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par DOHOON CHOI, SUBONG LIM et WISSAM RAJI

1. Introduction

The Eichler cohomology theory is a cohomology theory for Fuchsian groups, introduced by Eichler [4]. This gives an isomorphism between the direct sum of two spaces of cusp forms of integral weights with the cohomology group associated with the polynomial space. Since then, the Eichler cohomology theory has been extensively studied [5, 6, 11, 13] and it has many applications in number theory in connection with periods and period functions (for example, see [1, 10, 12]).

For half-integral weights Knopp [8] defined a bigger space $P$, which is the space of holomorphic functions $g(z)$ on the upper half plane $\mathbb{H}$ satisfying
the following growth condition

$$|g(z)| < K(|z|^\rho + y^{-\sigma})$$

for some positive constants $K, \rho$ and $\sigma$, where $z = x + iy \in \mathbb{H}$. In [8] and [9] Knopp and Mawi proved that the space of cusp forms of a half-integral weight is isomorphic to the cohomology group associated with $\mathcal{P}$.

In this paper, we study more about Eichler cohomology and period functions associated with half-integral weight cusp forms. For this we use two important half-integral weight modular forms: the Dedekind eta function

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

and the theta function

$$\theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2},$$

where $q = e^{2\pi iz}$. We prove that $\eta$-multiplication gives an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group associated with $\eta \mathcal{P}$. It gives the following commutative diagram in which all arrows are isomorphisms

$$
\begin{array}{cccc}
S_{k,\chi \eta}(\text{SL}_2(\mathbb{Z})) & \longrightarrow & \check{H}^1_{2-k,\chi \eta}(\text{SL}_2(\mathbb{Z}), \mathcal{P}) \\
\times \eta \uparrow & & \downarrow \times \eta \\
S_{k-\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z})) & \longrightarrow & \check{H}^1_{\frac{3}{2}-k,\chi}(\text{SL}_2(\mathbb{Z}), \eta \mathcal{P}).
\end{array}
$$

We also prove the similar result for $\Gamma_0(4)$ using $\theta(z)$ instead of $\eta(z)$. Moreover, we show that there is an isomorphism between the direct sum of two spaces of cusp forms of half-integral weights and the cohomology group.

For an integer $k$ let $M_{k-\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z}))$ (resp. $S_{k-\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z}))$) be the space of holomorphic modular forms (resp. cusp forms) of weight $k - \frac{1}{2}$ with a multiplier system $\chi$ on $\text{SL}_2(\mathbb{Z})$. Let $\chi_\eta$ be the eta-multiplier system of weight $\frac{1}{2}$ on $\text{SL}_2(\mathbb{Z})$, i.e.,

$$\chi_\eta(\gamma) = \frac{\eta(\gamma z)}{\eta(z) (cz + d)^{1/2}}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. Precisely, the multiplier system $\chi_\eta$ is determined by (for example, see [7, Section 2.8])

$$\chi_\eta(\gamma) = \begin{cases}
    e(b/24) & \text{if } \gamma = \begin{pmatrix} 1 & b/2 \\ 0 & 1 \end{pmatrix}, \\
    e \left( \frac{a+d-3c}{24c} \right) - \frac{1}{2} s(d,c) & \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c > 0,
\end{cases}$$

where $e(x) = e^{2\pi ix}$. Here $s(d,c)$ denotes the Dedekind sum defined by

$$s(d,c) := \sum_{0 \leq n < c} \frac{n}{c} \psi \left( \frac{dn}{c} \right),$$
where $\psi(x) = x - \lfloor x \rfloor - \frac{1}{2}$.

It is known that the space $\mathcal{P}$ is preserved under the slash operator $|r, \psi \gamma$ for any real $r$, multiplier system $\psi$ of weight $r$ and $\gamma \in \text{SL}_2(\mathbb{Z})$ (for example, see [8, Section 4]). Using the $\text{SL}_2(\mathbb{Z})$-module $\mathcal{P}$, we consider another $\text{SL}_2(\mathbb{Z})$-module

\[ \eta \mathcal{P} = \{ \eta(f)(z) \mid f(z) \in \mathcal{P} \}. \]

In this case, $\text{SL}_2(\mathbb{Z})$ acts on $\eta \mathcal{P}$ by

\[ (\eta \mathcal{P})(f(z)) = \eta(f)(z) \]

for $\eta \in \text{SL}_2(\mathbb{Z})$, where $\psi$ is a multiplier system of weight $2 - k$. Let $\hat{H}^1_{\frac{1}{2} - k, \psi \chi} \mathcal{P}$ be the parabolic Eichler cohomology group of weight $\frac{1}{2} - k$ with a multiplier system $\psi \chi$ on $\text{SL}_2(\mathbb{Z})$ associated with the module $\eta \mathcal{P}$ (for the precise definition see Section 3). Then we have an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group.

