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A note on the regularity of the Diophantine pair 
\{k, 4k \pm 4\}

par Bo HE, Keli PU, Rulin SHEN et Alain TOGBÉ

Abstract. Let \( \varepsilon \in \{\pm 1\} \) and let \( k \) be an integer such that \( k \geq 2 \) if \( \varepsilon = -1 \) and \( k \geq 1 \) if \( \varepsilon = 1 \). For positive integer \( d \), we prove that if the product of any two distinct elements of the set
\[ \{k, 4k + 4\varepsilon, 144k^3 + 240k^2\varepsilon + 124k + 20\varepsilon, d\} \]
augmented by 1 is a perfect square, then \( d = 9k + 6\varepsilon \) or
\[ d = 2304k^5 + 6144k^4\varepsilon + 6112k^3 + 2784k^2\varepsilon + 569k + 42\varepsilon. \]
Consequently, combining this result with a recent result of Filipin, Fujita and Togbé, we show that all Diophantine quadruples of the form \( \{k, 4k + 4\varepsilon, c, d\} \) are regular.

1. Introduction

A set \( \{a_1, a_2, \ldots, a_m\} \) of \( m \) positive integers is called a Diophantine \( m \)-tuple if \( a_ia_j + 1 \) is a perfect square for all \( i, j \) with \( 1 \leq i < j \leq m \). A folklore conjecture says that there does not exist a Diophantine quintuple. This conjecture was proved by the first, fourth authors and V. Ziegler [11]. Euler first proved that any Diophantine pair \( \{a, b\} \) can be extended to a Diophantine triple \( \{a, b, a + b + 2\sqrt{ab} + 1\} \). In 1979, Arkin, Hoggatt and
Strauss [1] showed that any Diophantine triple \( \{a, b, c\} \) can be extended to a Diophantine quadruple
\[
\left\{ a, b, c, a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)} \right\}.
\]
We call such a Diophantine quadruple regular. The following is a strong version of the folklore conjecture.

**Conjecture 1.1.** Any Diophantine quadruple is regular.

In 1969, by Baker and Davenport [2] who proved that the fourth element 120 in Fermat’s quadruple uniquely extends the Diophantine triple \( \{1, 3, 8\} \).

In 2004, Dujella [6] proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples. In 2014, Filipin, Fujita and Togbé [8], [9] studied the extendibility of some Diophantine pairs. They proved the following result.

**Theorem 1.2** (cf. [9, Theorem 1.4]). Let \( \{a, b\} \) be a Diophantine pair with \( a < b \leq 8a \) and \( r \) the positive integer satisfying \( ab + 1 = r^2 \). Define an integer \( c = c^{\tau}_{\nu} (\nu \in \{1, 2, \ldots\}, \tau \in \{\pm\}) \) by

\[
(1.1) \quad c^{\tau}_{\nu} = \frac{1}{4ab} \left\{ (\sqrt{b} + \tau \sqrt{a})^2 (r + \sqrt{ab})^{2\nu} + (\sqrt{b} - \tau \sqrt{a})^2 (r - \sqrt{ab})^{2\nu} - 2(a+b) \right\}.
\]

Suppose that \( \{a, b, c, d\} \) is a Diophantine quadruple with \( d > c^{\tau+1}_{\nu} \) and that \( \{a, b, c', c\} \) is not a Diophantine quadruple for any \( c' \) with \( 0 < c' < c^{\tau-1}_{\nu} \).

1. If \( b < 2a \), then \( c \leq c^{5}_{3} \).
2. If \( 2a \leq b \leq 8a \), then \( c \leq c^{+}_{2} \).

Let \( \varepsilon \in \{\pm1\} \) and let \( k \) be an integer such that \( k \geq 2 \) if \( \varepsilon = -1 \) and \( k \geq 1 \) if \( \varepsilon = 1 \). Define an integer \( c = c^{\tau}_{\nu} (\nu \in \{1, 2, \ldots\}, \tau \in \{\pm\}) \) by (1.2) with

\[
a = k, \quad \text{and} \quad b = 4k + 4\varepsilon.
\]

In [9], Filipin, Fujita and Togbé proved that

**Theorem 1.3** (cf. [9, Theorem 1.8]). If \( \{k, 4k + 4\varepsilon, c, d\} \) is a Diophantine quadruple with \( c^{+}_{2} \neq c < d \), then \( d = c^{\tau+1}_{\nu} \).

