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Résumé. Pour tous entiers \( k \geq 2 \) et \( n \neq 0 \), soit \( \nu_k(n) \) le plus grand entier positif \( e \) tel que \( k^e \) divise \( n \). De plus, soit \( (u_n)_{n \geq 0} \) une suite de Lucas non dégénérée telle que \( u_0 = 0 \), \( u_1 = 1 \) et \( u_{n+2} = au_{n+1} + bu_n \), pour certains entiers \( a \) et \( b \). Shu et Yao ont montré que, pour tout nombre premier \( p \), la suite \( \nu_p(u_{n+1})_{n \geq 0} \) est \( p \)-régulière. Medina et Rowland ont déterminé le rang de \( \nu_p(F_{n+1})_{n \geq 0} \), où \( F_n \) est le \( n \)-ième nombre de Fibonacci.

Nous montrons que si \( k \) et \( b \) sont premiers entre eux, alors \( \nu_k(u_{n+1})_{n \geq 0} \) est une suite \( k \)-régulière. Si de plus \( k \) est un nombre premier, nous déterminons aussi le rang de cette suite. En outre, nous donnons des formules explicites pour \( \nu_k(u_n) \), généralisant un théorème précédent de Sanna concernant les valuations \( p \)-adiques des suites de Lucas.

Abstract. For integers \( k \geq 2 \) and \( n \neq 0 \), let \( \nu_k(n) \) denote the greatest nonnegative integer \( e \) such that \( k^e \) divides \( n \). Moreover, let \( (u_n)_{n \geq 0} \) be a nondegenerate Lucas sequence satisfying \( u_0 = 0 \), \( u_1 = 1 \), and \( u_{n+2} = au_{n+1} + bu_n \), for some integers \( a \) and \( b \). Shu and Yao showed that for any prime number \( p \) the sequence \( \nu_p(u_{n+1})_{n \geq 0} \) is \( p \)-regular, while Medina and Rowland found the rank of \( \nu_p(F_{n+1})_{n \geq 0} \), where \( F_n \) is the \( n \)-th Fibonacci number.

We prove that if \( k \) and \( b \) are relatively prime then \( \nu_k(u_{n+1})_{n \geq 0} \) is a \( k \)-regular sequence, and for \( k \) a prime number we also determine its rank. Furthermore, as an intermediate result, we give explicit formulas for \( \nu_k(u_n) \), generalizing a previous theorem of Sanna concerning \( p \)-adic valuations of Lucas sequences.

1. Introduction

For integers \( k \geq 2 \) and \( n \neq 0 \), let \( \nu_k(n) \) denote the greatest nonnegative integer \( e \) such that \( k^e \) divides \( n \). In particular, if \( k = p \) is a prime number then \( \nu_p(\cdot) \) is the usual \( p \)-adic valuation. We shall refer to \( \nu_k(\cdot) \) as the \( k \)-adic valuation, although, strictly speaking, for composite \( k \) this is not
a “valuation” in the algebraic sense of the term, since it is not true that \( \nu_k(mn) = \nu_k(m) + \nu_k(n) \) for all integers \( m, n \neq 0 \).

Valuations of sequences with combinatorial meanings have been studied by several authors (see, e.g., [4, 6, 7, 8, 9, 10, 12, 14, 15, 18]). To this end, an important role is played by the family of \( k \)-regular sequences, which were first introduced and studied by Allouche and Shallit [1, 2, 3] with the aim of generalizing the concept of automatic sequences.

Given a sequence of integers \( s(n)_{n \geq 0} \), its \( k \)-kernel is defined as the set of subsequences

\[
\ker_k(s(n)_{n \geq 0}) := \{ s(k^e n + i)_{n \geq 0} : e \geq 0, 0 \leq i < k^e \}.
\]

Then \( s(n)_{n \geq 0} \) is said to be \( k \)-regular if the \( \mathbb{Z} \)-module \( \langle \ker_k(s(n)_{n \geq 0}) \rangle \) generated by its \( k \)-kernel is finitely generated. In such a case, the rank of \( s(n)_{n \geq 0} \) is the rank of this \( \mathbb{Z} \)-module.

