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par NICOLA MAZZARI

Résumé. Nous prouvons que la dimension cohomologique de la catégorie des 1-motifs de Laumon à isogénie près sur un corps de caractéristique nulle est $\leq 1$. En conséquence, cela implique le même résultat pour la catégorie des structures de Hodge formelles de niveau $\leq 1$ (sur $\mathbb{Q}$).

Abstract. We prove that the category of Laumon 1-motives up to isogenies over a field of characteristic zero is of cohomological dimension $\leq 1$. As a consequence this implies the same result for the category of formal Hodge structures of level $\leq 1$ (over $\mathbb{Q}$).

1. Introduction

In [6] P. Deligne defined a 1-motive over a field $k$ as $\text{Gal}(k^{\text{sep}}|k)$-equivariant morphism $[u : X \to G(k^{\text{sep}})]$ where $X$ is a free $\text{Gal}(k^{\text{sep}}|k)$-module and $G$ is a semi-abelian algebraic group over $k$. They form a category that we shall denote by $\mathcal{M}_1,k$ or $\mathcal{M}_1$.

Deligne’s definition was motivated by Hodge theory. In fact the category of 1-motives over the complex numbers is equivalent, via the so called Hodge realization functor, to the category $\text{MHS}_1$ of mixed Hodge structures of level $\leq 1$. It is known the the category $\text{MHS}_1$ is of cohomological dimension 1 (see [5]) and the same holds for $\mathcal{M}_{1,\mathbb{C}}$.

F. Orgogozo proved more generally that for any field $k$, the category $\mathcal{M}_{1,k} \otimes \mathbb{Q}$ is of cohomological dimension $\leq 1$ (see [14, Prop. 3.2.4]).

Over a field of characteristic 0 it is possible to define the category $\mathcal{M}_{1,k}^a$ of Laumon 1-motives generalizing that of Deligne 1-motives (See [11]). In [3] L. Barbieri-Viale generalized the Hodge realization functor to Laumon 1-motives. He defined the category $\text{FHS}_1$ of formal Hodge structures of level $\leq 1$ containing $\text{MHS}_1$ and proved that $\text{FHS}_1$ is equivalent to the category of Laumon 1-motives over $\mathbb{C}$ (compatibly with the Hodge realization).

In this paper we prove that the category of Laumon 1-motives up to isogenies is of cohomological dimension 1.
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2. Laumon 1-motives

Let $k$ be a (fixed) field of characteristic zero. Let $\text{Sch}_k$ be the category of schemes over $k$ and $\text{Aff}_k$ be the full sub-category of affine schemes. According to [1, Exp. IV §6.3] the fppf topology on $\text{Sch}_k$ is the one generated by: the families of jointly surjective open immersions in $\text{Sch}_k$; the finite families of jointly surjective, flat, of finite presentation and quasi-finite morphisms in $\text{Aff}_k$.

Let $\text{Ab}_k$ be the category of abelian sheaves on $\text{Aff}_k$ w.r.t. the fppf topology. We will consider both the category of commutative group schemes and that of formal group schemes (over $k$) as full sub-categories of $\text{Ab}_k$. We denote by $\bar{k}$ the algebraic closure of $k$.

Definition. A Laumon 1-motive over $k$ (or an effective free 1-motive over $k$, cf. [2, 1.4.1]) is the data of

1. a (commutative) formal group $F$ over $k$, such that $\text{Lie} F$ is a finite dimensional $k$-vector space and $F(\bar{k}) = \lim_{k' \to k} F(k')$ is a finitely generated and torsion-free $\text{Gal}(\bar{k}/k)$-module;
2. a connected commutative algebraic group scheme $G$ over $k$;
3. a morphism $u : F \to G$ in the category $\text{Ab}_k$.

Note that we can consider a Laumon 1-motive (over $k$) $M = [u : F \to G]$ as a complex of sheaves in $\text{Ab}_k$ concentrated in degree 0, 1.

It is known that any formal $k$-group $F$ splits canonically as product $F^o \times F_{\text{ét}}$, where $F^o$ is the identity component of $F$ and is a connected formal $k$-group, and $F_{\text{ét}} = F/F^o$ is étale. Moreover, $F_{\text{ét}}$ admits a maximal subgroup scheme $F_{\text{tor}}$, étale and finite, such that the quotient $F_{\text{ét}}/F_{\text{tor}} = F_{\text{fr}}$ is constant of the type $\mathbb{Z}^r$ over $\bar{k}$. One says that $F$ is torsion-free if $F_{\text{tor}} = 0$.