However, it remains the case of the Diophantine triple
\[
\{a, b, c^{+}_{2}\} = \{k, 4k + 4\varepsilon, 144k^3 + 240k^2\varepsilon + 124k + 20\varepsilon\}.
\]

Note that
\[
c^{+}_{1} = 9k + 6\varepsilon, \quad c^{+}_{3} = 2304k^5 + 6144k^4\varepsilon + 6112k^3 + 2784k^2\varepsilon + 569k + 42\varepsilon,
\]
such that \( \{a, b, c^{+}_{1}, c^{+}_{2}\} \) and \( \{a, b, c^{+}_{2}, c^{+}_{3}\} \) are both regular Diophantine quadruples. In this paper, we will show the following result.
Theorem 1.4. If \( \{k, 4k + 4\varepsilon, c_2^+, d\} \) is a Diophantine quadruple with \( c_2^+ < d \), then \( d = c_3^+ \).

Therefore, combining Theorem 1.3 and Theorem 1.4, we show that the Diophantine quadruples

\[
\{k, 4k \pm 4, c, d\}
\]

are regular. Moreover, with earlier works of Fujita [10], Bugeaud, Dujella and Mignotte [3] on Diophantine pairs \( \{k - 1, k + 1\} \), we have

Corollary 1.5. Any Diophantine quadruple which contains at least two elements in \( \{k - 1, k + 1, 4k\} \) is regular.

This also extends a result of Dujella [4] on the Diophantine triple \( \{k - 1, k + 1, 4k\} \). It is interesting to mention that in this paper we study the extension of a Diophantine pair \( \{a, b\} \) to a Diophantine triple \( \{a, b, c\} \) with \( c = c_2^+ \). In general, it was very difficult to consider

\[
c = c_2^+ = 4r(r \pm a)(b \pm r).
\]

This was done by Bugeaud, Dujella and Mignotte [3] when the pair is \( \{k - 1, k + 1\} \). In [11], we have defined an operator on Diophantine triples by

\[
\partial(\{a, b, c\}) = \{a, b, d_-(a, b, c)\}, \quad \text{for} \quad a < b < c,
\]

where

\[
d_-(a, b, c) = a + b + c + 2abc - 2\sqrt{(ab + 1)(ac + 1)(bc + 1)}
\]

and the degree of a given Diophantine triple is the number of iterations of \( \partial \)-operators to arrive at an Euler triple (a triple with \( c = a + b + 2r \)). For example, when \( c = c_2^+ \) as in (1.2), the triple \( \{a, b, c\} = \{a, b, c_2^+\} \) has just degree \( \nu - 1 \). In particular, even though we remove the additional condition \( b \leq 8a \), the form \( \{a, b, c_2^+\} \) gives all Diophantine triples of degree 1.

The success here is due to the use of new congruences and a linear form in two logarithms. Moreover, the technique used for the proof of Theorem 1.4 can be used in the study of triples with \( \text{deg}(a, b, c) = 1 \). Not only in some special case like \( \{a, b\} = \{k - 1, k + 1\}, \{k, 4k \pm 4\}, \{A^2k + 2A, (A + 1)^2k + 2(A + 1)\} \), but also in general.

2. Preliminaries

Suppose that \( \{a, b, c, d\} \) is a Diophantine quadruple with \( a < b < c < d \). Then, there exist positive integers \( x, y, z \) such that \( ad + 1 = x^2, bd + 1 = y^2, cd + 1 = z^2 \). Eliminating \( d \) from these relations, we obtain

\[
\begin{align*}
ay^2 - bx^2 &= a - b, \\
az^2 - cx^2 &= a - c, \\
bz^2 - cy^2 &= b - c.
\end{align*}
\]
Assume that $a < b \leq 8a$. If $\gcd(a, b) = 1$, then [8, Lemma 4.1] implies that the positive solutions of the Diophantine equation (2.1) are given by
\[(2.4) \quad y\sqrt{a} + x\sqrt{b} = (\lambda \sqrt{a} + \sqrt{b})(r + \sqrt{ab})^l, \quad \lambda \in \{\pm 1\}, \quad l \geq 0, \quad (l \text{ odd}).\]

Thus, we may write
\[(2.5) \quad p_0 = 1, \quad p_1 = r + \lambda a, \quad p_{l+2} = 2rp_{l+1} - p_l, \]
\[(2.6) \quad V_0 = \lambda, \quad V_1 = b + \lambda r, \quad V_{l+2} = 2rV_{l+1} - V_l.\]

Moreover, by Lemma 1 in [6] the positive solutions of Diophantine equations (2.2) and (2.3) are respectively given by
\[(2.7) \quad z\sqrt{a} + x\sqrt{c} = (z_0 \sqrt{a} + x_0 \sqrt{c})(s + \sqrt{ac})^m, \quad m \geq 0,\]
\[(2.8) \quad z\sqrt{b} + y\sqrt{c} = (z_1 \sqrt{b} + y_1 \sqrt{c})(t + \sqrt{bc})^n, \quad n \geq 0,\]
where $m, n$ are non-negative integers, and $(z_0, x_0), (z_1, y_1)$ are fundamental solutions of (2.2), (2.3), respectively. We have $z = v_m = w_n$, where
\[(2.9) \quad v_0 = z_0, \quad v_1 = s_0z_0 + cx_0, \quad v_{m+2} = 2sv_{m+1} - v_m, \]
\[(2.10) \quad w_0 = z_1, \quad w_1 = tz_1 + cy_1, \quad w_{n+2} = 2tw_{n+1} - w_n.\]