Allouche and Shallit provided many examples of regular sequences. In particular, they showed that the sequence of \( p \)-adic valuations of factorials \( \nu_p(n!)_{n \geq 0} \) is \( p \)-regular [1, Example 9], and that the sequence of 3-adic valuations of sums of central binomial coefficients

\[
\nu_3\left( \sum_{i=0}^{n} \binom{2i}{i} \right)_{n \geq 0}
\]

is 3-regular [1, Example 23]. Furthermore, for any polynomial \( f(x) \in \mathbb{Q}[x] \) with no roots in the natural numbers, Bell [5] proved that the sequence \( \nu_p(f(n))_{n \geq 0} \) is \( p \)-regular if and only if \( f(x) \) factors as a product of linear polynomials in \( \mathbb{Q}[x] \) times a polynomial with no root in the \( p \)-adic integers.

Fix two integers \( a \) and \( b \), and let \( (u_n)_{n \geq 0} \) be the Lucas sequence of characteristic polynomial \( f(x) = x^2 - ax - b \), i.e., \( (u_n)_{n \geq 0} \) is the integral sequence satisfying \( u_0 = 0 \), \( u_1 = 1 \), and \( u_{n+2} = au_{n+1} + bu_n \), for each integer \( n \geq 0 \). Assume also that \( (u_n)_{n \geq 0} \) is nondegenerate, i.e., \( b \neq 0 \) and the ratio \( \alpha/\beta \) of the two roots \( \alpha, \beta \in \mathbb{C} \) of \( f(x) \) is not a root of unity.

Using \( p \)-adic analysis, Shu and Yao [16, Corollary 1] proved the following result.

**Theorem 1.1.** For each prime number \( p \), the sequence \( \nu_p(u_{n+1})_{n \geq 0} \) is \( p \)-regular.

In the special case \( a = b = 1 \), i.e., when \( (u_n)_{n \geq 0} \) is the sequence of Fibonacci numbers \( (F_n)_{n \geq 0} \), Medina and Rowland [11] gave an algebraic proof of Theorem 1.1 and also determined the rank of \( \nu_p(F_{n+1})_{n \geq 0} \). Their result is the following.

**Theorem 1.2.** For each prime number \( p \) the sequence \( \nu_p(F_{n+1})_{n \geq 0} \) is \( p \)-regular. Precisely, for \( p \neq 2, 5 \) the rank of \( \nu_p(F_{n+1})_{n \geq 0} \) is \( \alpha(p) + 1 \), where \( \alpha(p) \) is the least positive integer such that \( p \mid F_{\alpha(p)} \), while for \( p = 2 \) the rank is 5, and for \( p = 5 \) the rank is 2.
In this paper, we extend Theorem 1.1 to $k$-adic valuations with $k$ relatively prime to $b$; and we generalize Theorem 1.2 to nondegenerate Lucas sequences. Let $\Delta := a^2 + 4b$ be the discriminant of $f(x)$. Also, for each positive integer $m$ relatively prime to $b$ let $\tau(m)$ denote the rank of appariation of $m$ in $(u_n)_{n \geq 0}$, i.e., the least positive integer $n$ such that $m \mid u_n$ (which is well-defined, see, e.g., [13]).

Our first two results are the following.

**Theorem 1.3.** If $k \geq 2$ is an integer relatively prime to $b$, then the sequence $\nu_k(u_{n+1})_{n \geq 0}$ is $k$-regular.

**Theorem 1.4.** Let $p$ be a prime number not dividing $b$, and let $r$ be the rank of $\nu_p(u_{n+1})_{n \geq 0}$.

- If $p \mid \Delta$ then:
  - $r = 2$ if $p \in \{2, 3\}$ and $\nu_p(u_p) = 1$, or if $p \geq 5$;
  - $r = 3$ if $p \in \{2, 3\}$ and $\nu_p(u_p) \neq 1$.

