Dragan STANKOV

A necessary and sufficient condition for an algebraic integer to be a Salem number


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Abstract. We present a necessary and sufficient condition for a root greater than unity of a monic reciprocal polynomial of an even degree at least four, with integer coefficients, to be a Salem number. This condition requires that the minimal polynomial of some power of the algebraic integer has a linear coefficient that is relatively large. We also determine the probability that an arbitrary power of a Salem number, of certain small degrees, satisfies this condition.

1. Introduction

A Salem number is a real algebraic integer $\tau > 1$ of degree at least four, conjugate to $\tau^{-1}$, all of whose conjugates, excluding $\tau$ and $\tau^{-1}$, are unimodal i.e., lie on $|z| = 1$. The corresponding minimal polynomial $P(x)$ of degree $d$ of these numbers, called a Salem polynomial, is (self-)reciprocal, that is $x^dP(1/x) = P(x)$. Since $P(x)$ is self-reciprocal and irreducible it must have even degree. It is well known [7] that $\tau^n$ should also be a Salem number of degree $d$ for any natural $n$. Fractional parts of $\tau^n$ are dense in the unit interval $[0, 1]$, but are not uniformly distributed [1, 8]. Salem numbers have appeared in quite different areas of mathematics (number theory, harmonic analysis, knot theory, etc.). Throughout, when we speak about a conjugate, the minimal polynomial or the degree of an algebraic number we mean over the field of the rationals $\mathbb{Q}$.
In [9] Vieira, extending a result of Lakatos and Losonczi [3], presented a sufficient condition for a self-reciprocal polynomial to have a fixed number of roots on the complex unit circle $U = \{z \in \mathbb{C} : |z| = 1\}$. Let $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$ be a $d$-th degree self-reciprocal polynomial. If the inequality

$$|a_d - l| > \frac{1}{2} \left( \frac{d}{d - 2} \right) \sum_{k=0,k \neq l,d-l}^{d} |a_k|, \quad l < d/2$$

holds, then $p(z)$ has exactly $d - 2l$ roots on $U$ and these roots are simple.

Here we present, in a sense, a result which lies in the opposite direction of a special case of this theorem. Namely, we shall prove the following

**Theorem 1.1.** A real algebraic integer $\tau > 1$ is a Salem number if and only if its minimal polynomial $P(x)$ is reciprocal of even degree $d \geq 4$, and there is $n \in \mathbb{N}$, $n \geq 2$ such that $\tau^n$ has the minimal polynomial $P_n(x) = 1 + a_{1,n} x + a_{2,n} x^2 + \cdots + a_{d-1,n} x^{d-1} + x^d$, which is also reciprocal of degree $d$, and satisfies the condition

$$|a_{d-1,n}| > \frac{1}{2} \left( \frac{d}{d - 2} \right) \left( 2 + \sum_{k=2}^{d-2} |a_{k,n}| \right).$$

Notice that the condition (1.2) is the special case when $l = 1$ of the condition (1.1) applied to $P_n(x)$.

We present a method, easy for implementation, for the calculation of the coefficients of $P_n(x)$ starting with $P(x)$ without determination of its roots. We can use the companion matrix $C$ of a monic polynomial $P(x) = x^d + a_{d-1} x^{d-1} + a_{d-2} x^{d-2} + \cdots + a_0$ defined as

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-2} & -a_{d-1} \end{bmatrix}_{d \times d}$$

It is well known [4, 5] that $P(x)$ is the characteristic polynomial of $C$ so the root $\lambda$ of $P(x)$ is an eigenvalue of $C$. If $v$ is an eigenvector of $C$ associated with $\lambda$ then $C^n v = C^{n-1} C v = C^{n-2} \lambda v = \cdots = \lambda^n v$. Thus $C^n$ should have an eigenvalue $\lambda^n$ and the characteristic polynomial of $C^n$ must be $P_n(x)$, i.e. $P_n(x) = \text{det}(xI - C^n)$. It is easy to show that $v = [1 \, \lambda \, \lambda^2 \, \ldots \, \lambda^{d-1}]^T$.