We may also write $x = q_m$, $y = W_n$, where
\[(2.11) \quad q_0 = x_0, \quad q_1 = s_0x_0 + az_0, \quad q_{m+2} = 2sq_{m+1} - q_m, \]
\[(2.12) \quad W_0 = y_1, \quad W_1 = ty_1 + bz_1, \quad W_{n+2} = 2tw_{n+1} - W_n.\]

In our case,
\[a = k, \quad b = 4k + 4s, \quad c = c^+_2 = 144k^3 + 240s^2 + 124k + 20s,\]
\[r = 2k + s, \quad s = 12k^2 + 10sk + 1, \quad t = 24k^2 + 32sk + 9.\]

We have some special relations in our case.

**Lemma 2.1.** If $(a, b, c) = (k, 4k + 4s, c_2^+)$, then $s \equiv t \equiv -1$ (mod $2r$) and $c \equiv 0$ (mod $4r$).

**Proof.** The results directly come from
\[s+1 = 2(2k+s)(3k+s) = 2r(3k+s), \quad t+1 = 2(2k+s)(6k+5s) = 2r(6k+5s),\]
and
\[c = 4(2k+s)(3k+s)(6k+5s) = 4r(3k+s)(6k+5s).\]

The following result is just Lemma 3.1 of [9].

**Lemma 2.2** ([9, Lemma 3.1(4)]). If $(a, b, c) = (k, 4k + 4s, c_2^+)$, then $v_{2m+1} \neq w_{2n}$ and $v_{2m} \neq w_{2n+1}$. Moreover, there are two types of fundamental solution to equation (2.2) and (2.3):

1. If $v_{2m} = w_{2n}$, then $z_0 = z_1 = \lambda_1 \in \{\pm 1\}$.
2. If $v_{2m+1} = w_{2n+1}$, then $z_0 = \lambda_2 t$ and $z_1 = \lambda_2 s$ with $\lambda_2 \in \{\pm 1\}$. 

We prove the following results.

**Lemma 2.3.** We have \( \lambda = 1 \). Moreover,

1. If \( v_{2m} = w_{2n} \), then \( l \) is even.
2. If \( v_{2m+1} = w_{2n+1} \), then \( l \) is odd.

**Proof.** By Lemma 2.2, when \( v_{2m} = w_{2n} \), then \( |z_1| = 1 \) implies \( y_1 = 1 \). When \( v_{2m+1} = w_{2n+1} \), the fact \( |z_1| = s \) provides \( y_1 = r \). From (2.12) and \( t \equiv 1 \pmod{b} \), we have

\[
(W_n \mod b)_{n \geq 0} = \begin{cases} 
(1,1,1,\ldots), & \text{if } v_{2m} = w_{2n}, \\
(r,r,r,\ldots), & \text{if } v_{2m+1} = w_{2n+1}.
\end{cases}
\]

On the other hand, from (2.6), we have

\[
(V_l \mod b)_{l \geq 0} = (\lambda,\lambda r,\lambda,\lambda r,\ldots).
\]

Since \( y = V_l = W_n \), consider the two cases. Therefore, the lemma is proved. \( \square \)

**Lemma 2.4.** We have

1. If \( v_{2m} = w_{2n} \), then \( 2m \equiv 2n \equiv 0 \pmod{r} \) or \( m \equiv -4n \equiv -2\varepsilon\lambda_1 \pmod{r} \).
2. If \( v_{2m+1} = w_{2n+1} \), then \( 2m + 1 \equiv 2n + 1 \equiv \pm 1 \pmod{r} \).

**Proof.** In our proof, we will use the congruences \( s \equiv t \equiv -1 \pmod{r} \) and \( c \equiv 0 \pmod{4r} \) (cf. Lemma 2.1).

**Case (1).** We have \( v_{2m} = w_{2n} \). From (2.4), we have

\[
y\sqrt{a} + x\sqrt{b} = (\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2l} \\
\equiv (\sqrt{a} + \sqrt{b})(2r^2 - 1 + 2r\sqrt{ab})^l \\
\equiv \pm(\sqrt{a} + \sqrt{b}) \pmod{2r}.
\]

Thus, by (2.5) we deduce

\[
x = p_{2l} \equiv \pm 1 \pmod{2r}.
\]

