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Degeneration of the Kummer sequence in characteristic \( p > 0 \)


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par Yuji TSUNO

Abstract. We study a deformation of the Kummer sequence to the radicial sequence over an $\mathbb{F}_p$-algebra, which is somewhat dual for the deformation of the Artin-Schreier sequence to the radicial sequence, studied by Saidi. We also discuss some relations between our sequences and the embedding of a finite flat commutative group scheme into a connected smooth affine commutative group schemes, constructed by Grothendieck.

Introduction

Let $p$ be a prime number. The Artin-Schreier sequence

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{G}_{a,\mathbb{F}_p} \xrightarrow{F-I} \mathbb{G}_{a,\mathbb{F}_p} \rightarrow 0$$

has an important role in algebraic geometry in characteristic $p$. Indeed we obtain a description of cyclic extensions of degree $p$ over a field of characteristic $p$ or more generally of cyclic coverings of a variety over a field of characteristic $p$, applying the theory of Galois cohomology or étale cohomology to the Artin-Schreier sequence.

Mohamed Saidi [4] studies the degeneration of cyclic coverings of a curve over a ring of characteristic $p$, using the exact sequence

$$0 \rightarrow N_A \rightarrow \mathbb{G}_{a,A} \xrightarrow{F-\mu I} \mathbb{G}_{a,A} \rightarrow 0,$$

where $A$ is an $\mathbb{F}_p$-algebra and $\mu \in A$. When $\mu = 0$, we obtain an exact sequence

$$0 \rightarrow \mathbb{G}_{a,A} \xrightarrow{F} \mathbb{G}_{a,A} \rightarrow 0,$$
called the radical sequence.

As is well known, the Cartier dual of $\mathbb{Z}/p\mathbb{Z}$ is isomorphic to $\mu_{p,A}$, the group scheme of $p$-th roots of unity, and $\alpha_{p,A}$ is auto-dual for the Cartier duality. Hence the Cartier dual of $N_A$ is a deformation of $\mu_{p,A}$ to $\alpha_{p,A}$.

On the other hand, we have an exact sequence
\[ 0 \to \mu_{p,A} \to \mathbb{G}_m,A \xrightarrow{F} \mathbb{G}_m,A \to 0, \]
called the Kummer sequence. It would be interesting to consider an analogue of the sequence which combines the Artin-Schreier sequence and the radical sequence.

The main results of this article are the following theorems:

**Theorem 1** (= Theorem 2.6.) Let $A$ be an $\mathbb{F}_p$-algebra and $\lambda \in A$. Put $N_A = \text{Ker}[F - \lambda^{p-1}I : G_{a,A} \to G_{a,A}]$. Then there exists an exact sequence of group $A$-schemes:
\[ 0 \to N_A^\vee \to G_A^{(\lambda)} \xrightarrow{F} G_A^{(\lambda^p)} \to 0. \]

**Theorem 2** (= Theorem 2.9.) Let $A$ be an $\mathbb{F}_p$-algebra and $\lambda \in A$. Put $N_A = \text{Ker}[F - \lambda(p-1)/2I : G_{a,A} \to G_{a,A}]$. Then there exists an exact sequence of group $A$-schemes:
\[ 0 \to N_A^\vee \to G_{B/A} \xrightarrow{F} G_{B/A} \to 0. \]

(For the notation, see Section 1. We owe the description of the group scheme $G_{B/A}$ to Waterhouse-Weisfeiler [14].)

Now we explain the contents of the article. In Section 1, we recall needed facts on group schemes. In Section 2, after giving a precise description of the Cartier dual of $N_A$, we prove Theorem 1 and Theorem 2. The exact sequence (1) gives a deformation of the Kummer sequence to the radical sequence. Moreover, applying the cohomology theory of group schemes, we obtain an analogue of the classical Kummer theory:

**Corollary 1** (= Corollary 2.11.) Under the assumption of Theorem 1, suppose that Spec $S$ has a structure of $N_A^\vee$-torsor over Spec $R$. If $R$ is a local ring or $\lambda$ is nilpotent, then there exists a morphism Spec $R \to G_A^{(\lambda^p)}$ such that the square
\[ \begin{array}{ccc}
\text{Spec } S & \longrightarrow & G_A^{(\lambda)} \\
\downarrow & & \downarrow F \\
\text{Spec } R & \longrightarrow & G_A^{(\lambda^p)}
\end{array} \]
is cartesian.

