac{du}{u}
\]

and exchange the sum and the integral

\[
s_1 = \sum_{d \leq V_0} \mu(d)\chi(d)e(d) \int_1^{\frac{y}{d}} \sum_{u \leq h \leq \frac{y}{d}} \chi(h) \frac{du}{u}
\]

\[
= \int_1^{y} \sum_{d \leq \min(V_0, y/u)} \mu(d)\chi(d)e(d) \sum_{u \leq h \leq \frac{y}{d}} \chi(h) \frac{du}{u}.
\]
Now since for \( d > y/u \) the internal sum over \( h \) is empty we drop the condition \( d \leq y/u \) and get

\[
s_1 = \int_1^y \left( \sum_{d \leq V} \mu(d) \chi(d) \eta(d) \sum_{u \leq h \leq y/d} \chi(h) \right) \frac{du}{u}
\]

\[
= \int_1^y \left( \sum_{d \leq V} \mu(d) \chi(d) \sum_{u \leq h \leq y/d} \chi(h) \right) \frac{du}{u}
\]

\[
+ \frac{1}{\log(V_0/V)} \int_1^y \left( \sum_{V \leq d \leq V_0} \mu(d) \chi(d) \log \frac{V_0}{d} \sum_{u \leq h \leq y/d} \chi(h) \right) \frac{du}{u},
\]

where in the last equality we used the definition of \( \eta \) given in (2.2). Denote the summands \( \sigma_1 \) and \( \sigma_2 \). Then for \( \sigma_1 \) we have

\[
|\sigma_1| \leq \sum_{d \leq V} \max_{1 \leq u \leq y} \left| \sum_{u \leq h \leq y/d} \chi(h) \right| \int_1^y \frac{du}{u} \leq (\log y) \sum_{d \leq V} \max_{1 \leq u \leq y} \left| \sum_{u \leq h \leq y/d} \chi(h) \right|.
\]

If \( q = 1 \), then we only have the trivial character modulo \( q \) and

\[
|\sigma_1| \leq y(\log y) \sum_{d \leq V} \frac{1}{d} \leq x(\log xV)^2.
\]

If \( q > 1 \) and \( \chi \) is a primitive character mod \( q \), we use the Pólya–Vinogradov inequality (see [5, Theorem 2] for explicit results): for all \( x, y \) we have

\[
(2.3) \quad \left| \sum_{x \leq n \leq y} \chi(n) \right| < \sqrt{q} \log q.
\]

Then

\[
|\sigma_1| < (\log y) \sum_{d \leq V} \max_{1 \leq u \leq y} q^{1/2} \log q < q^{1/2} V(\log xV)^2.
\]

Further, for \( \sigma_2 \) we put the absolute value inside the integral to get

\[
|\sigma_2| \leq \frac{\log y}{\log(V_0/V)} \sum_{V \leq d \leq V_0} \log \frac{V_0}{d} \max_{1 \leq u \leq y} \left| \sum_{u \leq h \leq y/d} \chi(h) \right|
\]

and work with the internal sum in the same way as in \( \sigma_1 \) to get \( |\sigma_2| \leq x(\log xV_0)^2 \) for \( q = 1 \) and \( |\sigma_2| \leq q^{1/2} V_0(\log xV_0)^2 \) for \( q > 1 \).
On inserting the estimate for $s_1$ in (2.1) we obtain

$$
S_1 \leq \left( \max_{y \leq x} |s_1| + \sum_{1 < q \leq Q} \frac{q}{\varphi(q)} \sum_{y \leq x}^{*} \max |s_1| \right)
$$

(2.4)

$$
\leq \left( (x \log x V_0)^2 + V_0 (\log x V_0)^2 \sum_{1 < q \leq Q} \frac{q^2}{\varphi(q)} \sum_{1 < q \leq Q}^{*} 1 \right)
$$

$$
\leq (x + Q^{5/2} V_0) (\log x V_0)^2.
$$

**Type II sums.** Now we work with $s_2$ and want to use dyadic decomposition. Write

$$
s_2 = \sum_{c \leq y, d \leq U} \Lambda(c) \mu(d) \eta(d) \chi(cdr) = \sum_{c \leq y, d \leq t} \Lambda(c) \mu(d) \eta(d) \chi(ct)
$$

$$
= \sum_{c \leq w} + \sum_{w < c \leq U} = s_2' + s_2'',
$$

where we introduced a new parameter $w$, that should be smaller than $U$ and will be chosen later. We deal first with the small range of the parameter $c$ in $s_2$, namely with $s_2'$. Write

$$
s_2' = \sum_{c \leq w} \Lambda(c) \chi(c) \sum_{t \leq y/c} \sum_{d | t} \mu(d) \eta(d) \chi(t).
$$