Using this method we are able, for a Salem number $\tau$, to find at least one $n$ such that the minimal polynomial $P_n(x)$ of $\tau^n$ satisfies condition (1.2). In Table 1.1 we present examples of Salem numbers and the smallest $n$ which we have found. The forth example in the table is the root of the Lehmer polynomial which is the smallest known Salem number. We notice that $n$...
becomes large as $d$ increases. It would be interesting to find at least one $n$ for all small Salem numbers in the Mossinghoff’s list [6].

The relative frequency of $n$ such that the minimal polynomial $P_n(x)$ of $\tau^n$ satisfies (1.2), which we approximated for $d = 4$ (2.12) and for $d = 6$ (2.13), significantly decreases when $d$ increases. One might ask what is the probability of fulfilling the condition (1.2) for an arbitrary power of the root. We determined the exact value of the probability for $d = 4$, $6$ and we approximated the probability for $d = 8$, $10$.

**Theorem 1.2.** Let $\tau$ be a Salem number of degree $d$, $n, n_0 \in \mathbb{N}$ and let $P_n(x)$ be the minimal polynomial of $\tau^n$. Let $p_d$ denotes the limit of the probability that coefficients of $P_n(x)$ satisfy (1.2) when $n \geq n_0$ is randomly chosen, as $n_0$ approaches infinity. Then:

1. $p_4$ is equal to $1/3$ and,  
2. 

\[
(1.3) \quad p_6 = \frac{4}{\pi^2} \left[ \int_{\arccos(\sqrt{\frac{19}{6}})}^{\arccos(\sqrt{19})} \left( \arccos \left( \frac{-5 - 6 \cos t}{6 + 6 \cos t} \right) - (\pi - t) \right) dt \\
+ \int_{\arccos(\sqrt{\frac{19}{6}})}^{\arccos(\sqrt{30})} \left( \arccos \left( \frac{1 - 6 \cos t}{6 - 6 \cos t} \right) - (\pi - t) \right) dt \right] \\
= 0.0717258 \ldots
\]

Furthermore, we have approximated the probabilities for $d = 8$ and $d = 10$ using a numerical method and have got $p_8 \approx 0.012173$, $p_{10} \approx 0.0018$. These results suggest that $p_d$ decreases approximately five times when $d$ is increased by two.

If $P(x)$ is monic, reciprocal, with integer coefficients then $P_n(x)$ is a periodic sequence of polynomials if and only if $P(x)$ is the product of cyclotomic polynomials. In fact, if $P_n(x)$ is a periodic sequence, among these polynomials there are only finitely many distinct ones. Then the set of roots of these polynomials is also finite, and all the powers $\alpha, \alpha^2, \alpha^3, \ldots$ of a root $\alpha$ of $P(x)$ are in this set. Therefore for some $p$, $q$, $\alpha^p = \alpha^q$, $p \neq q$. Since $\alpha \neq 0$ it follows that $\alpha^{p-q} = 1$. Vice versa, if $P(x)$ is the product of cyclotomic polynomials then all its roots are roots of 1 so the set of its powers is finite and the set of coefficients $a_{k,n}$ for $k = 1, 2, \ldots, d - 1$, $n = 1, 2, \ldots$ of $P_n(x)$ is also finite. Thus $P_n(x)$ is a periodic sequence of polynomials.

**Acknowledgments.** The author would like to thank the unknown referee for valuable comments that resulted in an improvement of this paper, especially for the calculation of the smallest $n$ of the last seven Salem numbers in the Table 1.1, marked with *, which the referee generously permitted to be added in the Table.
Table 1.1. Salem number $\tau$ and the smallest $n$ such that the minimal polynomial $P_n(x)$ of $\tau^n$ satisfies (1.2)