By a theorem of Chevalley any connected algebraic group scheme $G$ is the extension of an abelian variety $A$ by a linear $k$-group scheme $L$ that is product of its maximal sub-torus $T$ with a vector $k$-group scheme $V$. See [7] for more details on algebraic and formal groups.

Definition. A morphism of Laumon 1-motives is a commutative square in the category $\text{Ab}_k$. We denote by $\mathcal{M}_1^a = \mathcal{M}_{1,k}^a$ the category of Laumon $k$-1-motives, i.e. the full sub-category of $C^b(\text{Ab}_k)$ whose objects are Laumon 1-motives. We denote by $\mathcal{M}_1$ the full sub-category of $\mathcal{M}_1^a$ whose objects are Deligne 1-motives (over $k$) [6, §10.1.2].

Proposition 2.1. The category $\mathcal{M}_1^a$ of Laumon 1-motives (over $k$) is an additive category with kernels and co-kernels. In particular let $(f, g)$ be a
morphism from $M = [u : F \to G]$ to $M' = [u' : F' \to G']$ (i.e. $u'f = gu$), then

(2.1) \[ \text{Ker}(f, g) = [u^* \text{Ker}(g)^o \to \text{Ker}(g)^o] \]

and

(2.2) \[ \text{Coker}(f, g) = [\text{Coker}(f)_{fr} \to \text{Coker}(g)] \]

Proof. See [11, Prop. 5.1.3].

□

Remark. The category of connected algebraic groups is fully embedded in $\mathcal{M}_1^a$ and it is not abelian. So the category of Laumon 1-motives is not abelian too. In fact consider a surjective morphism of connected algebraic groups $g : G \to G'$. Then $\text{Ker}(g)$ is not necessarily connected and the canonical map (in the category of connected algebraic groups)

\[ \text{Coim}(g) = G/\text{Ker}(g)^o \to \text{Im}(g) = G' \]

is not an isomorphism in general.

According to [14] we define the category $\mathcal{M}_1^a \otimes \mathbb{Q}$ of Laumon 1-motives up to isogenies: the objects are the same of $\mathcal{M}_1^a$; the Hom groups are $\text{Hom}_{\mathcal{M}_1^a}(M, M') \otimes_{\mathbb{Z}} \mathbb{Q}$.

Remark. Note that a morphism $(f, g) : M \to M'$ is an isogeny (i.e. an isomorphism in $\mathcal{M}_1^a \otimes \mathbb{Q}$) if and only if $f$ is injective with finite co-kernel and $g$ is surjective with finite kernel.

Proposition 2.2. The category of Laumon 1-motives up to isogenies is abelian.

Proof. By construction $\mathcal{M}_1^a \otimes \mathbb{Q}$ is an additive category. Let $(f, g) : M \to M'$ be a morphism of Laumon 1-motives. We know that the group $\pi_0(\text{Ker}(g)) = \text{Ker}(g)/\text{Ker}(g)^o$ is a finite group scheme, hence there exists an integer $n$ such that the following diagram commutes in $\text{Ab}_k$

\[ \begin{array}{ccc}
\text{Ker}(f) & \xrightarrow{n \cdot u} & \text{Ker}(g)^o \\
\downarrow & & \downarrow \circ 0 \\
\text{Ker}(g)^o & \xrightarrow{\pi_0(\text{Ker}(g))} & \text{Ker}(g)^o \\
\end{array} \]

Then $n \cdot u$ factors through $\text{Ker}(g)^o$ and it is easy to check that $\text{Ker}((f, g)) = [(u^* \text{Ker}(g)^o \to \text{Ker}(g)^o)]$ is isogenous to $[\text{Ker}(f) \to \text{Ker}(g)]$.

It follows that $\text{Coim}(f, g)$ is isogenous to $[(F/\text{Ker}(f))_{fr} \to G/\text{Ker}(g)]$. As $G/\text{Ker}(g)^o \to G/\text{Ker}(g)$ is an isogeny we get that the canonical map $\text{Coim}(f, g) \to \text{Im}(f, g)$ is an isogeny too.