Furthermore, the exact sequence (2) is a quadratic twist of (1), that is, after the base change by the quadratic extension $A[\sqrt{\lambda}]/A$, the sequence
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(2) is isomorphic to a sequence of the form (1). We have also a similar assertion as above:

**Corollary 2** (= Corollary 2.17.) Under the assumption of Theorem 2, suppose that Spec $S$ has a structure of $N_A^\vee$-torsor over Spec $R$. If $R$ is a local ring or $\lambda$ is nilpotent, then there exists a morphism Spec $R \to G_{\tilde{B}/A}$ such that the square

\[
\begin{array}{ccc}
\text{Spec } S & \longrightarrow & G_{B/A} \\
\downarrow & & \downarrow F \\
\text{Spec } R & \longrightarrow & G_{\tilde{B}/A}
\end{array}
\]

is cartesian.

In Section 3, we compare our sequences and the exact sequence constructed by Grothendieck. In fact,

**Theorem 3** (= Theorem 3.12.) Let $A$ be an $\mathbb{F}_p$-algebra and $\lambda \in A$. Put $N_A = \text{Ker}[F - \lambda^{p-1}I : \mathbb{G}_{a,A} \to \mathbb{G}_{a,A}]$. Then there exist commutative diagrams of group schemes

\[
\begin{array}{ccc}
N_A^\vee & \longrightarrow & \prod_{N_A/A} \mathbb{G}_{m,N_A} \\
\downarrow & & \downarrow \tilde{\chi} \\
N_A & \longrightarrow & \mathcal{G}_A^{(\lambda)}
\end{array}
\]

and

\[
\begin{array}{ccc}
N_A^\vee & \longrightarrow & \mathcal{G}_A^{(\lambda)} \\
\downarrow & & \downarrow \tilde{\sigma} \\
N_A & \longrightarrow & \prod_{N_A/A} \mathbb{G}_{m,N_A}.
\end{array}
\]

**Theorem 4** (= Theorem 3.15.) Let $p$ be a prime number $> 2$, $A$ an $\mathbb{F}_p$-algebra and $\lambda \in A$. Put $N_A = \text{Ker}[F - \lambda^{p-1}I : \mathbb{G}_{a,A} \to \mathbb{G}_{a,A}]$. Then there exist commutative diagrams of group schemes

\[
\begin{array}{ccc}
N_A^\vee & \longrightarrow & \prod_{N_A/A} \mathbb{G}_{m,N_A} \\
\downarrow & & \downarrow \tilde{\chi} \\
N_A & \longrightarrow & G_{B/A}
\end{array}
\]
and
\[
\begin{array}{ccc}
N_A \rightarrow & G_{B/A} \\
\downarrow \iota & \downarrow \tilde{\sigma} \\
N_A \rightarrow & \prod_{N_A/A} \mathbb{G}_{m,N_A}.
\end{array}
\]

It should be mentioned that the argument in Section 3 is an analogue of the statement for the unit group schemes of group algebras, developed in Suwa [10] after Serre [7, Ch.IV, 8].

**Notation.** Throughout the article, \( p \) denotes a prime number and \( \mathbb{F}_p \) denotes the finite field of order \( p \). Unless otherwise indicated, \( F \) denotes the Frobenius endomorphism.

For a scheme \( X \) and a commutative group scheme \( G \) over \( X \), \( H^*(X,G) \) denotes the cohomology group with respect to the fppf-topology. It is known that, if \( G \) is smooth over \( X \), the fppf-cohomology group coincides with the étale cohomology group (Grothendieck [2], III.11.7). By the abbreviation, \( H^*(R,G) \) denotes \( H^*(\mathrm{Spec} R, G) \) when \( R \) is a ring.