- If $p \nmid \Delta$ then:
  - $r = 5$ if $p = 2$ and $\nu_2(u_6) \neq \nu_2(u_3) + 1$;
  - $r = \tau(p) + 1$ if $p > 2$, or if $p = 2$ and $\nu_2(u_6) = \nu_2(u_3) + 1$.

Note that Theorem 1.2 follows easily from our Theorem 1.4, since in the case of Fibonacci numbers $b = 1$, $\Delta = 5$, $\nu_2(F_3) = 1$, $\nu_2(F_6) = 3$, and $\tau(p) = \alpha(p)$.

As a preliminary step in the proof of Theorem 1.3, we obtain some formulas for the $k$-adic valuation $\nu_k(u_n)$, which generalize a previous result of the second author. Precisely, Sanna [15] proved the following formulas for the $p$-adic valuation of $u_n$.

**Theorem 1.5.** If $p$ is a prime number such that $p \nmid b$, then

$$\nu_p(u_n) = \begin{cases} \nu_p(n) + \varrho_p(n) & \text{if } \tau(p) \mid n, \\ 0 & \text{if } \tau(p) \nmid n, \end{cases}$$

for each positive integer $n$, where

$$\varrho_2(n) := \begin{cases} \nu_2(u_3) & \text{if } 2 \nmid \Delta, 2 \nmid n, \\ \nu_2(u_6) - 1 & \text{if } 2 \nmid \Delta, 2 \mid n, \\ \nu_2(u_2) - 1 & \text{if } 2 \mid \Delta, \end{cases}$$

and

$$\varrho_p(n) = \varrho_p := \begin{cases} \nu_p(u_{\tau(p)}) & \text{if } p \nmid \Delta, \\ \nu_3(u_3) - 1 & \text{if } p \mid \Delta, p = 3, \\ 0 & \text{if } p \mid \Delta, p \geq 5, \end{cases}$$

for $p \geq 3$. 
Actually, Sanna’s result [15, Theorem 1.5] is slightly different but it quickly turns out to be equivalent to Theorem 1.5 using [15, Lemma 2.1(v), Lemma 3.1, and Lemma 3.2]. Furthermore, in Sanna’s paper it is assumed $\gcd(a, b) = 1$, but the proof of [15, Theorem 1.5] works exactly in the same way also for $\gcd(a, b) \neq 1$.

From now on, let $k = p_1^{a_1} \cdots p_h^{a_h}$ be the prime factorization of $k$, where $p_1 < \cdots < p_h$ are prime numbers and $a_1, \ldots, a_h$ are positive integers.

We prove the following generalization of Theorem 1.5.

**Theorem 1.6.** If $k \geq 2$ is an integer relatively prime to $b$, then

$$\nu_k(u_n) = \begin{cases} \nu_k(c_k(n)n) & \text{if } \tau(p_1 \cdots p_h) \mid n, \\ 0 & \text{if } \tau(p_1 \cdots p_h) \nmid n, \end{cases}$$

for any positive integer $n$, where

$$c_k(n) := \prod_{i=1}^{h} p_i^{\rho_p(n)}.$$

Note that Theorem 1.6 is indeed a generalization of Theorem 1.5. In fact, if $k = p$ is a prime number then obviously

$$\nu_p(c_p(n)n) = \nu_p(p^{\rho_p(n)}n) = \nu_p(n) + \rho_p(n),$$

for each positive integer $n$.

2. Preliminaries

In this section we collect some preliminary facts needed to prove the results of this paper. We begin with some lemmas on $k$-regular sequences.

**Lemma 2.1.** If $s(n)_{n \geq 0}$ and $t(n)_{n \geq 0}$ are two $k$-regular sequences, then $(s(n) + t(n))_{n \geq 0}$ and $s(n)t(n)_{n \geq 0}$ are $k$-regular too. Precisely, if $A$ is a finite set of generators of $\langle \ker_k(s(n)_{n \geq 0}) \rangle$ and $B$ is a finite set of generators of $\langle \ker_k(t(n)_{n \geq 0}) \rangle$, then $A \cup B$ is a set of generators of $\langle \ker_k((s(n)+t(n))_{n \geq 0}) \rangle$.