This is enough to prove that the category $\mathcal{M}_1^a \otimes \mathbb{Q}$ is abelian. □
Remark. One can also define the category $^t\mathcal{M}_1^a$ of 1-motives with torsion over $k$ (See [2, Def. 1.4.4]). We note that using the same arguments as in [4, C.7.3] it is easy to show that there is an equivalence of categories

$$\mathcal{M}_1^a \otimes \mathbb{Q} \xrightarrow{\sim} ^t\mathcal{M}_1^a \otimes \mathbb{Q}.$$ 

2.1. Weights. A Deligne 1-motive is endowed with an increasing filtration (of sub-1-motives) called the weight filtration ([6, §10.1.4]) defined as follows

$$W_i^M := \begin{cases} [X \to G] & i \geq 0 \\ [0 \to G] & i = -1 \\ [0 \to T] & i = -2 \\ [0 \to 0] & i \leq -3 \end{cases}; \quad \text{gr}_i^W M = \begin{cases} [X \to 0] & i = 0 \\ [0 \to A] & i = -1 \\ [0 \to T] & i = -2 \\ [0 \to 0] & \text{otherwise} \end{cases}.$$ 

According to [4, C.11.1] we extend the weight filtration to Laumon 1-motives.

Definition. Let $M = [u : F \to G]$ be a Laumon 1-motive. The weight filtration of $M$ is

$$W_{-3} = 0 \subset W_{-2} = [0 \to L] \subset W_{-1} = [0 \to G] \subset W_0 = M.$$ 

Remark.

(1) The morphisms of Laumon 1-motives are compatible w.r.t. the weight filtration. Also the weight filtration extends to a filtration on the objects of $\mathcal{M}_1^a \otimes \mathbb{Q}$.

(2) Let $\text{Mod}_k^f$ be the category of finite dimensional $k$-vector spaces. The full sub-category of $\mathcal{M}_1^a \otimes \mathbb{Q}$ of Laumon 1-motives of weight 0 is equivalent to the category $\text{Mod}_k^f \times \text{Rep}_\mathbb{Q}(\text{Gal}(\overline{k}/k))$ via the functor $F \mapsto (\text{Lie}(F), F(k) \otimes \mathbb{Q})$.

3. Cohomological dimension

3.1. Extensions. Let $A$ be any abelian category (we don’t suppose it has enough injective objects), then we can define its derived category $D(A)$ and the group of n-fold extension classes $\text{Ext}_A^n(A, B) := \text{Hom}_{D(A)}(A, B[n])$, $A, B \in A$. As usual we identify this group with the group of classes of Yoneda extensions, i.e. the set of exact sequences

$$0 \to B \to E_1 \to \cdots \to E_n \to A \to 0$$

modulo congruences (See [10] or [9]).

We will use the two following well-known facts about extensions and filtrations.

(1) Let $W_{-2} \subset W_{-1} \subset W_0 = W$ be a filtration of $W \in A$. We have the following exact sequences

$$\gamma : 0 \to W_{-2} \to W_{-1} \to W_0/W_{-2} \to W_0/W_{-1} \to 0$$
By the general facts on extensions (§3.1)

Assume that the objects of The category $\text{Mod}^w_{\text{pure}}$ of weight $(w)$ cohomological dimension 1.

Proof. Theorem 3.1. $\text{Mod}^w_{\text{pure}}$ checks that $\text{Ext}^A$ the Yoneda product of two classes (checking that if $M, M'$ weight filtration). By point (2) above this formally reduces the problem to (a 1-motive is pure if it is isomorphic to one of its graded pieces w.r.t. the weight filtration). If $\text{Ext}^A_0(M, M') = 0$ for any $i, j$, then $\text{Ext}^A(A, B) = 0$. In fact assume for instance that $B$ has a 3 steps filtration $0 \subset W = W_0 = B$: then we have the canonical exact sequences

$$0 \rightarrow W_1M' \rightarrow M' \rightarrow \text{gr}^W_0 M' \rightarrow 0$$

$$0 \rightarrow W_2M' \rightarrow W_1M' \rightarrow \text{gr}^W_1 M' \rightarrow 0$$

By applying $\text{Hom}(A, -)$ we get two long exact sequences

$$\cdots \text{Ext}^2(A, W_1B') \rightarrow \text{Ext}^2(A, B) \rightarrow \text{Ext}^2(M, \text{gr}^W_0 B) \cdots$$

$$\cdots \text{Ext}^2(A, \text{gr}^W_1 B) \rightarrow \text{Ext}^2(A, W_1B) \rightarrow \text{Ext}^2(A, \text{gr}^W_1 B) \cdots$$

from this follows that we can reduce to prove $\text{Ext}^2(A, \text{gr}^W_1 B) = 0$. This process can be easily adapted to the general case.