**Theorem 1.1.** Let $k$ be an integer and $\chi$ be a multiplier system of weight $k - \frac{1}{2}$ on $\text{SL}_2(\mathbb{Z})$. Then the space $S_{k - \frac{1}{2}, \chi}(\text{SL}_2(\mathbb{Z}))$ is isomorphic to the cohomology group $\hat{H}^1_{\frac{1}{2} - k, \chi} \mathcal{P}$.

Note that if $\chi$ is a multiplier system then $\chi \overline{\chi}$ is a trivial multiplier system so that it can be a multiplier system of any even integral weight. Let $k$ be an integer with $k \geq 2$. For a multiplier system $\chi$ of weight $k - \frac{1}{2}$ we let $\chi^2 = \chi \overline{\chi}$. Note that $\overline{\chi}$ is a multiplier system of weight $k - \frac{1}{2}$ on $\text{SL}_2(\mathbb{Z})$ because $\chi \overline{\chi}$ is a trivial multiplier system and it can be a multiplier system of weight $2k$. Let $P_{k - 2}$ be the space of polynomials of $z$ of degree $\leq k - 2$. Note that it is a $\text{SL}_2(\mathbb{Z})$-module for which the action of $\text{SL}_2(\mathbb{Z})$ on $P_{k - 2}$ is given by the slash operator of weight $2 - k$

\[ (f|_{2-k, \psi \gamma})(z) := (cz + d)^{k - 2} \psi(\gamma f(\gamma z)) \]

for $\gamma = \frac{(a, b)}{c, d} \in \text{SL}_2(\mathbb{Z})$, where $\psi$ is a multiplier system of weight $2 - k$ on $\text{SL}_2(\mathbb{Z})$. Then $\eta P_{k - 2}$ is a $\text{SL}_2(\mathbb{Z})$-module by the slash operator $| \frac{5}{2} - k, \psi \chi |$ as in (1.2). Let $\hat{H}^1_{\frac{1}{2}, \chi} \mathcal{P}$ be the parabolic Eichler cohomology group of weight $\frac{5}{2} - k$ with a multiplier system $\psi \chi$ on $\text{SL}_2(\mathbb{Z})$ associated with the module $\eta P_{k - 2}$ (for the precise definition see Section 3). Then we have an isomorphism between the direct sum of two spaces of cusp forms of half-integral weights and the cohomology group.

**Theorem 1.2.** Let $k$ be an integer with $k \geq 2$ and $\chi$ be a multiplier system of weight $k - \frac{1}{2}$ on $\text{SL}_2(\mathbb{Z})$. Then the space $S_{k - \frac{1}{2}, \chi}(\text{SL}_2(\mathbb{Z})) \oplus S_{k - \frac{1}{2}, \chi}(\text{SL}_2(\mathbb{Z}))$ is isomorphic to $\hat{H}^1_{\frac{1}{2}, \chi} \mathcal{P}$.
On the other hand, if we use the theta function $\theta(z)$, we can obtain similar results for $\Gamma_0(4)$. Let $\chi_\theta$ be the theta-multiplier system of weight $\frac{1}{2}$ on $\Gamma_0(4)$, i.e.,

$$\chi_\theta(\gamma) = \frac{\theta(\gamma z)}{\theta(z)(cz + d)^{\frac{1}{2}}}$$

for $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(4)$. Note that $\chi_\theta$ is given by $\chi_\theta(\gamma) = \left( \frac{c}{d} \right) \bar{\epsilon}_d$ for $\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(4)$ (for example, see [7, Section 2.8]), where $(\frac{c}{d})$ denotes the extended quadratic residue symbol and

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \text{ (mod 4)}, \\ i & \text{if } d \equiv -1 \text{ (mod 4)}. \end{cases}$$

Then we have the following result, which is analogous to Theorem 1.1.

**Theorem 1.3.** Let $k$ be an integer and $\chi = \chi_\theta^{2k-1}$ be a multiplier system of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$. Then the space $S_{k-\frac{1}{2}}(\chi(\Gamma_0(4)))$ is isomorphic to $\tilde{H}^{1}_{\frac{1}{2}-k,\chi}(\Gamma_0(4), \theta P)$.

If we let $\chi^*_\theta := \chi\chi_\theta^2$ for a multiplier system $\chi$ of weight $k - \frac{1}{2}$, then we see that $\chi^*_\theta$ is also a multiplier system of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$. As in the case of $\text{SL}_2(\mathbb{Z})$ we obtain the following theorem.

**Theorem 1.4.** Let $k$ be an integer with $k \geq 2$ and $\chi = \chi_\theta^{2k-1}$ be a multiplier system of weight $k - \frac{1}{2}$ on $\Gamma_0(4)$. Then the space $S_{k-\frac{1}{2}}(\chi(\Gamma_0(4))) \oplus S_{k-\frac{1}{2}}(\chi^*_\theta(\Gamma_0(4)))$ is isomorphic to $\tilde{H}^{1}_{\frac{1}{2}-k,\chi^*_\theta}(\Gamma_0(4), \theta P_{k-2})$.