Since we have the bound

$$
\left| \sum_{c | t} \Lambda(c) \mu(d) \eta(d) \chi(t) \right| \leq \sum_{c | t} \Lambda(c) = \log t,
$$

then proceeding as for $s_1$ via the Pólya–Vinogradov inequality and using the fact that $cd = t \leq w V_0$ we get

$$
(2.5)
$$

$$
|S_2'| \leq (x + Q^{5/2} w V_0) (\log(x w V_0))^2,
$$

where the $x$ term comes from the contribution of $q = 1$ and $Q^{5/2} w V_0$ from the remaining $q \neq 1$.

Next consider $s_2''$. Writing $s_2''$ as a dyadic sum we have

$$
s_2'' = \sum_{M = 2^a}^{w/2} \sum_{w/2 < M \leq U} \sum_{M < c \leq 2M} \sum_{t \leq y/c} \sum_{d | t} \Lambda(c) \mu(d) \eta(d) \chi(ct).
$$
Denote the contributions of $s'_2$ and $s''_2$ to $S_2$ by $S'_2$ and $S''_2$ respectively. Inserting the above into (2.1) and using the triangle inequality we get

$$S''_2 \leq \sum_{\substack{M=2^a \atop w/2 < M \leq U}} \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)} \max_{y \leq x} \left| \sum_{w < c < U} \sum_{M < c \leq 2M} \sum_{t \leq y/c} \Lambda(c) \mu(d) \eta(d) \chi(ct) \right|.$$ 

By the large sieve inequality given in Lemma 1.8 we get

$$S''_2 \ll \sum_{\substack{M=2^a \atop w/2 < M \leq U}} (M' + Q^2)^{\frac{1}{12}} \left( K' + Q^2 \right)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}} \sigma_2(M)^{\frac{1}{2}} \log \left( \frac{2x}{M} \min(U, 2M) \right),$$

where $M'$ and $K'$ are the number of terms in sums over $c$ and $t$ respectively and

$$\sigma_1(M) = \sum_{\substack{w < c < U \atop M < c \leq 2M}} \Lambda(c)^2,$$

$$\sigma_2(M) = \sum_{t \leq y/M} \left| \sum_{d | t} \mu(d) \eta(d) \right|^2,$$

and $\log(2x/M \min(U, 2M)) \ll \log x$. By Chebyshev’s estimate we have $\sigma_1(M) \ll M \log U$ and using the bounds $M' \leq M, K' \leq x/M$ we obtain

$$S''_2 \ll (\log x) (\log U)^{\frac{1}{2}} \sum_{\substack{M=2^a \atop w/2 < M \leq U}} (M + Q^2)^{\frac{1}{2}} \left( \frac{x}{M} + Q^2 \right)^{\frac{1}{2}} M^{\frac{1}{2}} \sigma_2(M)^{\frac{1}{2}}.$$

To bound $\sigma_2(M)$ we apply Corollary 1.7 and get

$$\sigma_2(M) \ll \frac{y}{M \log(V_0/V)}.$$

On putting it together we obtain

$$S''_2 \ll (\log x) (\log U)^{\frac{1}{2}} x^{\frac{1}{2}} \sum_{\substack{M=2^a \atop \frac{1}{2} w < M \leq U}} (M + Q^2)^{\frac{1}{2}} \left( \frac{x}{M} + Q^2 \right)^{\frac{1}{2}} \left( \log \frac{V_0}{V} \right)^{-\frac{1}{2}}$$

$$\ll (\log x) \left( \frac{(\log U)^{\frac{1}{2}}}{(\log(V_0/V))^{\frac{1}{2}}} (\log(Uw)) \left( x + Qxw^{-\frac{1}{2}} + Q^2x^{\frac{1}{2}} + U^{\frac{1}{2}}Qx^{\frac{1}{2}} \right) \right),$$