From (2.7) and Lemma 2.1, we obtain

\[
z\sqrt{a} + x\sqrt{c} = (\lambda_1\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^{2m} \\
\equiv (\lambda_1\sqrt{a} + \sqrt{c})(2ac + 1 + 2s\sqrt{ac})^m \\
\equiv (\lambda_1\sqrt{a} + \sqrt{c})(1 - 2\sqrt{ac})^m \\
\equiv (\lambda_1\sqrt{a} + \sqrt{c})(1 - 2m\sqrt{ac}) \\
\equiv \lambda_1\sqrt{a} + (1 - 2\lambda_1am)\sqrt{c} \pmod{2r}.
\]

Thus, from (2.11) we get

\[
x = q_{2m} \equiv 1 - 2\lambda_1am \pmod{2r}.
\]
Using (2.14) and (2.15), we have $\pm 1 \equiv 1 - 2\lambda_1 am \pmod{2r}$. This implies $2\lambda_1 am \equiv 0, 2 \pmod{r}$. Since $a = k$, $r = 2k + \varepsilon$, then $2a \equiv -\varepsilon \pmod{r}$. Thus, we have

$$m \equiv 0, -2\varepsilon\lambda_1 \pmod{r}.$$  

(2.16)

Similarly, from (2.13) we have

$$y = V_{2l} \equiv \pm 1 \pmod{2r}.$$  

(2.17)

Equation (2.8) and Lemma 2.1 imply

$$z\sqrt{b} + y\sqrt{c} = (\lambda_1 \sqrt{b} + \sqrt{c})(t + \sqrt{bc})^{2n} \equiv (\lambda_1 \sqrt{b} + \sqrt{c})(2bc + 1 + 2t\sqrt{bc})^{n} \equiv (\lambda_1 \sqrt{b} + \sqrt{c})(1 - 2\sqrt{bc})^{n} \equiv (\lambda_1 \sqrt{b} + \sqrt{c})(1 - 2n\sqrt{bc}) \equiv \lambda_1 \sqrt{b} + (1 - 2\lambda_1 bn)\sqrt{c} \pmod{2r}.$$  

Thus, we get

$$y = W_{2n} \equiv 1 - 2\lambda_1 bn \pmod{2r}.$$  

(2.18)

From (2.17) and (2.18), we have $\pm 1 \equiv 1 - 2\lambda_1 bn \pmod{2r}$. It follows that $\lambda_1 bn \equiv 0, 1 \pmod{r}$. By $b = 4k + 4\varepsilon$, $r = 2k + \varepsilon$, we have $b \equiv 2\varepsilon \pmod{2r}$.

$$2n \equiv 0, \varepsilon\lambda_1 \pmod{r}.$$  

(2.19)

Combining (2.16) and (2.19), the first part of the lemma is proved.

**Case (2).** Now, we consider $v_{2m+1} = w_{2n+1}$. It has been shown by Lemma 2.3 that $l$ is odd. From (2.4), we have

$$y\sqrt{a} + x\sqrt{b} = (\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2l+1} \equiv (\sqrt{a} + \sqrt{b})(\sqrt{ab})^{2l+1} \equiv (-1)^l(\sqrt{a} + \sqrt{b})\sqrt{ab} \equiv (-1)^l b\sqrt{a} + (-1)^l a\sqrt{b} \pmod{r}.$$  

(2.20)

Thus, we see that

$$x = p_{2l+1} \equiv (-1)^l a \pmod{r}.$$  

(2.21)

From (2.7) and Lemma 2.1, we have

$$z\sqrt{a} + x\sqrt{c} = (\lambda_2 t\sqrt{a} + r\sqrt{c})(s + \sqrt{ac})^{2m+1} \equiv -\lambda_2 \sqrt{a}(-1 + \sqrt{ac})^{2m+1} \equiv -\lambda_2 \sqrt{a}(-1 + (2m + 1)\sqrt{ac}) \equiv \lambda_2 \sqrt{a} - \lambda_2 (2m + 1)a\sqrt{c} \pmod{2r}.$$  

(2.22)
Thus, we have
\[ x = q_{2m+1} \equiv -\lambda_2(2m+1)a \quad (\text{mod } r). \]
Using (2.21) and (2.22), we deduce that \((2m+1)a \equiv (-1)^{l+1}\lambda_2a \quad (\text{mod } r).\)
Since \(\gcd(a,r) = 1\), thus we get
\[ 2m + 1 \equiv (-1)^{l+1}\lambda_2 \quad (\text{mod } r). \]