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\tau$</th>
<th>Coefficients</th>
<th>$n$</th>
</tr>
</thead>
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<tr>
<td>1.</td>
<td>4</td>
<td>1 $-1$ $-1$</td>
<td>9</td>
</tr>
<tr>
<td>2.</td>
<td>6</td>
<td>1 $-1$ $0$ $-1$</td>
<td>14</td>
</tr>
<tr>
<td>3.</td>
<td>8</td>
<td>1 $0$ $0$ $-1$ $-1$</td>
<td>72</td>
</tr>
<tr>
<td>4.</td>
<td>10</td>
<td>1 $1$ $0$ $-1$ $-1$ $-1$</td>
<td>605</td>
</tr>
<tr>
<td>5.</td>
<td>10</td>
<td>1 $0$ $0$ $0$ $-1$ $-1$</td>
<td>53</td>
</tr>
<tr>
<td>6.</td>
<td>10</td>
<td>1 $0$ $-1$ $0$ $-1$</td>
<td>240</td>
</tr>
<tr>
<td>7.</td>
<td>10</td>
<td>1 $0$ $-1$ $0$ $-1$ $-1$</td>
<td>43</td>
</tr>
<tr>
<td>8.*</td>
<td>10</td>
<td>1 $0$ $-1$ $0$ $1$ $1$</td>
<td>1367</td>
</tr>
<tr>
<td>9.*</td>
<td>12</td>
<td>1 $1$ $-1$ $1$ $0$ $0$ $-1$</td>
<td>5894</td>
</tr>
<tr>
<td>10.*</td>
<td>14</td>
<td>1 $0$ $0$ $-1$ $1$ $0$ $0$ $-1$</td>
<td>61739</td>
</tr>
<tr>
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<tr>
<td>14.*</td>
<td>16</td>
<td>1 $-1$ $0$ $0$ $0$ $0$ $0$ $-1$</td>
<td>68667</td>
</tr>
</tbody>
</table>

2. Proofs of Theorems

In order to prove Theorem 1.1 we shall use a theorem of Kronecker [1, Theorem 4.6.4], which is a consequence of Weyl’s theorems [2]. Suppose $\alpha = (\alpha_k)_{1 \leq k \leq p} \in \mathbb{R}^p$ has the property that the real numbers $1, \alpha_1, \ldots, \alpha_p$ are $\mathbb{Q}$-linearly independent, and let $\mu$ denote an arbitrary vector in $\mathbb{R}^p$, $N$ an integer and $\varepsilon$ a positive real number. Then Kronecker’s theorem states that there exists an integer $n > N$ such that $\| n\alpha_k - \mu_k \| < \varepsilon$, $(k = 1, \ldots, p)$ where $\|x\| = \min \{|x - m| : m \in \mathbb{Z}\}$ is the distance from $x$ to the nearest integer.

Proof of Theorem 1.1.

Necessity. Suppose that $\tau > 1$ is a Salem number. The essence of the proof is to show that there is $n$ such that each of $d - 2$ unimodal roots of $P_n(x)$ could be arbitrarily close to exactly one root of $x^{d-2} + 1$ (see [10, Lemma 2]) and to show that then the coefficients of $P_n(x)$ will satisfy the condition (1.2). It is obvious that roots of $x^{d-2} + 1$ are $\exp(\pm \frac{2\pi j}{d-2}i)$, $j = 0, 1, \ldots, d/2 - 2$. We denote conjugates of $\tau$ by

\[
\tau^{-1}, \exp(\pm 2i\pi \omega_1), \ldots, \exp(\pm 2i\pi \omega_{d/2-1}).
\]

Numbers $1, \omega_1, \ldots, \omega_{d/2-1}$ are $\mathbb{Q}$-linearly independent [1, proof of Theorem 5.3.2]. According to the Kronecker’s theorem consider $(w_j)_{2 \leq j \leq d/2} \in \mathbb{R}^{d/2-1}$ with $\mu = (\frac{1/2+0}{d-2}, \frac{1/2+1}{d-2}, \ldots, \frac{1/2+d/2-2}{d-2})$. It is clear that for every
If $\tau$ is the seventh Salem number in Table 1.1 of degree 10 then conjugates of $\tau^{43}$ (represented with $\circ$) and of $\tau^{80}$ (represented with $+$), whose minimal polynomials $P_{43}(x), P_{80}(x)$ satisfy (2), are close to roots of $x^8 + 1$, vertices of the regular octagon.