For an \( A \)-algebra \( B, \prod_B \) denotes the Weil restriction functor.

**List of group schemes**
- \( \mathbb{G}_{a,A} \): the additive group scheme over \( A \)
- \( \mathbb{G}_{m,A} \): the multiplicative group scheme over \( A \)
- \( \mu_{n,A} : \text{Ker}[n : \mathbb{G}_{m,A} \to \mathbb{G}_{m,A}] \)
- \( \alpha_{p,A} : \text{Ker}[F : \mathbb{G}_{a,A} \to \mathbb{G}_{a,A}] \) when \( A \) is of characteristic \( p \)
- \( G^{(\lambda)}_A \): recalled in 1.2
- \( G_{B/A} \): defined in 1.3

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**1. Preliminaries**

**Definition 1.1.** Let \( A \) be a ring. The additive group scheme \( \mathbb{G}_{a,A} \) over \( A \) is defined by
\[
\mathbb{G}_{a,A} = \text{Spec} A[T]
\]
with
(a) the multiplication: \( T \mapsto T \otimes 1 + 1 \otimes T \),
(b) the unit: \( T \mapsto 0 \),
(c) the inverse: \( T \mapsto -T \).

On the other hand, the multiplicative group scheme \( \mathbb{G}_{m,A} \) over \( A \) is defined by
\[
\mathbb{G}_{m,A} = \text{Spec } A[T, \frac{1}{T}]
\]
with
(a) the multiplication: \( T \mapsto T \otimes T \),
(b) the unit: \( T \mapsto 1 \),
(c) the inverse: \( T \mapsto 1/T \).

**Definition 1.2.** Let \( A \) be a ring and \( \lambda \in A \). A commutative group scheme \( \mathcal{G}^{(\lambda)}_A \) over \( A \) is defined by
\[
\mathcal{G}^{(\lambda)}_A = \text{Spec } A[T, \frac{1}{1 + \lambda T}]
\]
with
(a) the multiplication: \( T \mapsto T \otimes 1 + 1 \otimes T + \lambda T \otimes T \),
(b) the unit: \( T \mapsto 0 \),
(c) the inverse: \( T \mapsto -T/(1 + \lambda T) \).

A homomorphism \( \alpha^{(\lambda)} : \mathcal{G}^{(\lambda)}_A \to \mathbb{G}_{m,A} \) of group schemes over \( A \) is defined by
\[
U \mapsto \lambda T + 1 : A[U, \frac{1}{U}] \to A[T, \frac{1}{1 + \lambda T}].
\]
If \( \lambda \) is invertible in \( A \), then \( \alpha^{(\lambda)} \) is an isomorphism. On the other hand, if \( \lambda = 0 \), \( \mathcal{G}^{(\lambda)}_A \) is nothing but the additive group scheme \( \mathbb{G}_{a,A} \).

**Definition 1.3.** Let \( A \) be a ring and \( \lambda \in A \). Put \( B = A[\sqrt{\lambda}] = A[t]/(t^2 - \lambda) \). Then the functor from \( A \)-algebras to groups \( R \mapsto (R \otimes_A B)^\times \) is represented by the group scheme
\[
\prod_{B/A} \mathbb{G}_{m,B} = \text{Spec } A[U, V, \frac{1}{U^2 - \lambda V^2}]
\]
with the multiplication
\[
U \mapsto U \otimes U + \lambda V \otimes V, \ V \mapsto U \otimes V + V \otimes U.
\]
Moreover, the canonical injection \( R^\times \to (R \otimes_A B)^\times \) is represented by the homomorphism of group schemes
\[
i : \mathbb{G}_{m,A} = \text{Spec } A[T, \frac{1}{T}] \to \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec } A[U, V, \frac{1}{U^2 - \lambda V^2}]
\]
defined by
\[
U \mapsto T, \ V \mapsto 0.
\]
On the other hand, the norm map $\text{Nr} : (R \otimes_A B)^{\times} \rightarrow R^{\times}$ is represented by the homomorphism of group schemes

$$
\text{Nr} : \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec } A[U,V, \frac{1}{U^2 - \lambda V^2}] \rightarrow \mathbb{G}_{m,A} = \text{Spec } A[T, \frac{1}{T}]
$$
defined by

$$
U \mapsto U^2 - \lambda V^2.
$$

We define a group scheme $U_{B/A}$ over $A$ by

$$
U_{B/A} = \text{Ker}[\text{Nr} : \prod_{B/A} \mathbb{G}_{m,B} \rightarrow \mathbb{G}_{m,A}].
$$