where we applied the bound

$$\sum_{\substack{M=2^a \atop w/2 < M \leq U}} 1 \leq \frac{\log(2U/w)}{\log 2}.$$
We continue with an estimate for $S_3$ that uses the large sieve inequality (Lemma 1.8) and properties of $\eta(\cdot)$ from Lemma 1.6. Writing $s_3$ as a dyadic sum we have

$$s_3 = \sum_{M=2^\alpha}^{U/2 < M \leq x/V} \sum_{U < m \leq x/V} \sum_{V < k \leq x/M} \sum_{M < m \leq 2M} \sum_{mk \leq y} a_m c_k \chi(mk),$$

where $a_m = \Lambda(m)$ and $c_k = \sum_{d \mid k, d > V} \mu(d)(1 - \eta(d))$. Inserting the above into (2.1) and using the triangle inequality we get

$$S_3 \leq \sum_{M=2^\alpha}^{U/2 < M \leq x/V} \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^{\ast} \max_{y \leq x} \left| \sum_{U < m \leq x/V} \sum_{V < k \leq x/M} \sum_{M < m \leq 2M} \sum_{mk \leq y} a_m c_k \chi(mk) \right| .$$

Now we apply the large sieve inequality given in Lemma 1.8 to get

$$S_3 \ll \sum_{M=2^\alpha}^{U/2 < M \leq x/V} (M' + Q^2)^{\frac{1}{2}} (K' + Q^2)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}}$$

$$\times \sigma_2(M)^{\frac{1}{2}} \log \left( \frac{2x}{M} \min \left( \frac{x}{V}, 2M \right) \right),$$

where

$$\sigma_1(M) = \sum_{V < k \leq x/M} |c_k|^2, \quad \sigma_2(M) = \sum_{U < m \leq x/V} \sum_{M < m \leq 2M} |a_m|^2.$$

and $M'$ and $K'$ denote the number of terms in the sums over $m$ and $k$, respectively. Clearly, $\log(2x/M \min(x/V, 2M)) \ll \log x$. From the definition of $M'$ and $N'$ we conclude

$$M' = \min(2M, x/V) - \max(M + 1, U + 1) \leq M,$$

$$K' = x/M - (V + 1) + 1 \leq x/M.$$

By Chebyshev’s estimate we have an upper bound

$$\sigma_2(M) \leq \sum_{m \leq 2M} \Lambda(m)^2 \leq \psi(2M) \log 2M \ll M \log M.$$

Thus by the above we conclude

$$S_3 \ll (\log x) \sum_{M=2^\alpha}^{U/2 < M \leq x/V} (M + Q^2)^{\frac{1}{2}} \left( \frac{x}{M} + Q^2 \right)^{\frac{1}{2}} (M \log M)^{\frac{1}{2}} \sigma_1(M)^{\frac{1}{2}}.$$

Further, we use

$$M(M + Q^2)(x/M + Q^2) = Mx + Q^2x + M^2Q^2 + MQ^4.$$
and \( \log M \leq \log(x/V) \). to get

\[
S_3 \ll (\log x) \left( \log \frac{x}{V} \right)^{\frac{1}{2}} \sum_{M=2^a \atop U/2 < M \leq x/V} (Mx + Q^2x + M^2Q^2 + MQ^4)^{\frac{1}{2}} \sigma_1'(M)^{\frac{1}{2}}.
\]

By the definition of \( \eta \) given in (2.2) we have

\[
1 - \eta(d) = 1 - \frac{\log(V_0/d)}{\log(V_0/V)} = \frac{\log(d/V)}{\log(V_0/V)} \quad V \leq d \leq V_0.
\]

On applying Corollary 1.6 we obtain

\[
\sigma_1(M) = \sum_{V < k \leq x/M} \left( \sum_{d|k} \mu(d) - \sum_{d|k} \mu(d) \eta(d) \right)^2 \ll \frac{x}{M \log(V_0/V)}
\]

that implies

\[
S_3 \ll \frac{\log x}{(\log(V_0/V))^{\frac{1}{2}}} \left( \log(x/V) \right)^{\frac{1}{2}} \sum_{M=2^a \atop U/2 < M \leq x/V} \left( x^2 + \frac{Q^2x^2}{M} + MQ^2x + Q^4x \right)^{\frac{1}{2}}.
\]