There exists an arbitrarily large integer $n$ such that

$$\left| n\omega_j - \frac{1/2 + j - 1}{d - 2} \right| < \varepsilon \pmod{1} \quad (j = 1, 2, \ldots, d/2 - 1).$$

Since a coefficient of a polynomial is a continuous function of its roots, for every $\varepsilon > 0$ there exists an arbitrarily large integer $n$ such that the minimal polynomial

$$P_n(x) = (x - \tau^n)(x - \tau^{-n})$$

$$= x^{d-2} + 1 + \sum_{j=1}^{d/2-2} \epsilon_j (x^{d-2-j} + x^j) + \epsilon_{d/2-1} x^{d/2-1},$$

of the Salem number $\tau^n$ satisfies $|\epsilon_k| < \varepsilon$, $k = 1, \ldots, d/2 - 1$. We denote

$$-\tau^n - \tau^{-n} = T$$
\[ P_n(x) = (x^2 + T \, x + 1) \]
\[ \cdot \left( x^{d-2} + 1 + \sum_{j=1}^{d/2-2} \epsilon_j (x^{d-2-j} + x^j) + \epsilon_{d/2-1} x^{d/2-1} \right) \]
\[ = x^d + 1 + (T + \epsilon_1) (x^{d-1} + x) + (\epsilon_2 + T \epsilon_1 + 1) (x^{d-2} + x^2) \]
\[ + \sum_{j=3}^{d/2-1} (\epsilon_j + T \epsilon_{j-1} + \epsilon_{j-2}) (x^{d-j} + x^j) \]
\[ + (2 \epsilon_{d/2-2} + T \epsilon_{d/2-1}) x^{d/2}. \]

Now we consider the coefficients of \( P_n(x) \) to show they satisfy the condition (1.2). It is obvious that \( |a_{d-1,n}| = |T + \epsilon_1| \geq |T| - |\epsilon| \). We need to estimate
\[
\frac{1}{2} \left( \frac{d}{d-2} \right) \left( 2 + \sum_{k=2}^{d-2} |a_{k,n}| \right) \]
\[ = \frac{1}{2} \left( \frac{d}{d-2} \right) \left( 2 + 2|\epsilon_2 + T \epsilon_1 + 1| + 2 \sum_{j=3}^{d/2-1} |\epsilon_j + T \epsilon_{j-1} + \epsilon_{j-2}| \right) \]
\[ + |2 \epsilon_{d/2-2} + T \epsilon_{d/2-1}| \]
\[ \leq \frac{1}{2} \left( \frac{d}{d-2} \right) \left( 2 + 2 \epsilon_2 + 2 |T| \epsilon_1 + 2 + 2 \sum_{j=3}^{d/2-1} (\epsilon_j + |T| \epsilon_{j-1} + \epsilon_{j-2}) \right) \]
\[ + 2 \epsilon_{d/2-2} + |T| \epsilon_{d/2-1} \]
\[ \leq \frac{1}{2} \left( \frac{d}{d-2} \right) \left( 4 + (2d - 8) \epsilon + (d - 3)|T| \epsilon \right). \]

So the condition (1.2) will be satisfied if
\[ |T| - |\epsilon| > \frac{1}{2} \left( \frac{d}{d-2} \right) \left( 4 + (2d - 8) \epsilon + (d - 3)|T| \epsilon \right), \]
which is equivalent to
\[ (2.5) \quad \frac{(2d - 4)|T| - 4d}{(d^2 - 3d)|T| + 2d^2 - 6d - 4} > \epsilon, \quad d \geq 4. \]

Since \( |T| = r^n + r^{-n} \) tends to \( \infty \) as \( n \to \infty \) it is obvious that the left side of (2.5) tends to \( D := \frac{2d-4}{d(d-3)} \) as \( n \to \infty \). The determination of \( n \) such that coefficients of \( P_n(x) \) satisfies (1.2) has to be done in following four steps:

(i) we choose \( \epsilon \) such that \( D > \epsilon > 0; \)
(ii) we choose an integer \( N \) such that (2.5) will be fulfilled for all \( n \geq N \); (iii) we chose an \( \varepsilon > 0 \) such that if each of \( d - 2 \) unimodal roots of a \( P_n(x) \) is at the distance \( < \varepsilon \) in modulus of exactly one root of \( x^{d-2} + 1 \) then \( |\alpha_k| < \varepsilon \), \( k = 1, \ldots, d/2 - 1 \) is fulfilled in (2.3); (iv) we chose \( n \geq N \) such that (2.2) is fulfilled.

Sufficiency. Suppose that \( \tau > 1 \) is a real algebraic integer with conjugates \( \tau_1 = \tau, \tau_2, \ldots, \tau_d \) over \( \mathbb{Q} \) such that \( \tau^n \) has the minimal polynomial \( P_n(x) \) which is also reciprocal of degree \( d \), and satisfies the condition (1.2). If \( \tau \) is a conjugate of \( \tau' \) then \( \tau^n \) is a conjugate of \( \tau'^n \). Since the minimal polynomial \( P_n(x) \) of \( \tau^n \) is of degree \( d \) so \( \tau_1^n, \tau_2^n, \ldots, \tau_d^n \) must be different numbers and their product has to be 1 because \( P_n(x) \) is monic and reciprocal. The polynomial \( P_n(x) \) satisfies the condition (1.2) so it satisfies the condition (1.1) of Vieira’s theorem where \( l = 1 \). According to the theorem there are \( d - 2 \) roots of \( P_n(x) \) on the boundary of the unit disc \( |z| = 1 \). Since they occur in conjugate complex pairs their product is equal to 1. It follows that \( \tau^{-n} \) should be a conjugate of \( \tau^n \) which allow us to conclude that \( \tau^n \) is a Salem number. If \( |\tau^n| = 1 \) then \( |\tau'| = 1 \) thus it follows that there are \( d - 2 \) conjugates of \( \tau \) on the boundary of the unit disc. Finally, in the same manner as for \( \tau^n \), we conclude that \( \tau \) is also a Salem number.

Proof of Theorem 1.2.

(1). If we use (2.1) and denote \( D := \tau^n + 1/\tau^n \) \( (d = 4) \) we have

\[
P_n(x) = (x^2 - Dx + 1)
(x^2 - 2 \cos(2\pi n\omega_1)x + 1).
\]

We denote \( 2\pi\{n\omega_1\} \) by \( \theta_1 \) and \( 2\cos(\theta_1) \) by \( s_1 \) where \( \{ \cdot \} \) denotes the fractional part. Since \( n\omega_1 \) is uniformly distributed modulo one \( \theta_1 \) is uniformly distributed on \( [0, 2\pi] \). For \( d = 4 \) the condition (1.2) is reduced to \( |a_{3,n}| > 2 + |a_{2,n}| \). Since \( P_n(x) = (x^2 - Dx + 1)(x^2 - s_1x + 1) \) the condition becomes

(2.6) \[
|D - s_1| > 2 + |Ds_1 + 2|
\]

From the definition of \( D \) it is obvious that \( D \to \infty \) when \( n \to \infty \). Since \( |s_1| \leq 2 \) we have \( D + s_1 \to \infty \) so that \( |D - s_1| = |D + s_1| \) is equal, for every sufficiently large \( n \), to \( D + s_1 \). Finally (2.6) becomes \( D + s_1 > 2 + |Ds_1 + 2| \) i.e. \( D + s_1 - 2 > Ds_1 + 2 > -D - s_1 + 2 \). Solving this double inequality for \( s_1 \) we get

\[
-\frac{D}{D+1} < s_1 < \frac{D-4}{D-1}.
\]