Similarly, from (2.13) we have
\[ y = V_{2l+1} \equiv (-1)^l b \quad (\text{mod } r). \]
We see that equation (2.8) and Lemma 2.1 imply
\[ z\sqrt{b} + y\sqrt{c} = (\lambda_2s\sqrt{b} + r\sqrt{c})(t + \sqrt{bc})^{2n+1} \]
\[ \equiv -\lambda_2\sqrt{b}(-1 + \sqrt{bc})^{2n+1} \]
\[ \equiv -\lambda_2\sqrt{b}(-1 + (2n+1)\sqrt{bc}) \]
\[ \equiv \lambda_2\sqrt{b} - \lambda_2(2n+1)b\sqrt{c} \quad (\text{mod } r). \]
Thus, we have
\[ y = W_{2n+1} \equiv -\lambda_2(2n+1)b \quad (\text{mod } r). \]
From (2.24) and (2.25), we have \((-1)^l b \equiv -\lambda_2(2n + 1)b \quad (\text{mod } r).\) Since \(\gcd(b,r) = 1\), then
\[ 2n + 1 \equiv (-1)^{l+1}\lambda_2 \quad (\text{mod } r). \]
Therefore, from (2.23) and (2.26) we have
\[ 2m + 1 \equiv 2n + 1 \equiv (-1)^{l+1}\lambda_2 \equiv \pm 1 \quad (\text{mod } r). \]
This completes the proof of Lemma 2.4. \(\square\)

The following computational result can help us to have information about “very small” cases.

**Lemma 2.5** (cf. [9, Lemma 1.3(2)]). *Suppose that \(\{a,b,c,d\}\) is a Diophantine quadruple with \(a < b < c < d_+ < d\). If \(2a \leq b \leq 8a\), then \(b > 1.3 \cdot 10^5\).* Therefore, in order to proof our main theorem, we assume that \(k \geq 32499\).

**3. Proof of Theorem 1.4 for large \(k\)**

In this section, our goal is proof Theorem 1.4 for \(k \geq 7.84 \cdot 10^6\). Let us denote
\[ \alpha_1 = s + \sqrt{ac}, \quad \alpha_3 = \frac{\sqrt{b}(\sqrt{c} + \lambda_1\sqrt{a})}{\sqrt{a}(\sqrt{c} + \lambda_1\sqrt{b})}, \]
\[ \alpha_2 = t + \sqrt{bc}, \quad \alpha_4 = \frac{\sqrt{b}(r\sqrt{c} + \lambda_2\sqrt{a})}{\sqrt{a}(r\sqrt{c} + \lambda_2s\sqrt{b})}. \]
By formula (60) of [6], if $v_{m'} = w_{n'}$ has a solution with $m', n' > 0$, then we have

$$0 < m' \log \alpha_1 - n' \log \alpha_2 + \log \alpha_{3,4} < \frac{8}{3} a c \alpha_1^{-2m'}.$$

Define

$$\Lambda_1 = 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3, \quad \text{for } v_{2m} = w_{2n}$$

$$\Lambda_2 = (2m + 1) \log \alpha_1 - (2n + 1) \log \alpha_2 + \log \alpha_4, \quad \text{for } v_{2m+1} = w_{2n+1}.$$

Then, we have

$$0 < \Lambda_1 < \frac{8}{3} a c \alpha_1^{-4m} \quad \text{and} \quad 0 < \Lambda_2 < \frac{8}{3} a c \alpha_1^{-4m-2}.$$

We will transform the forms $\Lambda_{1,2}$ into linear forms in two logarithms in order to apply the following result due to Laurent that we recall. See Corollary 1 in [12]. For any non-zero algebraic number $\gamma$ of degree $D$ over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $A \prod_{j=1}^{D} (X - \gamma(j))$, we denote by

$$h(\gamma) = \frac{1}{D} \left( \log A + \sum_{j=1}^{D} \log \max \left( 1, |\gamma(j)| \right) \right)$$

its absolute logarithmic height.

**Lemma 3.1.** Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $\gamma_1 > 1$, $\gamma_2 > 1$, $\log \gamma_1$, $\log \gamma_2$ are real and positive, $b_1$ and $b_2$ are positive integers and

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$. Let

$$h_i \geq \max \left\{ h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D} \right\} \quad \text{for} \quad i = 1, 2$$

and

$$b' \geq \frac{|b_1|}{D h_2} + \frac{|b_2|}{D h_1}.$$

Then

$$\log |\Lambda| \geq -17.9 \cdot D^4 \left( \max \left\{ \log b' + 0.38, \frac{30}{D}, \frac{1}{2} \right\} \right)^2 h_1 h_2.$$

**Remark 3.2.** One can also use Theorem 2 of [12] to get a better result than the use of the above lemma. However, we still need to run a program of the Baker–Davenport reduction method. So we just choose this lemma.