**Proof.** See [1, Theorem 2.5].

**Lemma 2.2.** If $s(n)_{n \geq 0}$ is a $k$-regular sequence, then for any integers $c \geq 1$ and $d \geq 0$ the subsequence $s(cn + d)_{n \geq 0}$ is $k$-regular.

**Proof.** See [1, Theorem 2.6].

**Lemma 2.3.** Any periodic sequence is $k$-regular.

**Proof.** An ultimately periodic sequence is $k$-automatic for all $k \geq 2$, see [2, Theorem 5.4.2]. A $k$-automatic sequence is $k$-regular, see [1, Theorem 1.2].

The following lemma is essentially [1, Theorem 2.2(d) and remark (i) just below].
Lemma 2.4. Let \( s(n)_{n \geq 0} \) be a sequence of integers. If there exist some
\[
(2.1) \quad s_1 = s, s_2, \ldots, s_r \in \langle \ker_k(s(n)_{n \geq 0}) \rangle
\]
such that the sequences \( s_j(kn + i)_{n \geq 0} \), with \( 0 \leq i < k \) and \( 1 \leq j \leq r \),
are \( \mathbb{Z} \)-linear combinations of \( s_1, \ldots, s_r \), then \( s(n)_{n \geq 0} \) is \( k \)-regular and
\( \langle \ker_k(s(n)_{n \geq 0}) \rangle \) is generated by \( s_1, \ldots, s_r \).

Proof. It is sufficient to prove that \( s(k^e n + i)_{n \geq 0} \in \langle s_1, \ldots, s_r \rangle \) for all integers \( e \geq 0 \) and \( 0 \leq i < k^e \). In fact, this claim implies that \( \langle \ker_k(s(n)_{n \geq 0}) \rangle \subseteq (s_1, \ldots, s_r) \), while by (2.1) we have \( (s_1, \ldots, s_r) \subseteq \langle \ker_k(s(n)_{n \geq 0}) \rangle \), hence
\( \langle \ker_k(s(n)_{n \geq 0}) \rangle = (s_1, \ldots, s_r) \) and so \( s(n)_{n \geq 0} \) is \( k \)-regular. We proceed by induction on \( e \). For \( e = 0 \) the claim is obvious since \( s = s_1 \). Suppose \( e \geq 1 \) and that the claim holds for \( e - 1 \). We have \( i = k^{e-1}(j + i') \), for some integers \( 0 \leq j < k \) and \( 0 \leq i' < k^{e-1} \). Therefore, by the induction hypothesis,
\[
\begin{align*}
s(k^e n + i)_{n \geq 0} &= s(k^{e-1}(kn + j) + i')_{n \geq 0} \\
&\in \langle s_1(kn + j)_{n \geq 0}, \ldots, s_r(kn + j)_{n \geq 0} \rangle \\
&\subseteq \langle s_1, \ldots, s_r \rangle,
\end{align*}
\]
and the claim follows. \( \square \)

The next lemma is well-known; we give the proof just for completeness.

Lemma 2.5. The sequence \( \nu_k(n+1)_{n \geq 0} \) is \( k \)-regular of rank 2. Indeed, \( \langle \ker_k(\nu_k(n+1)_{n \geq 0}) \rangle \) is generated by \( \nu_k(n+1)_{n \geq 0} \) and the constant sequence \( (1)_{n \geq 0} \).