**Remark.** Knopp and Mawi also proved that there is an isomorphism between the space of cusp forms of a half-integral weight and the cohomology group. More precisely, in [9] Knopp and Mawi proved that

$$S_{k-\frac{1}{2}}(\Gamma) \cong \tilde{H}^{1}_{\frac{1}{2}-k}(\Gamma, \mathcal{P}),$$

where $\Gamma$ is a subgroup of $\text{SL}_2(\mathbb{Z})$, which is a Fuchsian group of the first kind with at least one parabolic class. If we combine this result with Theorem 1.1 and 1.3, then in the cases of $\text{SL}_2(\mathbb{Z})$ and $\Gamma_0(4)$ we obtain

$$\tilde{H}^{1}_{\frac{1}{2}-k}(\text{SL}_2(\mathbb{Z}), \mathcal{P}) \cong \tilde{H}^{1}_{\frac{1}{2}-k}(\text{SL}_2(\mathbb{Z}), \eta\mathcal{P})$$

and

$$\tilde{H}^{1}_{\frac{1}{2}-k}(\Gamma_0(4), \mathcal{P}) \cong \tilde{H}^{1}_{\frac{1}{2}-k}(\Gamma_0(4), \theta\mathcal{P}).$$

Note that $\mathcal{P}$ is the set of holomorphic functions which have polynomial growth at the boundary of $\mathbb{H}$. But elements of $\eta\mathcal{P}$ and $\theta\mathcal{P}$ should vanish at some boundary points because $\eta(z)$ and $\theta(z)$ decrease exponentially at some cusps. This implies that $\eta\mathcal{P}$ and $\theta\mathcal{P}$ are strictly smaller than $\mathcal{P}$.
Nevertheless, they give the cohomology groups isomorphic to the cohomology group associated with $\mathcal{P}$. It can be useful to analyze because we have smaller modules.

Now we look at some examples of period functions of a half-integral weight, which lies in the module of Theorem 1.2 or 1.4.

**Example.** (1) Suppose that $k$ is an integer with $k \geq 2$. Let $f(z)$ be a cusp form in $S_{k-\frac{1}{2}} \chi(\text{SL}_2(\mathbb{Z}))$ with the Fourier expansion

$$
\sum_{m=0}^{\infty} a(m)e^{2\pi i (m+\kappa)z},
$$

where $\kappa$ is a real number in $[0, 1)$ such that $\chi(T) = e^{2\pi i \kappa}$ for $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We consider the period function of $f(z)$ with respect to the $\text{SL}_2(\mathbb{Z})$-module $\eta P_{k-2}$

$$
F(z) := \int_0^{i\infty} f(\tau)\eta(\tau)\eta(z)(\tau - z)^{k-2}d\tau.
$$

If we compute this integral, then we have

$$
F(z) = \eta(z) \sum_{n=0}^{k-2} (-1)^n z^n \eta^{k-n-1} \int_0^{i\infty} \sum_{n=0}^{\infty} c(n)e^{-2\pi(n+\kappa')v}v^{k-2-n}dv,
$$

where $c(n) = \sum_{m=0}^{n} a(m)b(n-m)$ and $b(n)$ is defined by

$$
b(n) := \left| \left\{ l \in \mathbb{Z} \mid \frac{3l^2 - l}{2} = n \text{ and } l \equiv 0 \pmod{2} \right\} \right| - \left| \left\{ l \in \mathbb{Z} \mid \frac{3l^2 - l}{2} = n \text{ and } l \equiv 1 \pmod{2} \right\} \right|,
$$

and $\kappa' = \kappa + \frac{1}{24}$. If we continue to compute $F(z)$, then we have

$$
F(z) = \eta(z) \sum_{n=0}^{k-2} (-1)^n \left( \frac{i}{2\pi} \right)^{k-1-n} \Gamma(k-n-1)L(f\eta, k-n-1) z^n,
$$

where $L(f\eta, s) = \sum_{m=0}^{\infty} \frac{c(m)}{(m+\kappa')^s}$ is the $L$-function associated with the cusp form $f(z)\eta(z)$. Therefore, $F(z)$ is a product of $\eta(z)$ and a polynomial of degree $k - 2$. If $f(z)\eta(z)$ is a newform, then this polynomial satisfies a good property, i.e., a ratio of coefficients of even (resp. odd) degrees are rational over the algebraic number field $\mathbb{Q}(c(0), \ldots, c(n), \ldots)$ by the Periods Theorem [12].