We will consider two cases: $v_{2m} = w_{2n}$ and $v_{2m+1} = w_{2n+1}$. 
Even case, i.e. \( v_{2m} = w_{2n} \). By Lemma 2.4(1), if \( v_{2m} = w_{2n} \) has a solution, then \( 2m \equiv 2n \equiv 0 \, (\text{mod} \, r) \) or \( m \equiv -4n \equiv -2\varepsilon \lambda_1 \, (\text{mod} \, r) \). So we set

\[
2m = m_1 r - 4\mu_1 \quad \text{and} \quad 2n = n_1 r + \mu_1,
\]

with some positive integers \( m_1, n_1 \) and \( \mu_1 \in \{0, \pm 1\} \). Then, we rewrite \( \Lambda_1 \) into the form

\[
\Lambda_1 = (m_1 r - 4\mu_1) \log \alpha_1 - (n_1 r + \mu_1) \log \alpha_2 + \log \alpha_3
\]

\[
(3.2)
\]

In order to apply Lemma 3.1, we set

\[
D = 4, \quad b_1 = 1, \quad b_2 = r, \quad \gamma_1 = \frac{(\alpha_1^4 \alpha_2)^{\mu_1}}{\alpha_3}, \quad \gamma_2 = \frac{\alpha_1^{m_1}}{\alpha_2^{m_2}}.
\]

The multiplicative independence of \( \gamma_1 \) and \( \gamma_2 \) is easy to check, so we omit it. To ensure that \( \log \gamma_1 \) and \( \log \gamma_2 \) are positive, if \( \log \gamma_1 < 0 \) and \( \log \gamma_2 < 0 \), we use \( 1/\gamma_1, 1/\gamma_2 \) instead of \( \gamma_1, \gamma_2 \), respectively. Then, we work on \( -\Lambda_1 \) and exchange the indexes. Or, if one of \( \log \gamma_i \) \((i = 1, 2)\) is negative and the other is positive, then we have a contradiction to

\[
4 < 5 \log \alpha_1 - 1 < |\log(\alpha_1^4 \alpha_2) - |\log \alpha_3||
\]

\[
\leq |\log \gamma_1| < |\Lambda_1| < \frac{8}{3} a c \alpha_1^{-4 m} \leq \frac{1}{6 a c},
\]

for \( \mu_1 = \pm 1 \) or

\[
\frac{1}{4} < \left( 1 - \sqrt{\frac{a}{b}} \right) \cdot \frac{\sqrt{c}}{\sqrt{c} + \sqrt{a}} = \frac{\sqrt{bc} - \sqrt{ac}}{\sqrt{bc} + \sqrt{ab}} < \log \left( 1 + \frac{\sqrt{bc} - \sqrt{ac}}{\sqrt{ac} + \sqrt{ab}} \right)
\]

\[
= \log \frac{\sqrt{b}(\sqrt{c} + \sqrt{a})}{\sqrt{a}(\sqrt{c} + \sqrt{b})} \leq |\log \alpha_3| = |\log \gamma_1| < |\Lambda_1| < \frac{1}{6 a c},
\]

for \( \mu_1 = 0 \), where we used \( |\log \alpha_3| < 1 \) and \( \log(1 + x) > \frac{x}{1 + x} \) for \( x > -1 \).

We have \( h(\alpha_1) = \frac{1}{2} \log \alpha_1, \ h(\alpha_2) = \frac{1}{2} \log \alpha_2 \). Since the absolute values of the conjugates of \( \alpha_3 \) greater than one are

\[
\frac{\sqrt{b}(\sqrt{c} + \sqrt{a})}{\sqrt{a}(\sqrt{c} + \sqrt{b})}, \quad \frac{\sqrt{b}(\sqrt{c} + \sqrt{a})}{\sqrt{a}(\sqrt{c} - \sqrt{b})}, \quad \frac{\sqrt{b}(\sqrt{c} - \sqrt{a})}{\sqrt{a}(\sqrt{c} + \sqrt{b})}, \quad \frac{\sqrt{b}(\sqrt{c} - \sqrt{a})}{\sqrt{a}(\sqrt{c} - \sqrt{b})},
\]

then

\[
h(\alpha_3) \leq \frac{1}{4} \log \left( (ac - ab) \cdot \frac{b^2}{a^2} \cdot \frac{(c - a)^2}{(c - b)^2} \right) < \frac{1}{2} \log(bc) < \log \alpha_2.
\]
It follows that
\[(3.3) \quad h(\gamma_1) \leq 4h(\alpha_1) + h(\alpha_2) + h(\alpha_3) < 2 \log \alpha_1 + \frac{1}{2} \log \alpha_2 + \log \alpha_2 < 3.5 \log \alpha_2.\]

Moreover, we have

\[|\log_\gamma_1| \leq 4 \log \alpha_1 + \log \alpha_2 + |\log \alpha_3| < 5 \log \alpha_2 + 1.\]