Since

\[
\sum_{M=2^a \atop U/2 < M \leq x/V} 1 \leq \frac{\log(2x/V)}{\log 2},
\]

then

\[
(2.7) \quad S_3 \ll \frac{\log x}{(\log(V_0/V))^{\frac{1}{2}}} \left( \log(x/V) \right)^{\frac{3}{2}} \left( x + QxU^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2x^{\frac{1}{2}} \right).
\]

Finally we have to adjust the parameters \( U, V, V_0, w \) using all our previous estimates \( S_0 \ll UQ^2, (2.4), (2.5), (2.6) \) and (2.7) We take \( U = V \) and get

\[
(2.8) \quad S = \sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q) \text{ max } y \leq x} |\psi(y, \chi)| \ll R(x, Q, w, V, V_0)G(x, w, V, V_0),
\]

where

\[
(2.9) \quad R(x, Q, w, V, V_0) = x + Q^\frac{5}{2}(V + wV_0) + Qxw^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2x^{\frac{1}{2}} + V^\frac{1}{2}Qx^{\frac{1}{2}},
\]

\[
(2.10) \quad G(x, w, V, V_0) = x + Q^\frac{5}{2}(V + wV_0) + Qxw^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + Q^2x^{\frac{1}{2}} + V^\frac{1}{2}Qx^{\frac{1}{2}}.
\]
(2.10) \( G(x, Q, w, V, V_0) = \max \left\{ (\log x V_0)^2, (\log(x w V_0))^2, \right. \\
(\log V w) \frac{(\log V)^{1/2}}{(\log(V/V))^{1/2}} \log x, \\
(\log(x/V))^2 \frac{\log x}{(\log(V/V))^{1/2}} \right\}. \)

Now we specify \( V \) and \( V_0 \). We introduce a parameter \( 0 < \alpha < 1/2 \) to be chosen later and subdivide into two ranges

1. \( x^\alpha \leq Q \leq x^{1/2}, \)
2. \( Q \leq x^\alpha. \)

In what follows we let \( \beta_i, \gamma_i, \delta_i \in (0, 1), i = 1, 2 \) be chosen later. If \( x^\alpha \leq Q \leq x^{1/2} \), then we set \( V = x^{\beta_1}/Q \). We choose \( V_0 = x^{\delta_1}/Q \) and \( w = x^{\gamma_1}/Q \). Then putting that into (2.9) we get

(2.11) \( R(x, Q) \ll x + Q^{3/2} x^{\beta_1} + Q^{1/2} x^{\gamma_1 + \delta_1} + Q^{3/2} x^{1 - \gamma_1 \alpha - \beta_1} + Q^{3/2} x^{1 - \beta_1 / 2} + Q^{2} x^{3/2} + Q x^{4/3} + Q^{1/2} x^{1 + \beta_1 / 2}. \)

If \( Q \leq x^\alpha \), we let \( V = x^{\beta_2}, V_0 = x^{\delta_2}, w = x^{\gamma_2} \). On inserting it into (2.9) we obtain

(2.12) \( R(x, Q) \ll x + Q^{2} x^{\beta_2} + Q^{5/2} x^{\beta_2} + Q^{5/2} x^{\gamma_2 + \delta_2} + Q x^{1 - \gamma_2 / 2} + Q x^{1 - \beta_2 / 2} + Q^{2} x^{3/2} + Q x^{4/3} + Q^{1/2} x^{1 + \beta_2 / 2}. \)

Let \( 0 < \varepsilon < 1/14 \). We keep in mind conditions \( \alpha < 1/2, \gamma_1 < \beta_1, \delta_1 > \beta_1 \) and put

\( \alpha = 3/7 + \varepsilon, \beta_1 = 4/7, \gamma_1 = 4/7 - \varepsilon, \delta_1 = 4/7 + 5\varepsilon/2. \)

Then, substitution into (2.11) gives

\( R(x, Q) \ll x + Q^{3/2} x^{4/7} + Q^{3/2} x^{8/7 + \varepsilon/2} + Q^{3/2} x^{5/7 + \varepsilon} + Q^{3/2} x^{5/7} + Q^{2} x^{3/2} + Q^{1/2} x^{11/14}. \)