(2) Suppose that $k$ is an integer with $k \geq 2$. Let

$$
f(z) = \sum_{m=1}^{\infty} a(m)e^{2\pi i mz} \in S_{k-\frac{1}{2}} \chi(\Gamma_0(4)),
$$
where $\chi = \chi^{2k-1}_\theta$. We consider the period function of $f(z)$ with respect to the $\Gamma_0(4)$-module $\theta P_{k-2}$

$$G(z) := \int_0^{i\infty} f(\tau)\theta(\tau)\theta(z)(\tau - z)^{k-2}d\tau.$$  

Then this integral can be divided into two parts

$$G(z) = G_1(z) + G_2(z),$$

where

$$G_1(z) := \int_0^{i\infty} f(\tau)\theta(z)(\tau - z)^{k-2}d\tau$$

and

$$G_2(z) := \int_0^{i\infty} f(\tau)\left(2 \sum_{m=1}^{\infty} e^{2\pi im^2\tau}\right)\theta(z)(\tau - z)^{k-2}d\tau.$$

After some computations we see that

$$G_1(z) = \theta(z) \sum_{n=0}^{k-2} (-1)^n \left(\frac{i}{2\pi}\right)^{k-n-1} \Gamma(k - n - 1) L(f, k - n - 1) z^n$$

and

$$G_2(z) = 2\theta(z) \sum_{n=0}^{k-2} (-1)^n \left(\frac{i}{2\pi}\right)^{k-n-1} \Gamma(k - n - 1) L(f', k - n - 1) z^n.$$

Here, the function $f'(z)$ is defined by

$$f'(z) = \sum_{m=1}^{\infty} b(m)e^{2\pi imz},$$

where $b(m) = \sum_{t < \sqrt{m}} a(m - t^2)$. This type of Fourier coefficients appeared in the papers [2] and [14].

The remainder of this paper is organized as follows. In Section 2 we recall the definition of modular forms and in Section 3 we introduce the Eichler cohomology group. Finally, in Section 4 we prove the main results: Theorem 1.1, 1.2, 1.3 and 1.4.

2. Modular forms

In this section we review the definition of modular forms and cusp forms. Let $\Gamma$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$. Let $k \in \frac{1}{2}\mathbb{Z}$ and $\chi$ be a (unitary) multiplier system in weight $k$ on $\Gamma$. Thus $\chi(\gamma)$ is a complex number independent of $z$ such that

(1) $|\chi(\gamma)| = 1$ for all $\gamma \in \Gamma$, 

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Fourier expansion at (2.2)

for all Definition. Suppose the above notations, we review the following definitions. Here, \( \lambda \) fixes other than phic function \( c \) cusp lies on the real axis. Let \( \Gamma \) generates in \( \Gamma \). Each of these classes corresponds to a cyclic subgroup of parabolic elements leaving fixed a parabolic cusp on the boundary of \( \mathbb{H} \). Such a parabolic cusp lies on the real axis. Let \( q_1, \ldots, q_t \) be the inequivalent parabolic cusps other than \( i \infty \) on the boundary of \( \mathbb{H} \) and let \( \Gamma_j \) be the cyclic subgroup of \( \Gamma \) fixing \( q_j \), \( 1 \leq j \leq t \). Let \( A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in SL_2(\mathbb{Z}) \) such that \( A_j(q_j) = \infty \). Then \( \lambda_j > 0 \) is chosen so that

\[
Q_j := A_j^{-1} \begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix} A_j
\]
generates \( \Gamma_j \). For \( 1 \leq j \leq t \), put \( \chi(Q_j) = e^{2\pi i \kappa_j}, \ 0 \leq \kappa_j < 1 \). If a holomorphic function \( f(z) \) satisfies \( (f|_{k,\chi}Q_j)(z) = f(z) \), then \( f(z) \) has the following Fourier expansion at \( q_j \)

\[
(c_j z + d_j)^{-k} f(A_j^{-1} z) = \sum_{n=-\infty}^{\infty} a_j(n) e^{2\pi i (n+\kappa_j) z}/\lambda_j.
\]

Here, \( \lambda_j \) is a positive real number called the width of the cusp \( q_j \). Using the above notations, we review the following definitions.

**Definition.** Suppose \( f(z) \) is holomorphic in \( \mathbb{H} \) and satisfies the functional equation

\[
(f|_{k,\chi}\gamma)(z) = f(z)
\]

for all \( \gamma \in \Gamma \).

(1) If \( f(z) \) has only terms with \( n + \kappa \geq 0 \) in (2.1) and \( n + \kappa_j \geq 0, \ 1 \leq j \leq t \), in (2.2), then \( f(z) \) is called a holomorphic modular form. The set of holomorphic modular forms is denoted by \( M_{k,\chi}(\Gamma) \).
(2) If \( f(z) \in M_{k,\chi}(\Gamma) \) and has only terms with \( n + \kappa > 0 \), \( n + \kappa_j > 0 \) in the expansions (2.1), (2.2), respectively, then \( f(z) \) is called a cusp form. The collection of cusp forms in \( M_{k,\chi}(\Gamma) \) is denoted by \( S_{k,\chi}(\Gamma) \).

3. The Eichler cohomology

In this section, we define the Eichler cohomology group. Let \( k \in \frac{1}{2} \mathbb{Z} \) and \( \Gamma \) be a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Let \( \chi \) be a multiplier system of weight \( k \) on \( \Gamma \). If \( \mathcal{M} \) is a vector space of functions on \( \mathbb{H} \) and is preserved under the slash operator \( |_{k,\chi} \), we can form the Eichler cohomology group associated with \( \mathcal{M} \) as follows.

