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RéSUMÉ. On construit un exemple concret d’une famille à un paramètre de variétés lisses, projectives, et géométriquement intègres sur un sous-schéma ouvert de $\mathbb{P}^1_\mathbb{Q}$, de sorte qu’il y ait précisément une fibre rationnelle sans point rationnel. Ceci rend explicite une construction de Poonen.

Abstract. We construct a concrete example of a 1-parameter family of smooth projective geometrically integral varieties over an open subscheme of $\mathbb{P}^1_\mathbb{Q}$ such that there is exactly one rational fiber with no rational points. This makes explicit a construction of Poonen.

1. Introduction

We construct a family of smooth projective geometrically integral surfaces over an open subscheme of $\mathbb{P}^1_\mathbb{Q}$ with the following curious arithmetic property: there is exactly one $\mathbb{Q}$-fiber with no rational points. Our proof makes explicit a non-effective construction of Poonen [6, Prop. 7.2], thus giving “an extreme example of geometry not controlling arithmetic” [6, p.2]. We believe that this is the first example of its kind.

Theorem 1.1. Define $P_0(x) := (x^2 - 2)(3 - x^2)$ and $P_\infty(x) := 2x^4 + 3x^2 - 1$. Let $\pi : X \to \mathbb{P}^1_\mathbb{Q}$ be the Châtelet surface bundle over $\mathbb{P}^1_\mathbb{Q}$ given by

$$y^2 + z^2 = \left(6u^2 - v^2\right)^2 P_0(x) + \left(12v^2\right)^2 P_\infty(x),$$

where $\pi$ is projection onto $(u : v)$. Then $\pi(X(\mathbb{Q})) = \mathbb{A}^1_\mathbb{Q}(\mathbb{Q})$.

Note that the degenerate fibers of $\pi$ do not lie over $\mathbb{P}^1(\mathbb{Q})$ so the family of smooth projective geometrically integral surfaces mentioned above contains all $\mathbb{Q}$-fibers.

The non-effectivity in [6, Prop. 7.2] stems from the use of higher genus curves and Faltings’ theorem. (This is described in more detail in [6, §9]).

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We circumvent the use of higher genus curves by an appropriate choice of $P_{\infty}(x)$.

2. Background

This information can be found in [6, §3.5, and 6]. We review it here for the reader’s convenience.

Let $E$ be a rank 3 vector sheaf on a $k$-variety $B$. A conic bundle $C$ over $B$ is the zero locus in $\mathbb{P}E$ of a nowhere vanishing zero section $s \in \Gamma(\mathbb{P}E, \text{Sym}^2(E))$. A diagonal conic bundle is a conic bundle where $E = L_1 \oplus L_2 \oplus L_3$ and $s = s_1 + s_2 + s_3, s_i \in \Gamma(\mathbb{P}E, L_i \otimes^2)$.

Now let $\alpha \in k^\times$, and let $P(x) \in k[x]$ be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle $X$ given by $B = \mathbb{P}^1, L_1 = O, L_2 = O, L_3 = O(2), s_1 = 1, s_2 = -\alpha, s_3 = -w^4P(x/w)$. This smooth conic bundle contains the affine hypersurface $y^2 - \alpha z^2 = P(x) \subset A^3$ as an open subscheme. We say that $X$ is the Châtelet surface given by

$$y^2 - \alpha z^2 = P(x).$$

Note that since $P(x)$ is not identically zero, $X$ is an integral surface.

A Châtelet surface bundle over $\mathbb{P}^1$ is a flat proper morphism $V \to \mathbb{P}^1$ such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let $P, Q \in k[x, w]$ be linearly independent homogeneous polynomials of degree 4 and let $\alpha \in k^\times$. Let $V$ be the diagonal conic bundle over $\mathbb{P}^1_{(a:b)} \times \mathbb{P}^1_{(u:w)}$ given by $L_1 = O, L_2 = O, L_3 = O(1, 2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2P + b^2Q)$. By composing $V \to \mathbb{P}^1 \times \mathbb{P}^1$ with the projection onto the first factor, we realize $V$ as a Châtelet surface bundle. We say that $V$ is the Châtelet surface bundle given by

$$y^2 - \alpha z^2 = a^2P(x) + b^2Q(x),$$

where $P(x) = P(x, 1)$ and $Q(x) = Q(x, 1)$. We can also view $a, b$ as relatively prime, homogeneous, degree $d$ polynomials in $u, v$ by pulling back by a suitable degree $d$ map $\phi: \mathbb{P}^1_{(u:v)} \to \mathbb{P}^1_{(a:b)}$.