where we used

\( Q^{3/2} x^{4/7} \leq Q^{2} x^{3/7 - 3/14 - \varepsilon} < Q^{2} x^{3/2}, \)
\( Q^{3/2} x^{8/7 + \varepsilon/2} \leq Q^{2} x^{8/7 + \varepsilon/2 - 3/14 - \varepsilon} = Q^{2} x^{1/2}, \)
\( Q^{3/2} x^{5/7 + \varepsilon} \leq Q^{2} x^{5/7 + \varepsilon/2 - 3/14 - \varepsilon} = Q^{2} x^{1/2}, \)
\( Q^{3/2} x^{5/7} \leq Q^{2} x^{5/7 - 1/4} = Q^{2} x^{1/2} - \varepsilon < Q^{2} x^{1/2}, \)
\( Q^{1/2} x^{9/14} \leq Q^{2} x^{9/14 - 3/14 (3/7 + \varepsilon)} < Q^{2} x^{1/2}. \)
Similarly to satisfy $\gamma_2 < \beta_2$, $\delta_2 > \beta_2$ we put
\[
\beta_2 = 1/7, \quad \gamma_2 = 1/7 - \varepsilon, \quad \delta_2 = 1/7 + \varepsilon/2
\]
in (2.12) to obtain
\[
R(x, Q) \ll x + Q^5 x^{1/7} + Q^5 x^{2/7 - \delta_2} + Q x^{13/14 + \varepsilon/2} + Q x^{13} + Q^2 x^{3/2} + Q x^4
\]
\[
\ll x + Q^2 x^{1/2} + Q x^{13/14 + \varepsilon/2},
\]
where we used
\[
Q^5 x^{1/7} \leq Q^2 x^{4/7} + Q x^{13/14 + \varepsilon/2} < Q^2 x^{3/2},
\]
\[
Q^5 x^{2/7 - \delta_2} \leq Q^2 x^{5/7 - \delta_2 + \varepsilon/2} = Q^2 x^{1/2}.
\]

Now we bound $G(x, Q, w, V, V_0)$ given by (2.10). We notice that with our choice of parameters above $\log(V_0/V) \gg_\varepsilon \log x$, where the implied constant depends on $\beta_i, \delta_i$. Thus $\log \ll_\varepsilon (\log x)^2$. Finally, we have
\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(q)}^* \max_{y \leq x} |\psi(y, \chi)| \ll_\varepsilon (x + Q^2 x^{1/2} + Q x^{13/14 + \varepsilon/2})(\log x)^2.
\]
The power $13/14 + \varepsilon/2$ is optimal here. Indeed, let us show first that $\alpha > 3/7$. The system
\[
\begin{cases}
Q^2 x^{\gamma_1 + \delta_1} \leq Q^2 x^{1/2}, \\
Q^2 x^{1 - \gamma_1 + \delta_2} \leq Q^2 x^{1/2},
\end{cases}
\]
brings us to
\[
\begin{cases}
\gamma_1 + \delta_1 - \frac{3\alpha}{2} \leq 1/2, \\
1 - \gamma_1 - \frac{\alpha}{2} \leq 1/2.
\end{cases}
\]
Solving this we obtain $\delta_1 \leq 5\alpha/2 - 1/2$. Further since $Q^2 x^{1 - \gamma_1 + \delta_2} \leq Q^2 x^{1/2}$, we get
\[
1 - \alpha \leq \beta_1 < \delta_1 \leq 5\alpha/2 - 1/2.
\]

Thus $\alpha > 3/7$. We use that to obtain the fact that the term $Q x^A$ has $A > 13/14$. Since $Q^2 x^{\gamma_2 + \delta_2} \leq Q^2 x^{1/2}$, we get $\gamma_2 + \delta_2 \leq 1/2 - \alpha/2 < 2/7$. The inequality $Q x^{1 - \gamma_2 + \delta_2} \leq Q x^A$ gives us $\delta_2 > \beta_2 \geq 2(1 - A)$. Similarly for $\gamma_2$ we obtain $\gamma_2 \geq 2(1 - A)$ because of the term $Q x^{1 - \gamma_2 + \delta_2} \leq Q x^A$. Combining all of this we get $4(1 - A) < \gamma_2 + \delta_2 < 2/7$ and thus $A > 13/14$.

3. Proofs of Theorem 1.2 and Corollary 1.4

The proof is essentially the same and differs only in the last step. It is classical and almost identical to the one in [1] and [13], hence we only sketch it here. Let $y \geq 2,(a, q) = 1$ and $\chi_0$ be the principal character modulo $q$. 