Proof. For all nonnegative integers \( n \) and \( i < k \) we have
\[
\nu_k(kn + i + 1) = \begin{cases} 
1 + \nu_k(n + 1) & \text{if } i = k - 1, \\
0 & \text{if } i < k - 1.
\end{cases}
\]
Therefore, putting \( s_1 = \nu_k(n+1)_{n \geq 0} \) and \( s_2 = (1 + \nu_k(n+1))_{n \geq 0} \) in Lemma 2.4, we obtain that \( \langle \ker_k(\nu_k(n+1)_{n \geq 0}) \rangle \) is generated by \( \nu_k(n+1)_{n \geq 0} \) and \( (1 + \nu_k(n+1))_{n \geq 0} \), hence it is also generated by \( \nu_k(n+1)_{n \geq 0} \) and \( (1)_{n \geq 0} \), which are obviously linearly independent. Thus \( \nu_k(n+1)_{n \geq 0} \) is \( k \)-regular of rank 2. \( \square \)

Now we state a lemma that relates the \( k \)-adic valuation of an integer with its \( p_i \)-adic valuations. The proof is quite straightforward and we leave it to the reader.

Lemma 2.6. We have
\[
\nu_k(m) = \min_{i=1,\ldots,h} \left\lfloor \frac{\nu_{p_i}(m)}{a_i} \right\rfloor,
\]
for any integer \( m \geq 2 \).
We conclude this section with two lemmas on the rank of apparition \( \tau(n) \).

**Lemma 2.7.** For each prime number \( p \) not dividing \( b \),

\[
\tau(p) \mid p - (-1)^{p-1} \left( \frac{\Delta}{p} \right),
\]

where \( \left( \frac{\cdot}{p} \right) \) denotes the Legendre symbol. In particular, if \( p \mid \Delta \) then \( \tau(p) = p \).

**Proof.** The case \( p = 2 \) is easy. For \( p > 2 \) see [17, Lemma 1]. \( \square \)

**Lemma 2.8.** If \( m \) and \( n \) are two positive integers relatively prime to \( b \), then

\[
\tau(\text{lcm}(m,n)) = \text{lcm}(\tau(m),\tau(n)).
\]

**Proof.** See [13, Theorem 1(a)]. \( \square \)

3. Proof of Theorem 1.6

Thanks to Lemma 2.6, we know that

\[
(3.1) \quad \nu_k(u_n) = \min_{i=1,\ldots,h} \left\lfloor \frac{v_{p_i}(u_n)}{a_i} \right\rfloor.
\]

Moreover, from Lemma 2.8 it follows that

\[
\tau(p_1 \cdots p_h) = \text{lcm}\{\tau(p_1),\ldots,\tau(p_h)\}.
\]

Therefore, on the one hand, if \( \tau(p_1 \cdots p_h) \nmid n \) then \( \tau(p_i) \nmid n \) for some \( i \in \{1,\ldots,h\} \), so that by Theorem 1.5 we have \( v_{p_i}(u_n) = 0 \), which together with (3.1) implies \( v_k(u_n) = 0 \), as claimed.

On the other hand, if \( \tau(p_1 \cdots p_h) \mid n \) then \( \tau(p_i) \mid n \) for \( i = 1,\ldots,h \). Hence, from (3.1), Theorem 1.5, and Lemma 2.6, we obtain

\[
\nu_k(u_n) = \min_{i=1,\ldots,h} \left\lfloor \frac{v_{p_i}(n) + \varrho_{p_i}(n)}{a_i} \right\rfloor = \min_{i=1,\ldots,h} \left\lfloor \frac{v_{p_i}(c_k(n)n)}{a_i} \right\rfloor = v_k(c_k(n)n),
\]

so that the proof is complete.