3. Proof of Theorem 1.1

By [5], we know that the Châtelet surface

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violates the Hasse principle, i.e. it has $Q_v$-rational points for all completions $v$, but no $Q$-rational points. Thus, $\pi(X(\mathbb{Q})) \subseteq A_1^3(\mathbb{Q})$. Therefore, it remains to show that $X_{(u:1)}$, the Châtelet surface defined by

$$y^2 + z^2 = (6u^2 - 1)^2P_0(x) + 12^2P_\infty(x),$$

has a rational point for all $u \in \mathbb{Q}$. 

If $P_{(u,1)} := (6u^2 - 1)²P_0(x) + 12²P_\infty(x)$ is irreducible, then by [3], [4] we know that $X_{(u,1)}$ satisfies the Hasse principle. Thus it suffices to show that $P_{(u,1)}$ is irreducible and $X_{(u,1)}(\mathbb{Q}_v) \neq \emptyset$ for all $u \in \mathbb{Q}$ and all places $v$ of $\mathbb{Q}$.

3.1. Irreducibility. We prove that for any $u \in \mathbb{Q}$, the polynomial $P_{(u,1)}(x)$ is irreducible in $\mathbb{Q}[x]$ by proving the slightly more general statement, that for all $t \in \mathbb{Q}$

$$P_t(x) := (2x^4 + 3x^2 - 1) + t²(x^2 - 2)(3 - x²)$$
$$= x^4(2 - t²) + x²(3 + 5t²) + (-6t² - 1)$$

is irreducible in $\mathbb{Q}[x]$. We will use the fact that if $a, b, c \in \mathbb{Q}$ are such that $b² - 4ac$ and $ac$ are not squares in $\mathbb{Q}$ then $p(x) := ax^4 + bx^2 + c$ is irreducible in $\mathbb{Q}[x]$.

Let us first check that for all $t \in \mathbb{Q}$, $(3 + 5t²)^² - 4(2 - t²)(-6t² - 1)$ is not a square in $\mathbb{Q}$. This is equivalent to proving that the affine curve $C: w² = t^4 + 74t² + 17$ has no rational points. The smooth projective model, $\overline{C} : w² = t^4 + 74t²s² + 17s^4$ in weighted projective space $\mathbb{P}(1,1,2)$, has 2 rational points at infinity. Therefore $\overline{C}$ is isomorphic to its Jacobian. A computation in Magma shows that $\text{Jac}(\overline{C})(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [1]. Therefore, the points at infinity are the only 2 rational points of $\overline{C}$ and thus $C$ has no rational points.

Now we will show that $(-6t² -1)(2 - t²)$ is not a square in $\mathbb{Q}$ for any $t \in \mathbb{Q}$. As above, this is equivalent to determining whether $C': w² = (-6t² - 1)(2 - t²)$ has a rational point. Since 6 is not a square in $\mathbb{Q}$, this is equivalent to determining whether the smooth projective model, $\overline{C}'$, has a rational point. The curve $\overline{C}'$ is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in Magma shows that $\text{Jac}(\overline{C}')(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [1]. Thus $\#\overline{C}'(\mathbb{Q}) = 0$ or 2. If $(t, w)$ is a rational point of $\overline{C}'$, then $(\pm t, \pm w)$ is also a rational point. Therefore, $\#C(\mathbb{Q}) = 2$ if and only if there is a point with $t = 0$ or $w = 0$ and one can easily check that this is not the case.

3.2. Local solvability.

Lemma 3.1. For any point $(u : v) \in \mathbb{P}_\mathbb{Q}^1$, the Châtelet surface $X_{(u,v)}$ has $\mathbb{R}$-points and $\mathbb{Q}_p$-points for every prime $p$.