A collection \( \{ p_\gamma(z) \mid \gamma \in \Gamma \} \) of elements of \( \mathcal{M} \) is called a cocycle if it satisfies the following cocycle condition
\[
(3.1) \quad p_{\gamma_1 \gamma_2}(z) = p_{\gamma_2}(z) + (p_{\gamma_1}|_{k,\chi \gamma_2})(z)
\]
for \( \gamma_1, \gamma_2 \in \Gamma \). Then a coboundary is a collection \( \{ p_\gamma(z) \mid \gamma \in \Gamma \} \) such that
\[
p_\gamma(z) = (p|_{k,\chi \gamma})(z) - p(z)
\]
for \( \gamma \in \Gamma \) with \( p(z) \) a fixed element of \( \mathcal{M} \). Furthermore, a parabolic cocycle \( \{ p_\gamma(z) \mid \gamma \in \Gamma \} \) is a collection of elements of \( \mathcal{M} \) satisfying (3.1), in which for every parabolic element \( B \) in \( \Gamma \) there exists a fixed element \( Q_B(z) \in \mathcal{M} \) such that
\[
p_B(z) = (Q_B|_{k,\chi \Gamma})(z) - Q_B(z).
\]
Note that coboundaries are parabolic cocycles. The parabolic Eichler cohomology group \( \tilde{H}_{k,\chi}^1(\Gamma, \mathcal{M}) \) is defined as the vector space obtained by forming the quotient of the parabolic cocycles by the coboundaries.

4. Proof of Theorem 1.1, 1.2, 1.3 and 1.4

In this section we prove the main results of this paper. To prove Theorem 1.1 and 1.2 we need the following lemma.

**Lemma 4.1.** Let \( k \) be an integer with \( k \geq 2 \) and \( \chi \) be a multiplier system of weight \( k + \frac{1}{2} \) on \( \text{SL}_2(\mathbb{Z}) \). Then we have
\[
S_{k-\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z})) \cong S_{k,\chi \eta}(\text{SL}_2(\mathbb{Z})�)
\]

**Proof of Lemma 4.1.** First we define a map
\[
\phi_\eta : S_{k-\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z})) \to S_{k,\chi \eta}(\text{SL}_2(\mathbb{Z}))
\]
by \( \phi_\eta(f)(z) = f(z)\eta(z) \) for \( f(z) \in S_{k-\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z})) \). One can see that this map is injective. For surjectivity we note that
\[
\eta(z) = q^{\frac{1}{24}} + \ldots
\]
and \( \eta(z) \) has no zero in \( \mathbb{H} \).
Fourier expansion

\[ \chi \text{ and cusp form in } \Gamma \text{ according to the multiplier system} \]

then the form of Fourier expansion of \( \chi \) is as follows. For \( \Gamma \) with parameter \( i \in \mathbb{R} \mod 12\mathbb{Z} \) since it is the order of the abelianization of \( \text{SL}_2(\mathbb{Z}) \), where \( \chi_i = (\chi_\eta)^{2i} \). It is also known that the multiplier system is suitable for weight \( k \equiv i \mod 2 \) (for example, see [3, Section 2]). Therefore, we have that if \( g(z) \in S_k, \chi_\eta(\text{SL}_2(\mathbb{Z})) \) then \( \chi \chi_\eta \)

should be \( \chi_i \) for \( i \equiv k \mod 2 \). If \( g(z) \) has a Fourier expansion of the form

\[
g(z) = \sum_{n+\kappa>0} a(n)e^{2\pi i(n+\kappa)z},
\]

then the form of Fourier expansion of \( g(z) \) can be obtained by the Table 4.1 according to the multiplier system \( \chi_i \). It comes from the fact that \( \kappa = \chi_i(T) \) and \( \chi_i(T) = e^{2\pi i \frac{1}{12}} \) (for example, see [7, Section 2.8]). Therefore, \( \frac{g(z)}{\eta(z)} \) is a cusp form in \( S_{k-\frac{1}{2}}, \chi_\eta(\text{SL}_2(\mathbb{Z})) \) and hence the map \( \phi_\eta \) is surjective. \( \square \)