Now we can give a sketch of the proof of the main theorem: one first checks that $\text{Ext}^1_Q(M, M') = 0$ if $M, M'$ are pure of weights $w, w'$ (a 1-motive is pure if it is isomorphic to one of its graded pieces w.r.t. the weight filtration). By point (2) above this formally reduces the problem to checking that if $M, M', M''$ are pure respectively of weights 0, $-1, -2$, then the Yoneda product of two classes $(\gamma_1, \gamma_2) \in \text{Ext}^1_Q(M', M'') \times \text{Ext}^1_Q(M, M')$ is 0. Of course we may assume $\gamma_1$ and $\gamma_2$ integral. Then the point is that $\gamma_1$ and $\gamma_2$ glue into a 1-motive and we can conclude by (1) above.

3.2. Main result. From now on we call 1-motive a Laumon 1-motive (over $k$) and $\text{Ext}^1_Q(M, M')$ is the group of classes of $i$-fold extensions in $\mathcal{M}_1^q \otimes \mathbb{Q}$.

Theorem 3.1. The category $\mathcal{M}_1^q \otimes \mathbb{Q}$ (and in particular $\text{FHS}_1 \otimes \mathbb{Q}$) is of cohomological dimension 1.

Proof. By the general facts on extensions (§3.1 (2)) we can restrict to consider only pure motives $M = \text{gr}^W_w M$ and $M' = \text{gr}^W_w M'$ of weight $w$ and $w'$, respectively.

(Equal weights) If $w = w'$ we can show that $\text{Ext}^1_Q(M, M') = 0$. Let $0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0$ be an exact sequence in $\mathcal{M}_1^q \otimes \mathbb{Q}$, then also $E$ is pure of weight $w$. We have to consider 3 cases: first note the category $\text{Mod}^f_k \times \text{Rep}_Q(\text{Gal}(\bar{k}/k))$ is semi-simple by Maschke’s lemma [15, p. 47] and
so the claim holds for the weight zero case by point (2) of the remark in §2.1: also the category of abelian varieties up to isogenies (i.e. $w = -1$) is semi-simple by [13, p. 173]; the third case (weight $-2$) can be reduced to the first by Cartier duality (see [11, §5]) or proved explicitly.

(Different weights) Fix a 2-fold extension $\gamma \in \Ext^1_{\mathbb{Q}}(M, M')$ represented by

$$0 \to M' \to E_1 \to E_2 \to M \to 0$$

and let $E = \text{Ker}(E_2 \to M)$. Then we can write $\gamma = \gamma_1 \cdot \gamma_2$, where $\gamma_2 \in \Ext^1_{\mathbb{Q}}(M, E)$, $\gamma_1 \in \Ext^1_{\mathbb{Q}}(E, M')$. Using the canonical exact sequence induced by weights and the first part of the proof it is easy to reduce to the case $E = \text{gr}_{-1} E$, i.e. $E$ is an abelian variety.

If $w < w'$ then $\gamma_1$ is an extension of an abelian variety $E$ by $M'$ which is a formal group or an abelian variety. Then $\gamma_1 = 0$ (if $M'$ is a formal group we refer to [2, Lemma A.4.5]).

It remains to study what happens if $w > w'$. If $w$ or $w'$ is equal to $-1$ there is nothing to prove because $E$ is an abelian variety too. So the only case left is when $w = 0$ and $w' = -2$, i.e. $M = F[1]$, $M' = L[0]$. We want to reduce to the situation considered in §3.1 (1). Thus we have to show that there exists a 1-motive $N$ such that $\gamma_1 \in \Ext^1_{\mathbb{Q}}(E, L)$ is represented by $0 \to W_{-2} N \to W_{-1} N \to \text{gr}_{-1} N \to 0$; $\gamma_2 \in \Ext^1_{\mathbb{Q}}(F[1], E)$ is represented by $0 \to \text{gr}_{-1} N \to W_0 N/W_{-2} \to \text{gr}_0 N \to 0$.