Put \(T_{m_1} + K_{m_1} \sqrt{ac} := \alpha_1^{m_1}\), \(P_{n_1} + Q_{n_1} \sqrt{bc} := \alpha_2^{n_1}\). One can check that the leading coefficient of the irreducible polynomial of \(\frac{\alpha_1^{m_1}}{\alpha_2^{n_1}}\) is 1. If \(\alpha_1^{m_1} > \alpha_2^{n_1}\), then the absolute values of conjugates of \(\frac{\alpha_1^{m_1}}{\alpha_2^{n_1}}\) greater than one are

\[\frac{T_{m_1} + K_{m_1} \sqrt{ac}}{P_{n_1} + Q_{n_1} \sqrt{bc}}, \quad \frac{T_{m_1} + K_{m_1} \sqrt{ac}}{P_{n_1} - Q_{n_1} \sqrt{bc}}.\]

We deduce that \(h(\gamma_2) = \frac{m_1}{2} \log \alpha_1\). Similarly, if \(\alpha_1^{m_1} < \alpha_2^{n_1}\), then \(h(\gamma_2) = \frac{n_1}{2} \log \alpha_2\). By Lemma 2.5, we have \(r > 6.49 \cdot 10^4\). We use (3.1) and (3.2) to get

\[|\log \gamma_2| = \left|\frac{m_1}{2} \log \alpha_1 - \frac{n_1}{2} \log \alpha_2\right| < \frac{1}{2r} \left(\left|\log \gamma_1\right| + \frac{8}{3} \alpha c\alpha_1^{-4}\right) < \frac{1}{2r} \left(5 \log \alpha_2 + 1 + 0.001\right) < 0.001.\]

So we have

\[(3.4) \quad h(\gamma_2) < \frac{m_1}{2} \log \alpha_1 + 0.001.\]

We set

\[h_1 = 3.5 \log \alpha_2, \quad h_2 = \frac{m_1}{2} \log \alpha_1 + 0.001\]

and

\[\frac{b_1}{4h_1} + \frac{b_2}{4h_2} = \frac{r}{14 \log \alpha_2} + \frac{1}{2m_1 \log \alpha_1 + 0.004} < \frac{r}{14 \log \alpha_2} + 0.03 =: b'.\]

We have

\[b' > \frac{r}{14 \log \alpha_2} > \frac{2k - 1}{14 \log(48k^2 + 64k + 18)} > 188.\]

Applying Lemma 3.1, it results

\[\log |A_1| \geq -17.9 \cdot 4^4 (\log b' + 0.38)^2 h_1 h_2.\]

This and \(|A_1| < \frac{8}{3} \alpha c\alpha_1^{-4m}\) give

\[4m \log \alpha_1 < 17.9 \cdot 4^4 (\log b' + 0.38)^2 h_1 h_2 + \log \left(\frac{8}{3} \alpha c\right).\]

Then, we get

\[m < 17.9 \cdot 4^3 (\log b' + 0.38)^2 (3.5 \log \alpha_2) \left(\frac{m_1}{2} + 0.001\right) + 0.5.\]
As \(2m = m_1r - 4\mu_1 \geq m_1r - 4\), we have
\[
0.998r < 17.9 \cdot 4^3 (\log b' + 0.38)^2 (3.5 \log \alpha_2) + 5
\]
and so
\[
b' - 0.03 = \frac{r}{14 \log \alpha_2} < 286.974 (\log b' + 0.38)^2 + \frac{5.011}{14 \log \alpha_2}.
\]
We simplify it to have
\[
(3.5) \quad b' < 286.974 (\log b' + 0.38)^2 + 0.05.
\]
By a straightforward computation, we get \(b' < 33461.2\). Therefore, we get the inequality
\[
r < 468456.4 \log \alpha_2.
\]
Recall that \(r = 2k + \varepsilon\) and \(\alpha_2 = t + \sqrt{bc} < 2t = 2(24k^2 + 32\varepsilon k + 9)\), we have
\[
2k - 1 < 468456.4 \log(48k^2 + 64k + 18).
\]
This gives \(k < 8.38 \cdot 10^6\).