**Table 4.1**

<table>
<thead>
<tr>
<th>Multiplier system</th>
<th>Fourier expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_0 )</td>
<td>( a(1)q^1 + a(2)q^{\frac{1}{2}} + a(3)q^2 + \cdots )</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>( a(0)q^{\frac{1}{12}} + a(1)q^{1+\frac{1}{12}} + a(2)q^{2+\frac{1}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>( a(0)q^{\frac{1}{2}} + a(1)q^{1+\frac{1}{2}} + a(2)q^{2+\frac{1}{2}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>( a(0)q^{\frac{3}{12}} + a(1)q^{1+\frac{3}{12}} + a(2)q^{2+\frac{3}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td>( a(0)q^{\frac{4}{12}} + a(1)q^{1+\frac{4}{12}} + a(2)q^{2+\frac{4}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td>( a(0)q^{\frac{5}{12}} + a(1)q^{1+\frac{5}{12}} + a(2)q^{2+\frac{5}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_6 )</td>
<td>( a(0)q^{\frac{6}{12}} + a(1)q^{1+\frac{6}{12}} + a(2)q^{2+\frac{6}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_7 )</td>
<td>( a(0)q^{\frac{7}{12}} + a(1)q^{1+\frac{7}{12}} + a(2)q^{2+\frac{7}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_8 )</td>
<td>( a(0)q^{\frac{8}{12}} + a(1)q^{1+\frac{8}{12}} + a(2)q^{2+\frac{8}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_9 )</td>
<td>( a(0)q^{\frac{9}{12}} + a(1)q^{1+\frac{9}{12}} + a(2)q^{2+\frac{9}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_{10} )</td>
<td>( a(0)q^{\frac{10}{12}} + a(1)q^{1+\frac{10}{12}} + a(2)q^{2+\frac{10}{12}} + \cdots )</td>
</tr>
<tr>
<td>( \chi_{11} )</td>
<td>( a(0)q^{\frac{11}{12}} + a(1)q^{1+\frac{11}{12}} + a(2)q^{2+\frac{11}{12}} + \cdots )</td>
</tr>
</tbody>
</table>

**Proof of Theorem 1.1.** We define a map

\[
\mu : S_{k-\frac{1}{2}}, \chi(\text{SL}_2(\mathbb{Z})) \to \tilde{H}_{\frac{1}{2}, -k, \chi}^1(SL_2(\mathbb{Z}), \eta \mathcal{P})
\]

as follows. For \( \gamma \in \text{SL}_2(\mathbb{Z}) \) we define a non-holomorphic period function

\[
p_\gamma^*(g; z) := \int_{i\infty}^{\gamma^{-1}(i\infty)} g(\tau)\eta(\tau)\overline{\eta(z)}(\tau - z)^{k-2}d\tau
\]
for $g(z) \in S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z}))$. Actually, this function is real-analytic on $\mathbb{H}$. Then for $g(z) \in S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z}))$ we define $\mu(g)$ as the cohomology class in $\tilde{H}^1_{\frac{3}{2},-k,\chi}(SL_2(\mathbb{Z}),\eta P)$ containing a cocycle $\{p_\gamma^*(g;\tau)| \gamma \in SL_2(\mathbb{Z})\}$.

By Lemma 4.1 the Dedekind eta function $\eta(z)$ induces an isomorphism $\phi_\eta: S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z})) \to S_{k,\chi \chi}(SL_2(\mathbb{Z}))$.

By the Eichler cohomology theorem for arbitrary real weight modular forms (see Theorem 2.1 in [9]), the space $S_{k,\chi \chi}(SL_2(\mathbb{Z}))$ is isomorphic to the cohomology group $\tilde{H}^1_{2-k,\chi \chi}(SL_2(\mathbb{Z}),\mathcal{P})$. Moreover one can see that cohomology groups $\tilde{H}^1_{2-k,\chi \chi}(SL_2(\mathbb{Z}),\mathcal{P})$ and $\tilde{H}^1_{\frac{3}{2}-k,\chi}(SL_2(\mathbb{Z}),\eta P)$ are isomorphic by the mapping

\begin{equation}
\{p_\gamma(z)| \gamma \in SL_2(\mathbb{Z})\} \mapsto \{\eta(z)p_\gamma(z)| \gamma \in SL_2(\mathbb{Z})\}.
\end{equation}

Note that this isomorphism is formal and is not related to some specific property of $\eta(z)$. Since the map $\mu$ is a composition of the above three isomorphisms, the map $\mu$ is also an isomorphism. $\square$ 

**Proof of Theorem 1.2.** We define a map

$$\tilde{\mu}: S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z})) \oplus S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z})) \to \tilde{H}^1_{\frac{3}{2}-k,\chi}(SL_2(\mathbb{Z}),\eta P_{k-2})$$

as follows by using the period functions associated with cusp forms of half-integral weights. For $\gamma \in SL_2(\mathbb{Z})$ we define a holomorphic period function $p_\gamma(f;\tau) := \int_{\gamma^{-1}(i\infty)} f(\tau)\eta(\tau)\eta(z)(\tau - z)^{k-2}d\tau$ for $f(z) \in S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z}))$. Then for $f(z) \in S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z}))$ and $g(z) \in S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z}))$ we give a map

$$\tilde{\mu}(f,g) = \alpha(f) + \beta(g),$$

where $\alpha(f)$ is the cohomology class in $\tilde{H}^1_{\frac{3}{2}-k,\chi}(SL_2(\mathbb{Z}),\eta P_{k-2})$ containing a cocycle $\{p_\gamma(f;\tau)| \gamma \in SL_2(\mathbb{Z})\}$ and $\beta(g)$ is the cohomology class containing a cocycle $\{p_\gamma^*(g;\tau)| \gamma \in SL_2(\mathbb{Z})\}$.