We claim that $\gamma_1$ and $\gamma_2$ can be represented by extensions in the category Laumon-1-motives. In fact let

$$\gamma_1 : 0 \to L \xrightarrow{f \otimes n^{-1}} G \xrightarrow{g \otimes m^{-1}} E \to 0$$

be an extension in the category of 1-motives modulo isogenies: $f, g$ are morphism of algebraic groups, $n, m \in \mathbb{Z}$. Then consider the push-forward by $n^{-1}$ and the pull-back by $m^{-1}$, we get the following commutative diagram with exact rows in $\mathcal{M}_1^{a,fr} \otimes \mathbb{Q}$

```
0 → L → G → E → 0
```

The exactness of the last row is equivalent to the following: Ker $f$ is finite; let $(\text{Ker} g)^0$ be the connected component of Ker $g$, then Im $f \to (\text{Ker} g)^0$ is surjective with finite kernel $K$; $g$ is surjective. So after replacing $L, E$ with isogenous groups we have an exact sequence in $\mathcal{M}_1^{a,fr}$

$$0 \to L \to G \to E \to 0$$
Explicitly

\[
\begin{array}{ccccccccc}
0 & \rightarrow & L & \stackrel{f}{\rightarrow} & G & \stackrel{g}{\rightarrow} & E & \rightarrow & 0 \\
\downarrow & & \downarrow & \text{id} & \downarrow & \text{id} & & \\
0 & \rightarrow & L/\text{Ker}\ f & \rightarrow & G & \rightarrow & E & \rightarrow & 0 \\
\downarrow & \text{id} & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Im}\ f/K & \rightarrow & G & \rightarrow & E & \rightarrow & 0 \\
\downarrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & \text{Im}\ f/K & \rightarrow & G & \rightarrow & G/\text{(Ker}\ g)^0 & \rightarrow & 0
\end{array}
\]

With similar arguments we can prove that \( \gamma_2 \) is represented by an extension in the category \( \mathcal{M}^{a,fr}_1 \)

\[ 0 \rightarrow E \rightarrow N \rightarrow F[1] \rightarrow 0 \]

with \( N = [u : F \rightarrow E] \).

To conclude we have to prove that there is lifting \( u' : F \rightarrow G \). First suppose \( F = F_{\text{ét}} \): consider the long exact sequence

\[ \text{Hom}_{\text{Ab}}(F, G) \rightarrow \text{Hom}_{\text{Ab}}(F, E) \rightarrow \partial_{\text{Ext}}^1_{\text{Ab}}(F, L) \rightarrow \text{Ext}^1_{\text{Ab}}(F, L) \).

We can consider a (Galois) extension \( k'/k \) of finite degree \( d \) trivializing \( F \). By [12, Theorem 3.9] we get the vanishing of \( \text{Ext}^1_{\text{Ab}}(F_{k'}, L_{k'}) \). Recall that the multiplication by \( d \) on \( F \) can be written as the composition

\[ F \xrightarrow{\text{can}} \Pi_{k'/k} F_{k'} \xrightarrow{\text{tr}} F \]

where \( \Pi_{k'/k} \) is Weil restriction functor and \( \text{tr} \) is the trace map. This implies that \( \text{Ext}^1_{\text{Ab}}(F, L) \) is torsion, hence \( \partial u = 0 \) and the lift exists (up to isogeny).

In case \( F = F^{\circ} \) is a connected formal group we have a commutative diagram in \( \text{Ab}_k \)

\[
\begin{array}{ccc}
\widehat{G} & \xrightarrow{\widehat{\pi}} & \widehat{E} \\
\downarrow & & \downarrow u \\
G & \xrightarrow{\pi} & E
\end{array}
\]

where \( \widehat{\pi} \) is the formal completion at the origin of \( \hat{?} = G, E \). The formal completion is an exact functor so \( \widehat{\pi} \) is an epimorphism. The category of
connected formal groups is equivalent to $\text{Mod}^f_k$, thus we can choose a section $\sigma$ of $\hat{\pi}$. Then we can easily construct a (non canonical) lifting of $u$. \qed

References


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