4. Proof of Theorem 1.3

Clearly, if \( \Delta \) and \( k \) are fixed, then \( c_k(n) \) depends only on the parity of \( n \). Thus it follows easily from Theorem 1.6 that

\[
(4.1) \quad \nu_k(u_{n+1}) = \nu_k(c_k(1)(n + 1)) s(n) + \nu_k(c_k(2)(n + 1)) t(n),
\]

for each integer \( n \geq 0 \), where the sequences \( s(n)_{n \geq 0} \) and \( t(n)_{n \geq 0} \) are defined by

\[
s(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \mid n + 1, 2 \nmid n + 1, \\ 0 & \text{otherwise}, \end{cases}
\]

and

\[
t(n) := \begin{cases} 1 & \text{if } \tau(p_1 \cdots p_2) \nmid n + 1, 2 \mid n + 1, \\ 0 & \text{otherwise}. \end{cases}
\]
On the one hand, by Lemma 2.5 and Lemma 2.2, we know that both \( \nu_k(c_k(1)(n + 1))_{n \geq 0} \) and \( \nu_k(c_k(2)(n + 1))_{n \geq 0} \) are \( k \)-regular sequences. On the other hand, by Lemma 2.3, also the sequences \( s(n)_{n \geq 0} \) and \( t(n)_{n \geq 0} \) are \( k \)-regular, since obviously they are periodic.

In conclusion, using (4.1) and Lemma 2.1, we obtain that \( \nu_k(u_{n+1})_{n \geq 0} \) is a \( k \)-regular sequence.

### 5. Proof of Theorem 1.4

We generalize Medina and Rowland’s proof of Theorem 1.2. First, suppose that \( p \mid \Delta \). By Lemma 2.7 we have \( \tau(p) = p \). Moreover, it is clear that \( \varrho_p(n) = \varrho_p \) does not depend on \( n \). As a consequence, from Theorem 1.5 it follows easily that

\[
(5.1) \quad \nu_p(u_{n+1}) = \nu_p(n + 1) + s(n),
\]

for any integer \( n \geq 0 \), where the sequence \( s(n)_{n \geq 0} \) is defined by

\[
s(n) := \begin{cases} 
\varrho_p & \text{if } n + 1 \equiv 0 \mod p, \\
0 & \text{if } n + 1 \not\equiv 0 \mod p.
\end{cases}
\]

On the one hand, if \( p \in \{2,3\} \) and \( \nu_p(u_p) = 1 \), or if \( p \geq 5 \), then \( \varrho_p = 0 \). Thus \( s(n)_{n \geq 0} \) is identically zero and it follows by (5.1) and Lemma 2.5 that \( r = 2 \). On the other hand, if \( p \in \{2,3\} \) and \( \nu_p(u_p) \neq 1 \), then \( \varrho_p \neq 0 \). Moreover, for \( i = 0, \ldots, p - 1 \) we have

\[
s(pn + i) = \begin{cases} 
\varrho_p & \text{if } i = p - 1, \\
0 & \text{if } i \neq p - 1,
\end{cases}
\]

hence from Lemma 2.4 it follows that \( s(n)_{n \geq 0} \) is \( p \)-regular and that the module \( \langle \ker_p(s(n)_{n \geq 0}) \rangle \) is generated by \( s(n)_{n \geq 0} \) and \( (\varrho_p)_{n \geq 0} \). Therefore, by (5.1), Lemma 2.5, and Lemma 2.1, we obtain that \( \nu_p(u_{n+1})_{n \geq 0} \) is a \( p \)-regular sequence and that \( \langle \ker_p(\nu_p(u_{n+1})_{n \geq 0}) \rangle \) is generated by \( \nu_p(n + 1)_{n \geq 0} \), \( s(n)_{n \geq 0} \), and \( (1)_{n \geq 0} \), which are clearly linearly independent, hence \( r = 3 \).

Now suppose \( p \nmid \Delta \). By Lemma 2.7, we know that \( p \equiv \varepsilon \mod \tau(p) \), for some \( \varepsilon \in \{-1, +1\} \). Furthermore, if \( p = 2 \) then it follows easily that \( \tau(2) = 3 \). As a consequence, from Theorem 1.5 we obtain that

\[
(5.2) \quad \nu_p(u_{n+1}) = s(n) + t(n),
\]

for any integer \( n \geq 0 \), where the sequences \( s(n)_{n \geq 0} \) and \( t(n)_{n \geq 0} \) are defined by