By Lemma 4.1 we have an isomorphism

$$S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z})) \oplus S_{k-\frac{1}{2},\chi}(SL_2(\mathbb{Z})) \cong S_{k,\chi \chi}(SL_2(\mathbb{Z})) \oplus S_{k,\chi \chi}(SL_2(\mathbb{Z})).$$

By the Eichler cohomology theorem for integral weight modular forms (for example, see Corollary 1 in [6]), the space $S_{k,\chi \chi}(SL_2(\mathbb{Z})) \oplus S_{k,\chi \chi}(SL_2(\mathbb{Z}))$ is isomorphic to $\tilde{H}^1_{2-k,\chi \chi}(SL_2(\mathbb{Z}),P_{k-2})$. Moreover one can see that cohomology groups $\tilde{H}^1_{2-k,\chi \chi}(SL_2(\mathbb{Z}),P_{k-2})$ and $\tilde{H}^1_{\frac{3}{2}-k,\chi}(SL_2(\mathbb{Z}),\eta P_{k-2})$ are isomorphic by the same map as in (4.1). Combining the above three isomorphisms, one can see that the map $\tilde{\mu}$ is also an isomorphism. $\square$
Table 4.2

<table>
<thead>
<tr>
<th>Multiplier system</th>
<th>Fourier expansion at 1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi_1 )</td>
<td>( b_{1/2}(0)q^{3} + b_{1/2}(1)q^{5} + \cdots )</td>
</tr>
<tr>
<td>( \psi_2 )</td>
<td>( b_{1/2}(1)q^{1} + b_{1/2}(2)q^{2} + \cdots )</td>
</tr>
</tbody>
</table>

As in the proof of Theorem 1.1 and 1.2, we need the following lemma to prove Theorem 1.3 and 1.4.

**Lemma 4.2.** Let \( k \) be an integer with \( k \geq 2 \) and \( \chi = \chi_{\theta}^{2k-1} \) be a multiplier system of weight \( k - \frac{1}{2} \) on \( \Gamma_0(4) \). Then we have

\[
S_{k-\frac{1}{2}, \chi}(\Gamma_0(4)) \cong S_{k, \chi \theta}(\Gamma_0(4)).
\]

**Proof of Lemma 4.2.** We define a map

\[
\phi_{\theta} : S_{k-\frac{1}{2}, \chi}(\Gamma_0(4)) \to S_{k, \chi \theta}(\Gamma_0(4))
\]

by \( \phi_{\theta}(f)(z) = f(z)\theta(z) \) for \( f(z) \in S_{k-\frac{1}{2}, \chi}(\Gamma_0(4)) \). Then we can see that this map is injective. To show that it is surjective, we use the same argument in the proof of Lemma 4.1. Note that \( \Gamma_0(4) \) has three inequivalent cusps \( i\infty, 0 \) and \( \frac{1}{2} \) and \( \theta(z) \) has no zero in \( \mathbb{H} \cup \{i\infty, 0\} \). At the cusp \( \frac{1}{2} \), the theta function \( \theta(z) \) has a Fourier expansion of the form

\[
(4.2) \quad \sum_{n=0}^{\infty} a_{\frac{1}{2}}(n)e^{2\pi i(n+\frac{1}{2})z} = a_{\frac{1}{2}}(0)q^{\frac{1}{4}} + a_{\frac{1}{2}}(1)q^{\frac{5}{4}} + \cdots
\]

Let \( \gamma_{\frac{1}{2}} \) be the generator of \( \Gamma_{\frac{1}{2}} \) of the form \( \left( \begin{array}{cc} 2 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} 1 & \lambda_{\frac{1}{2}} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \), where \( \lambda_{\frac{1}{2}} \) is a positive integer. Then (4.2) comes from the fact that \( \kappa_{\frac{1}{2}} = \chi_{\theta}(\gamma_{\frac{1}{2}}) \), \( \gamma_{\frac{1}{2}} = \left( \begin{array}{cc} -1 & 1 \\ 4 & 3 \end{array} \right) \) and \( \lambda_{\frac{1}{2}} = 1 \). If \( \chi_{\chi_{\theta}} = \chi_{\theta}^{2k} \), then we have \( \chi_{\chi_{\theta}} = \psi_{i} \) for \( i \equiv k \pmod{2} \), where \( \psi_{i} = (\chi_{\theta})^{2i} \). If \( g(z) \in S_{k, \psi_{i}}(\Gamma_0(4)) \) has a Fourier expansion of the form at the cusp \( \frac{1}{2} \)

\[
g(z) = \sum_{n+\kappa > 0} b_{\frac{1}{2}}(n)e^{2\pi i(n+\kappa, \frac{1}{2})z},
\]

then the Fourier expansion of \( g(z) \) is given by Table 4.2 because we see that \( \kappa_{\frac{1}{2}} = \frac{1}{2} \) and \( \lambda_{\frac{1}{2}} = 1 \) from (4.2). Therefore, a function \( \frac{g(z)}{\theta(z)} \) is a cusp form in \( S_{k-\frac{1}{2}, \chi}(\Gamma_0(4)) \) and the map \( \phi_{\theta} \) is surjective, which completes the proof.