**Odd case, i.e.** \(v_{2m+1} = w_{2n+1}\). Also, from Lemma 2.4(2), if \(v_{2m+1} = w_{2n+1}\), then \(2m + 1 \equiv 2n + 1 \equiv \pm 1 \pmod{r}\). Let \(2m + 1 = m_2r + \mu_2, 2n + 1 = n_2r + \mu_2\), for some nonnegative integers \(m_2, n_2\) and \(\mu_2 \in \{\pm 1\}\). We have
\[
\Lambda_2 = (m_2r + \mu_2) \log \alpha_1 - (n_2r + \mu_2) \log \alpha_2 + \log \alpha_4
\]
\[
(3.6) \quad = \log \left( \alpha_4 \left( \frac{\alpha_1}{\alpha_2} \right)^{\mu_2} \right) - r \log \left( \frac{\alpha_2^{n_2}}{\alpha_1^{m_2}} \right).
\]
We set (by replacing \(\gamma_1\) and \(\gamma_2\) by their reciprocals, if necessary)
\[
D = 4, \quad b_1 = r, \quad b_2 = 1, \quad \gamma_1 = \frac{\alpha_2^{n_2}}{\alpha_1^{m_2}}, \quad \gamma_2 = \alpha_4 \left( \frac{\alpha_1}{\alpha_2} \right)^{\mu_2}.
\]
Similarly to the proof in the even case,
\[
(3.7) \quad h(\gamma_1) < \frac{m_2}{2} \log \alpha_1 + 0.001.
\]
Since the absolute values of conjugates of \(\alpha_4\) greater than one are
\[
\frac{\sqrt{b}(r\sqrt{c} + t\sqrt{a})}{\sqrt{a}(r\sqrt{c} + s\sqrt{b})}, \quad \frac{\sqrt{b}(r\sqrt{c} + t\sqrt{a})}{\sqrt{a}(r\sqrt{c} + s\sqrt{b})}, \quad \frac{\sqrt{b}(r\sqrt{c} - t\sqrt{a})}{\sqrt{a}(r\sqrt{c} - s\sqrt{b})},
\]
then
\[
h(\alpha_4) \leq \frac{1}{4} \log \left( \frac{a^2(c - b)^2 \cdot b^{3/2} \cdot c - a}{c - b} \cdot \frac{r\sqrt{c} + t\sqrt{a}}{r\sqrt{c} - s\sqrt{b}} \right)
\]
\[
< \frac{1}{4} \log \left( 4a^{1/2}b^{3/2}c^2r^2 \right) < \frac{3}{2} \log \alpha_2.
\]
So we get

\[ h(\gamma_2) \leq h(\alpha_1) + h(\alpha_2) + h(\alpha_4) \leq 2.5 \log \alpha_2. \]

One can see that the values of \( h(\gamma_i) \) are not exceeding those in the even case. Hence, after applying Lemma 3.1, we get that the upper bound of \( k \) is not exceeding \( 8.38 \cdot 10^6 \). We summarize it here.

**Proposition 3.3.** If \( \{k, 4k + 4\varepsilon, c_2^+, d\} \) is a Diophantine quadruple with \( c_2^+ < d \), then \( d = c_3^+ \) for \( k \geq 8.38 \cdot 10^6 \).

4. Final Computation

In order to deal with the remaining cases \( 32499 \leq k < 8.38 \cdot 10^6 \), we will use a Diophantine approximation algorithm called the Baker–Davenport reduction method. The following lemma is a slight modification of the original version of the Baker–Davenport reduction method (see [7, Lemma 5a]).

**Lemma 4.1.** Assume that \( M \) is a positive integer. Let \( p/q \) be the convergent of the continued fraction expansion of a real number \( \kappa \) such that \( q > 6M \) and let

\[ \eta = \|\mu q\| - M \cdot \|\kappa q\|, \]

where \( \| \cdot \| \) denotes the distance from the nearest integer. If \( \eta > 0 \), then the inequality

\[ 0 < J\kappa - K + \mu < AB^{-J} \]

has no solutions in integers \( J \) and \( K \) with

\[ \frac{\log (Aq/\eta)}{\log B} \leq J \leq M. \]

To apply the above lemma, we use

\[ \Lambda = m' \log \alpha_1 - n' \log \alpha_2 + \log \alpha_{3,4} \]

with

\[ \Lambda = \Lambda_1 = 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3, \quad \text{for} \quad v_{2m} = w_{2n}, \]

\[ \Lambda = \Lambda_2 = (2m + 1) \log \alpha_1 - (2n + 1) \log \alpha_2 + \log \alpha_4, \quad \text{for} \quad v_{2m+1} = w_{2n+1}. \]

We set

\[ J = m', \quad K = n', \quad \kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_{3,4}}{\log \alpha_2}. \]

Since \( 0 < \Lambda < \frac{8ac\alpha_1^{-2m'}}{3} \), then we take

\[ A = \frac{8ac/3}{\log \alpha_2}, \quad B = \alpha_1^2. \]
Before running the program, we need to determine the value of $M$. This is an absolute upper bound of $m'$. From formula (40) of [5], we have

$$\frac{m'}{\log m'} < 2.867 \cdot 10^{15} \log^2 c.$$ 

As $c \leq 144k^3 + 240k^2 + 124k + 20$ and $k < 8.38 \cdot 10^6$, we have $m' < 4 \cdot 10^{20} =: M$. We ran a GP program in 8 hours to check no more than $8 \cdot 8.38 \cdot 10^6$ cases. We obtained $m' \leq 2$. Thus we have

**Proposition 4.2.** If $\{k, 4k + 4\varepsilon, c_2^+, d\}$ is a Diophantine quadruple with $c_2^+ < d$, then $d = c_3^+$ for $k \leq 8.38 \cdot 10^6$.

Combining Proposition 3.3 and Proposition 4.2, we complete the proof of Theorem 1.4.

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