\[
s(n) := \begin{cases} 
\nu_p(n + 1) + v & \text{if } n + 1 \equiv 0 \mod \tau(p) \\
0 & \text{if } n + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]

and

\[
t(n) := \begin{cases} 
1 & \text{if } \tau(p_1 \cdots p_2) \mid n + 1, \ 2 \mid n + 1, \\
0 & \text{otherwise}.
\end{cases}
\]
with \( v := \nu_\nu(u_\tau(p)) \), and
\[
t(n) := \begin{cases} 
\nu_2(u_6) - \nu_2(u_3) - 1 & \text{if } p = 2, \ n + 1 \equiv 0 \mod 6, \\
0 & \text{otherwise}.
\end{cases}
\]

We shall show that \( s(n)_{n \geq 0} \) is a \( p \)-regular sequence of rank \( \tau(p) + 1 \). Let us define the sequences \( s_j(n)_{n \geq 0} \), for \( j = 0, \ldots, \tau(p) - 1 \), by
\[
s_j(n) := \begin{cases} 
1 & \text{if } n + j + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } n + j + 1 \not\equiv 0 \mod \tau(p).
\end{cases}
\]

On the one hand, for \( i = 0, \ldots, p - 2 \) we have
\[
s(pn + i) = \begin{cases} 
\nu_p(pn + i + 1) + v & \text{if } pn + i + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } pn + i + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= \begin{cases} 
v & \text{if } \varepsilon n + i + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } \varepsilon n + i + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= \begin{cases} 
v & \text{if } n + (\varepsilon(i + 1) - 1) + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } n + (\varepsilon(i + 1) - 1) + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= v \cdot s(\varepsilon(i+1)-1) \mod \tau(p)(n),
\]

since \( p \nmid i + 1 \) and consequently \( \nu_p(pn + i + 1) = 0 \).

On the other hand,
\[
(5.3) \quad s(pn + p - 1) = \begin{cases} 
\nu_p(pn + p) + v & \text{if } p(n + 1) \equiv 0 \mod \tau(p), \\
0 & \text{if } p(n + 1) \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= \begin{cases} 
\nu_p(n + 1) + v + 1 & \text{if } n + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } n + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= s(n) + s_0(n),
\]

since \( \nu_p(pn + p) = \nu_p(n + 1) + 1 \) and \( \gcd(p, \tau(p)) = 1 \).

Furthermore, for \( i = 0, \ldots, p - 1 \) and \( j = 0, \ldots, \tau(p) - 1 \),
\[
s_j(pn + i) = \begin{cases} 
1 & \text{if } pn + i + j + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } pn + i + j + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= \begin{cases} 
1 & \text{if } n + (\varepsilon(i + j + 1) - 1) + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } n + (\varepsilon(i + j + 1) - 1) + 1 \not\equiv 0 \mod \tau(p),
\end{cases}
\]
\[
= s(\varepsilon(i+j)+1)-1 \mod \tau(p)(n).
\]

Summarizing, the sequences \( s(pn + i)_{n \geq 0} \) and \( s_j(pn + i)_{n \geq 0} \), for \( 0 \leq i < p \) and \( 0 \leq j < \tau(p) \), are \( \mathbb{Z} \)-linear combinations of \( s(n)_{n \geq 0} \) and \( s_j(n)_{n \geq 0} \).
Moreover, for \( i = 0, \ldots, p^2 - 1 \) we have

\[
\begin{align*}
(5.4) \quad s_0(p^2n + i) &= \begin{cases} 
1 & \text{if } p^2n + i + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } p^2n + i + 1 \not\equiv 0 \mod \tau(p),
\end{cases} \\
&= \begin{cases} 
1 & \text{if } n + i + 1 \equiv 0 \mod \tau(p), \\
0 & \text{if } n + i + 1 \not\equiv 0 \mod \tau(p),
\end{cases} \\
&= s_{i \mod \tau(p)}(n),
\end{align*}
\]

hence, by (5.4) and (5.3), it follows that

\[
(5.5) \quad s_{i \mod \tau(p)}(n)_{n \geq 0} = s_0(p^2n + i)_{n \geq 0} \\
&= (p^3n + pi + p - 1)_{n \geq 0} - s(p^2n + i)_{n \geq 0} \\
&\in \langle \ker_p(s(n)_{n \geq 0}) \rangle.
\]