Define \( \psi'(y, \chi) = \psi(y, \chi) \) if \( \chi \neq \chi_0 \) and \( \psi'(y, \chi) = \psi(y, \chi) - \psi(y) \) otherwise. Then by orthogonality of characters modulo \( q \), we have
\[
\psi(y, q, a) - \frac{\psi(y)}{\varphi(q)} = \frac{1}{\varphi(q)} \sum_{\chi} \chi(a) \psi'(y, \chi).
\]

For a character \( \chi \) (mod \( q \)), we let \( \chi^* \) be the primitive character modulo \( q^* \) inducing \( \chi \). Thus
\[
\psi'(y, \chi^*) - \psi'(y, \chi) = \psi(y, \chi^*) - \psi(y, \chi) = \sum_{p^k \leq y} (\log p) (\chi^*(p^k) - \chi(p^k)).
\]

If \( p | q \) then \( (p^k, q^*) = 1 \), and hence \( \chi^*(p^k) = \chi(p^k) \). If \( p | q \) then \( \chi(p^k) = 0 \). Therefore
\[
|\psi'(y, \chi^*) - \psi'(y, \chi)| \leq \sum_{p^k \leq y} (\log p) \leq (\log y) \sum_{p | q} 1 \leq (\log qy)^2.
\]

Thus the quantity we want to average can be bounded as
\[
|\psi(y, q, a) - \frac{\psi(y)}{\varphi(q)}| \leq \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y, \chi)| \leq (\log qy)^2 + \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y, \chi^*)|.
\]

We have to take care just of the second term in the inequality above, since the first one is smaller than the desired bound. A primitive character \( \chi^* \) mod \( q^* \) induces characters of moduli \( dq^* \) and \( \psi'(y, \chi^*) = 0 \) for \( \chi \) principal, we observe
\[
(3.1) \quad \sum_{q \leq Q \atop l(q) > Q_1} \frac{1}{\varphi(q)} \sum_{q^* \mid q} \sum_{\chi(q^*)} \max_{2 \leq y \leq x} |\psi'(y, \chi)| \leq \sum_{q \leq Q \atop l(q) > Q_1} \sum_{\chi(q^*)} \max_{2 \leq y \leq x} |\psi'(y, \chi)| \sum_{k \leq Q/q^*} \frac{1}{\varphi(kq^*)}.
\]

Since \( q^* \leq Q \leq x^{1/2} \) and \( \varphi(k) \varphi(q^*) \leq \varphi(kq^*) \) we have
\[
\sum_{k \leq Q/q^*} \frac{1}{\varphi(kq^*)} \ll \frac{\log x}{\varphi(q^*)}.
\]

For \( q > 1 \) and \( \chi \) primitive character (mod \( q \)), we know that \( \chi \) is non-principal and \( \psi(y, \chi) = \psi'(y, \chi) \). Since we assumed \( Q_1 \geq 1 \) then we can
replace $\psi'(y, \chi)$ by $\psi(y, \chi)$ inside the internal sum in (3.1). Hence
\[
\sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{l(q) > Q_1} \sum_{q^* | q} \chi(q^*) \max_{2 \leq y \leq x} |\psi'(y, \chi)| \leq (\log x) \sum_{q \leq Q} \frac{1}{\varphi(q)} \sum_{l(q) > Q_1} \chi(q) \max_{2 \leq y \leq x} |\psi(y, \chi)|.
\]

For the right-hand side of the above apply the partial summation
\[
\sum_{Q_1 < q \leq Q} \frac{1}{\varphi(q)} \sum_{l(q) > Q_1} \chi(q) \max_{2 \leq y \leq x} |\psi(y, \chi)| = \frac{1}{Q} \sum_{q \leq Q} f(q) - \frac{1}{Q_1} \sum_{q \leq Q_1} f(q) + \int_{Q_1}^{Q} \left( \sum_{q \leq t} f(q) \right) \frac{dt}{t},
\]
where
\[f(q) = \frac{q}{\varphi(q)} \sum_{l(q) > Q_1} \chi(q) \max_{2 \leq y \leq x} |\psi(y, \chi)|\]
and use Proposition 1.3 to get
\[
\sum_{q \leq Q} f(q) \ll \varepsilon \left( x + Q^2 x^{\frac{1}{2}} + Q x^{\frac{13}{11} + \varepsilon} \right) (\log x)^2.
\]
This finishes the proof of Corollary 1.4. We apply Siegel–Walfisz theorem for the remaining moduli to get Theorem 1.2.

References


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