When \( n_0 \) tends to infinity we obtain \( -1 < s_1 < 1 \) i.e. \( -1/2 < \cos \theta_1 < 1/2 \). It follows that \( \pi/3 < \theta_1 < 2\pi/3 \) or \( 4\pi/3 < \theta_1 < 5\pi/3 \) so that the limit of the probability has to be \( p_4 = \frac{2\pi/3}{2\pi} = \frac{1}{3} \).
(2). Using (2.1) with \( d = 6 \) and the definition of \( D \) we have
\[
P_n(x) = (x^2 - Dx + 1)(x^2 - 2\cos(2\pi n\omega_1)x + 1)(x^2 - 2\cos(2\pi n\omega_2)x + 1).
\]
We denote \( \theta_1 := 2\pi \{n\omega_1\}, \theta_2 := 2\pi \{n\omega_2\} \). Coefficients of \( P_n(x) \) depend only on real parts of unimodal roots so that we can chose the complex conjugates from the upper half (complex) plane. Thus we define
\[
t_i = \begin{cases} \theta_i & \text{if } \theta_i \in (0, \pi); \\ 2\pi - \theta_i & \text{if } \theta_i \in (\pi, 2\pi), \end{cases}
\]
\( i = 1, 2 \).

Since \( n\omega_1, n\omega_2 \) are uniformly distributed modulo one \( \theta_1, \theta_2 \) are uniformly distributed on \([0, 2\pi]\) and \( t_1, t_2 \) are uniformly distributed on \([0, \pi]\). We denote
\[
s_1 := 2\cos(t_1), \quad s_2 := 2\cos(t_2).
\]
For \( d = 6 \) the condition (1.2) is reduced to \(|a_{5,n}| > \frac{6}{8}(2 + 2|a_{4,n}| + |a_{3,n}|)\). Since
\[
P_n(x) = (x^2 - Dx + 1)(x^2 - s_1x + 1)(x^2 - s_2x + 1)
\]
the condition becomes
\[
|\frac{1}{-D - s_1 - s_2}|
\]
\[
> \frac{6}{8}(2 + 2|Ds_1 + Ds_2 + s_1s_2 + 3| + |\frac{6}{2} - 2s_1 - 2s_2 - Ds_1s_2|).
\]
The main idea of the proof is to determine the region \( S \) in \( s_1Os_2 \) plane such that every point \((s_1, s_2)\) satisfies (2.9). Since \( D \rightarrow \infty \) when \( n \rightarrow \infty, \) \(|s_1| \leq 2, \ |s_2| \leq 2 \) we conclude that the left side in (2.9) \(|\frac{1}{-D - s_1 - s_2}|\) is equal, for every sufficiently large \( n \), to \( D + s_1 + s_2 \). We can find the boundary of \( S \) if we replace \( > \) in (2.9) with \( = \) and if we replace both \(|\cdot|\) on the right side with \( \pm(\cdot) \). There are four possibilities for replacing so we get four equations which we solve for \( s_2 \). We get rational functions \( s_2 = f_i(D, s_1) \) which tends to \( s_2 = F_i(s_1) \) when \( n_0 \rightarrow \infty, \ i = 1, 2, 3, 4: \)

\[
f_1(D, s_1) = \frac{10D + 10s_1 - 6Ds_1 - 24}{6D + 6s_1 - 3Ds_1 - 10},
\]
\[
f_2(D, s_1) = -\frac{2D + 2s_1 + 6Ds_1 + 24}{6D + 6s_1 + 3Ds_1 + 2},
\]
\[
f_3(D, s_1) = -\frac{10D + 10s_1 + 6Ds_1 + 12}{6D + 6s_1 + 3Ds_1 + 10},
\]
\[
f_4(D, s_1) = \frac{2D + 2s_1 - 6Ds_1 - 12}{6D + 6s_1 - 3Ds_1 - 2},
\]
\[
F_1(s_1) = \frac{10 - 6s_1}{6 - 3s_1},
\]
\[
F_2(s_1) = -\frac{2 + 6s_1}{6 + 3s_1},
\]
\[
F_3(s_1) = -\frac{10 + 6s_1}{6 + 3s_1},
\]
\[
F_4(s_1) = \frac{2 - 6s_1}{6 - 3s_1}.
\]
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The boundary of $S$ consists of parts of graphs of $F_i(s_1)$. We have to find intersection points of these graphs. Therefore we solve four equations:

$$\begin{align*}
F_1(s_1) &= F_2(s_1) \Rightarrow s_1 = 1/3 + \sqrt{19}/3, \\
F_1(s_1) &= F_3(s_1) \Rightarrow s_1 = \pm \sqrt{30}/3, \\
F_4(s_1) &= F_2(s_1) \Rightarrow s_1 = \pm \sqrt{6}/3, \\
F_4(s_1) &= F_3(s_1) \Rightarrow s_1 = -1/3 + \sqrt{19}/3.
\end{align*}$$

(2.10)

We have to determine the area of the region $T$ in $t_1 Ot_2$ plane such that for every point $(s_1, s_2) \in S$ there is unique $(t_1, t_2) \in T$ where

$$t_1 = \arccos(s_1/2), \quad t_2 = \arccos(s_2/2),$$

(2.11)

using (2.8). The ratio of the area of $T$ to the area of all possible values $(t_1, t_2)$, i.e. $\pi^2$, is equal to the probability $p_6$. Since $s_2 = F_i(s_1)$ it follows that $t_2 = \arccos(F_i(2 \cos(t_1))/2) =: G_i(t_1)$ using (2.8). For the determination of the area of $T$ it is convenient to show that $T$ has reflection symmetry across the line $t_2 = \pi - t_1$. Let the graph of $t_2 = G_i(t_1)$ be $\Gamma_i$. We claim that $\Gamma_1$ can be obtained by reflecting of $\Gamma_3$ about the line $t_2 = \pi - t_1$ i.e. if $(t_1, t_2) \in \Gamma_1$ then $(\pi - t_2, \pi - t_1) \in \Gamma_3$ (see Figure 2.2). Indeed, if $t_2 = G_1(t_1)$ then

$$G_3(\pi - t_2) = \arccos(F_3(2 \cos(\pi - t_2))/2)$$

$$= \arccos(F_3(-2 \cos(t_2))/2)$$

$$= \arccos(F_3(-2 \cos(G_1(t_1)))/2)$$

$$= \arccos(F_3(-F_1(s_1))/2)$$

$$= \arccos(-s_1/2)$$

$$= \arccos(-\cos(t_1))$$

$$= \arccos(\cos(\pi - t_1))$$

$$= \pi - t_1.$$

In the same manner we can show that $\Gamma_2$ is a reflection of $\Gamma_4$ in the line $t_2 = \pi - t_1$. Therefore $T$ consists of four congruent curve-triangles, each of them has the same area $A$ (see Figure 2.2). If we bring to mind the intersection points (2.10) and formulas (2.11) we find out the intersection points of graphs $\Gamma_i$, $i = 1, 2, 3, 4$ which are the limits of two definite integrals that occur in (1.3). We conclude that $A$ is equal to sum of these integrals (see Figure 2.2) and that $p_6 = 4A/\pi^2$ as it is claimed. \hfill $\square$

If we use the same method for the determination of $p_8$, $p_{10}$ etc. it requires multiple definite integrals applied on the regions with complicated boundaries. Thus it is much more convenient to use a numerical approach. For each pair of conjugate complex roots of a Salem polynomial we define a variable $t_i \in (0, \pi)$, as in (2.7) and $s_i$ as in (2.8) $i = 1, 2, \ldots, H$ where we
Figure 2.2. The event $T$ that a power of a Salem number of degree 6 has the minimal polynomial which satisfies the condition (1.2) is shaded in the figure. It consists of four congruent curve-triangles, each of them has the same area $A$ which is equal to the definite integral. Thus the probability of $T$ is $4A/\pi^2 = 0.0717258\ldots$

denoted $(d - 2)/2$ by $H$. Let $m \in \mathbb{N}$ and let $0 = t_{i,0}, t_{i,1}, \ldots, t_{i,m} = \pi$, $i = 1, 2, \ldots, H$ be nodes arranged consecutively with equal spacing $h = \pi/m$. Starting from