Since \( \tau(p) \mid p - \varepsilon \), we have

\[
\tau(p) \leq p - \varepsilon \leq p + 1 < p^2,
\]

hence by (5.5) we get that \( s_j(n)_{n \geq 0} \in \langle \ker_p(s(n)_{n \geq 0}) \rangle \), for \( 0 \leq j < \tau(p) \).

Therefore, in light of Lemma 2.4, we obtain that \( s(n)_{n \geq 0} \) is a \( p \)-regular sequence and that \( \langle \ker_p(s(n)_{n \geq 0}) \rangle \) is generated by \( s(n)_{n \geq 0} \) and \( s_j(n)_{n \geq 0} \), with \( j = 0, \ldots, \tau(p) - 1 \). It is straightforward to see that these last sequences are linearly independent, hence \( s(n)_{n \geq 0} \) has rank \( \tau(p) + 1 \).

If \( p > 2 \), or if \( p = 2 \) and \( \nu_2(u_6) = \nu_2(u_3) + 1 \), then \( t(n)_{n \geq 0} \) is identically zero, thus from (5.2) and the previous result on \( s(n) \) we find that \( r = \tau(p) + 1 \).

So it remains only to consider the case \( p = 2 \) and \( \nu_2(u_6) \neq \nu_2(u_3) + 1 \). Recall that in such a case \( \tau(2) = 3 \), and put \( d := \nu_2(u_6) - \nu_2(u_3) - 1 \). Obviously, the sequence \( t(2n)_{n \geq 0} \) is identically zero, while

\[
(5.6) \quad t(2n + 1) = \begin{cases} 
d & \text{if } 2n + 2 \equiv 0 \mod 6, \\
0 & \text{if } 2n + 2 \not\equiv 0 \mod 6,
\end{cases}
\]

Thus, again from Lemma 2.4, we have that \( t(n) \) is a 2-regular sequence and that \( \langle \ker_p(t(n)_{n \geq 0}) \rangle \) is generated by \( t(n)_{n \geq 0} \) and \( d \cdot s_j(n)_{n \geq 0} \), for \( j = 0, 1, 2 \).

In conclusion, by (5.2) and Lemma 2.1, we obtain that \( \nu_p(u_{n+1})_{n \geq 0} \) is a 2-regular sequence and that \( \langle \ker_p(\nu_p(u_{n+1})_{n \geq 0}) \rangle \) is generated by \( s(n) \), \( t(n) \), and \( s_j(n) \), for \( j = 0, 1, 2 \), which are linearly independent, hence \( r = 5 \). The proof is complete.
6. Concluding remarks

It might be interesting to understand if, actually, \( \nu_k(u_{n+1})_{n \geq 0} \) is \( k \)-regular for every integer \( k \geq 2 \), so that Theorem 1.3 holds even by dropping the assumption that \( k \) and \( b \) are relatively prime. A trivial observation is that if \( k \) and \( b \) have a common prime factor \( p \) such that \( p \nmid a \), then \( p \nmid u_n \) for all integers \( n \geq 1 \), and consequently \( \nu_k(u_{n+1})_{n \geq 0} \) is \( k \)-regular simple because it is identically zero. Thus the nontrivial case occurs when each of the prime factors of \( \gcd(b,k) \) divides \( a \).

Another natural question is if it is possible to generalize Theorem 1.4 in order to say something about the rank of \( \nu_k(u_{n+1})_{n \geq 0} \) when \( k \) is composite. Probably, the easier cases are those when \( k \) is squarefree, or when \( k \) is a power of a prime number.

We leave these as open questions to the reader.

References

On the $k$-regularity of the $k$-adic valuation of Lucas sequences

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