Proof. Let $a = 6u² - v²$ and let $b = 12u²$. We will refer to $a²P_0(x) + b²P_\infty(x)$ both as $P_{(a:b)}$ and $P_{(u,v)}$.

$\mathbb{R}$-points: It suffices to show that given $(u : v)$ there exists an $x$ such that

$$P_{(a:b)} = x^4(2b² - a²) + x²(3b² + 5a²) + (-6a² - b²)$$

is positive. If $2b² - a²$ is positive, then any $x$ sufficiently large will work. So assume $2b² - a²$ is negative. Then $\alpha = -\frac{(3b² + 5a²)}{2(2b² - a²)}$ is positive. We claim
$P_{(a:b)}(\sqrt{\alpha})$ is positive.

$$
P_{(a:b)}(\sqrt{\alpha}) = \alpha^2(2b^2 - a^2) + \alpha(3b^2 + 5a^2) + (-6a^2 - b^2)$$

$$= \frac{(3b^2 + 5a^2)^2}{4(2b^2 - a^2)} + \frac{-(3b^2 + 5a^2)^2}{2(2b^2 - a^2)} + (-6a^2 - b^2)$$

$$= \frac{1}{4(2b^2 - a^2)} \left( 4(2b^2 - a^2)(-6a^2 - b^2) - (3b^2 + 5a^2)^2 \right)$$

$$= \frac{1}{4(2b^2 - a^2)} \left( -17b^4 - 74a^2b^2 - a^4 \right)$$

Since $2b^2 - a^2$ is negative by assumption and $-17b^4 - 74a^2b^2 - a^4$ is always negative, we have our result.

**$\mathbb{Q}_p$-points:**

$p \geq 5$: Without loss of generality, let $a$ and $b$ be relatively prime integers. Let $\overline{X}_{(a:b)}$ denote the reduction of $X_{(a:b)}$ modulo $p$. We claim that there exists a smooth $\mathbb{F}_p$-point of $\overline{X}_{(a:b)}$ that, by Hensel’s lemma, we can lift to a $\mathbb{Q}_p$-point of $X_{(a:b)}$.

Since $P_{(a:b)}$ has degree at most 4 and is not identically zero modulo $p$, there is some $x \in \mathbb{F}_p$ such that $P_{(a:b)}(x)$ is nonzero. Now let $y, z$ run over all values in $\mathbb{F}_p$. Then the polynomials $y^2, P_{(a:b)}(x) - z^2$ each take $(p + 1)/2$ distinct values. By the pigeonhole principle, $y^2$ and $P_{(a:b)}(x) - z^2$ must agree for at least one pair $(y, z) \in \mathbb{F}_p^2$ and one can check that this pair is not $(0, 0)$. Thus, this tuple $(x, y, z)$ gives a smooth $\mathbb{F}_p$-point of $\overline{X}_{(a:b)}$. (The proof above that the quadratic form $y^2 + z^2$ represents any element in $\mathbb{F}_p$ is not new. For example, it can be found in [2, Prop 5.2.1].)

$p = 3$: From the equations for $a$ and $b$, one can check that for any $(u : v) \in \mathbb{P}_q^1$, $v_3(b/a)$ is positive. Since $\mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3$ is an unramified extension, it suffices to show that given $a, b$ as above, there exists an $x$ such that $P_{(a:b)}(x)$ has even valuation. Since $v_3(b/a)$ is positive, $v_3(2b^2 - a^2) = 2v_3(a)$. Therefore, if $x = 3^{-n}$, for $n$ sufficiently large, the valuation of $P_{(a:b)}(x)$ is $-4n + 2v_3(a)$ which is even.

$p = 2$: From the equations for $a$ and $b$, one can check that for any $(u : v) \in \mathbb{P}_q^1$, $v_2(b/a)$ is at least 2. Let $x = 0$ and $y = a$. Then we need to find a solution to $z^2 = a^2(-7 + (b/a)^2)$. Since $v_2(b/a) > 1$, $-7 + (b/a)^2 \equiv 1^2 \mod 8$. By Hensel’s lemma, we can lift this to a solution in $\mathbb{Q}_2$. 

$\Box$
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References


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