$$P_n(x) = (x^2 - Dx + 1) \prod_{i=1}^{H} (x^2 - 2\cos(t_i)x + 1)$$

we calculate the coefficients of $P_n(x)$ which obviously depend on $D$, $t_i$ so that there are the functions $A_{k,n}$ such that

$$a_{k,n} = A_{k,n}(D; t_1, t_2, \ldots, t_H), \quad k = 1, 2, \ldots, d - 1.$$

For $D$ fixed and for each $H$-tuple $(t_{1,j_1}, t_{2,j_2}, \ldots, t_{H,j_H})$ we calculate

$$a_{k,n} = A_{k,n}(D; t_{1,j_1}, t_{2,j_2}, \ldots, t_{H,j_H}), \quad j_i = 0, 1, \ldots, m,$$

and replace them into the condition (1.2). The number $N_c$ of all $H$-tuples, i.e. of all points of $\pi^H$, which satisfy this condition, divided with $(m + 1)^H$, the number of all $H$-tuples, approximates $p_d$. If we take a large $D = 10^9$ and a small $h \geq 0.002$ we get $p_8 \approx 0.012173$, $p_{10} \approx 0.0018$. Since there are four nested loops the calculation of $p_{10}$ requires much CPU time. Thus it
A condition for an algebraic integer to be a Salem number

Table 2.1. Coefficients of $P_{43}(x)$, $P_{80}(x)$ which satisfy (2) and of $P_{100}(x)$ which does not, where $P(x)$ is the minimal polynomial of the seventh Salem number in Table 1.1. The modulus of the linear coefficient of $P_{43}(x)$, $P_{80}(x)$ is relatively large.

<table>
<thead>
<tr>
<th>$P_{43}(x)$</th>
<th>$P_{80}(x)$</th>
<th>$P_{100}(x)$</th>
</tr>
</thead>
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<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

was necessary to improve our programm. We use the fact that all $H$-tuples which satisfy (1.2) are close to the point $P(\pi/H, 3\pi/H, \ldots, (d - 3)\pi/H)$ or to $H!$ points obtained by permuting the coordinates of $P$, because these coordinates are the arguments of the roots of $x^{d-2} + 1$. Therefore to get $N_c$ we have to check and count only points in a small region around the $P$ and then to multiply the number of them by $H!$.

We have also verified $p_4$ and $p_6$ experimentally. For the first Salem number in the Table 1.1 of degree 4 we have found that if $1 \leq n \leq 300$ then the coefficients of $P_n(x)$ satisfy (1.2) 98 times: for 9, 13, 16, 17, 20, 24, 27, 31, 35, 38, 42, 45, 46, 49, 53, 56, 57, 60, 64, 67, 68, 71, 75, 78, 79, 82, 86, 89, 93, 97, 100, 104, 107, 108, 111, 115, 118, 122, 126, 129, 130, 133, 137, 140, 141, 144, 148, 151, 155, 159, 162, 166, 169, 170, 173, 177, 180, 181, 184, 188, 191, 192, 195, 199, 202, 203, 206, 210, 213, 217, 221, 224, 228, 231, 232, 235, 239, 242, 243, 244, 246, 250, 253, 254, 257, 261, 264, 265, 268, 272, 275, 279, 283, 286, 290, 293, 294, 297, so that the relative frequency is

\[(2.12) \quad 98/300 \approx 0.33.\]

For the second Salem number in the Table 1.1 of degree 6 we have found that if $101 \leq n \leq 300$ then the event that $P_n(x)$ satisfies (1.2) occurs fourteen times: for $n = 116, 144, 157, 167, 187, 195, 206, 225, 238, 246, 257, 276, 287, 295$ so that the relative frequency is $14/200 = 0.07$. If $1001 \leq n \leq 1200$ then $P_n(x)$ satisfy (1.2) sixteen times: for $n = 1001, 1029, 1031, 1039, 1050, 1052, 1063, 1080, 1082, 1101, 1103, 1120, 1131, 1133, 1152, 1182$ with the relative frequency

\[(2.13) \quad 16/200 = 0.08.\]
References


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