**Proof of Theorem 1.3.** The proof is similar with the proof of Theorem 1.1. First we give a map

\[
\nu : S_{k-\frac{1}{2}, \chi}(\Gamma_0(4)) \to \tilde{H}_{\frac{1}{2}}^{1-k, \chi}(\Gamma_0(4), \theta P)
\]
by using the period function associated with cusp forms of a half-integral weight. For \( \gamma \in \Gamma_0(4) \) and \( g(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \) we put

\[
q_\gamma^*(g; z) := \int_{i\infty}^{\gamma-1(i\infty)} g(\tau)\theta(\tau)\overline{\theta(z)(\tau - z)^{k-2}}d\tau.
\]

Then for \( g(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \) we define \( \nu(g) \) as the cohomology class in \( \tilde{H}^1_{\frac{1}{2},\chi}(\text{SL}_2(\mathbb{Z}), \theta P) \) containing a cocycle \( \{q_\gamma^*(g; z) | \gamma \in \Gamma_0(4)\} \).

By Lemma 4.2 the theta function \( \theta(z) \) induces an isomorphism between \( S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \) and \( S_{k,\chi_0}(\Gamma_0(4)) \). We also have an isomorphism between the space \( S_{k,\chi_0}(\Gamma_0(4)) \) and the cohomology group \( \tilde{H}^1_{2-k,\chi_0}(\Gamma_0(4), \theta P) \) by Theorem 2.1 in [9]. Moreover one can see that \( \tilde{H}^1_{2-k,\chi_0}(\Gamma_0(4), \theta P) \) is isomorphic to \( \tilde{H}^1_{\frac{1}{2},\chi}(\Gamma_0(4), \theta P) \) by the mapping

\[
(4.3) \quad \{p_\gamma(z) | \gamma \in \Gamma_0(4)\} \mapsto \{\theta(z)p_\gamma(z) | \gamma \in \Gamma_0(4)\}.
\]

Then we obtain that the map \( \nu \) is an isomorphism since the map \( \nu \) is a composition of the above three isomorphisms, which completes the proof. \( \square \)

**Proof of Theorem 1.4.** The proof is similar with the proof of Theorem 1.2. We give a map

\[
\tilde{\nu} : S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \oplus S_{k-\frac{1}{2},\chi_0}(\Gamma_0(4)) \rightarrow \tilde{H}^1_{\frac{1}{2},\chi}(\Gamma_0(4), \theta P_{k-2})
\]

as follows. For \( \gamma \in \Gamma_0(4) \) we define

\[
q_\gamma(f; z) := \int_{i\infty}^{\gamma-1(i\infty)} f(\tau)\theta(\tau)\overline{\theta(z)(\tau - z)^{k-2}}d\tau
\]

for \( f(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \). Then for \( f(z) \in S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \) and \( g(z) \in S_{k-\frac{1}{2},\chi_0}(\Gamma_0(4)) \) we give a map

\[
\tilde{\nu}(f, g) = \alpha(f) + \beta(g),
\]

where \( \alpha(f) \) is the cohomology class in \( \tilde{H}^1_{\frac{1}{2},\chi}(\Gamma_0(4), \theta P_{k-2}) \) containing a cocycle \( \{q_\gamma(f; z) | \gamma \in \Gamma_0(4)\} \) and \( \beta(g) \) is the cohomology class containing a cocycle \( \{q_\gamma^*(g; z) | \gamma \in \Gamma_0(4)\} \).

Lemma 4.2 implies that the function \( \theta(z) \) induces the following isomorphism

\[
S_{k-\frac{1}{2},\chi}(\Gamma_0(4)) \oplus S_{k-\frac{1}{2},\chi_0}(\Gamma_0(4)) \cong S_{k,\chi}(\Gamma_0(4)) \oplus S_{k,\chi_0}(\Gamma_0(4)).
\]

By Corollary 1 in [6] the space \( S_{k,\chi}(\Gamma_0(4)) \oplus S_{k,\chi_0}(\Gamma_0(4)) \) is isomorphic to \( \tilde{H}^1_{2-k,\chi}(\Gamma_0(4), P_{k-2}) \). Moreover one can see that \( \tilde{H}^1_{2-k,\chi}(\Gamma_0(4), P_{k-2}) \) is isomorphic to \( \tilde{H}^1_{\frac{1}{2},\chi}(\Gamma_0(4), \theta P_{k-2}) \) by the same mapping as in (4.3).
Then since the map $\tilde{\nu}$ is a composition of the above three isomorphisms, the map $\tilde{\nu}$ is also an isomorphism.

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\section*{References}