More precisely

$$
U_{B/A} = \text{Spec } A[U,V]/(U^2 - \lambda V^2 - 1)
$$
with the multiplication

$$
U \mapsto U \otimes U + \lambda V \otimes V, \ V \mapsto U \otimes V + V \otimes U.
$$

If $2\lambda$ is invertible in $A$, then $U_{B/A}$ is an algebraic torus over $A$.
Moreover, we define a group scheme $G_{B/A}$ over $A$ by

$$
G_{B/A} = \text{Spec } A[X,Y]/(X^2 - \lambda Y^2 - Y)
$$
with
(a) the multiplication:

$$
X \mapsto X \otimes 1 + 1 \otimes X + 2\lambda X \otimes Y + 2\lambda Y \otimes X, \ Y \mapsto Y \otimes 1 + 1 \otimes Y + 2\lambda Y \otimes Y + 2X \otimes X;
$$
(b) the unit:

$$
X \mapsto 0, \ Y \mapsto 0;
$$
(c) the inverse:

$$
X \mapsto -X, \ Y \mapsto Y.
$$

**Remark 1.4.** We define a homomorphism of group $A$-schemes

$$
r : \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec } A[U,V, \frac{1}{U^2 - \lambda V^2}]
$$

$$
\rightarrow G_{B/A} = \text{Spec } A[X,Y]/(X^2 - \lambda Y^2 - Y)
$$
by

$$
X \mapsto \frac{UV}{U^2 - \lambda V^2}, \ Y \mapsto \frac{V^2}{U^2 - \lambda V^2}
$$

It is readily seen that the sequence

$$
0 \rightarrow \mathbb{G}_{m,A} \xrightarrow{i} \prod_{B/A} \mathbb{G}_{m,B} \xrightarrow{r} G_{B/A} \rightarrow 0
$$
is exact. If 2 is invertible in $A$, then $T \mapsto 2(X + \sqrt{\lambda}Y)$ defines an isomorphism over $B$:

$$\sigma : G_{B/A} \otimes_A B = \text{Spec} B[X,Y]/(X^2 - \lambda Y^2 - Y) \sim \mathcal{G}_B^{(\sqrt{\lambda})} = \text{Spec} B[T, \frac{1}{1 + \sqrt{\lambda}T}]$$

The inverse of $\sigma$ is given by

$$X \mapsto \frac{2T + \sqrt{\lambda}T^2}{4(1 + \sqrt{\lambda}T)}, \quad Y \mapsto \frac{T^2}{4(1 + \sqrt{\lambda}T)}$$

Furthermore,

$$U \mapsto 1 + 2\lambda Y, \quad V \mapsto 2X$$

define a homomorphism

$$\alpha : G_{B/A} = \text{Spec} A[X,Y]/(X^2 - \lambda Y^2 - Y) \rightarrow U_{B/A} = \text{Spec} A[U,V]/(U^2 - \lambda V^2 - 1)$$

If $2\lambda$ is invertible in $A$, then $\alpha$ is an isomorphism. Indeed, the inverse of $\alpha$ is given by

$$X \mapsto -\frac{V}{2}, \quad Y \mapsto -\frac{1 - U}{2\lambda}$$

2. Deformations of the Kummer sequence

Throughout this section, $A$ denotes an $\mathbb{F}_p$-algebra. We fix $\mu \in A$ and put $N_A = \text{Ker}[F - \mu I : G_{a,A} \rightarrow G_{a,A}]$ and $G = N_A^\vee$.

**Definition 2.1.** Let $A$ be an $\mathbb{F}_p$-algebra and $\mu \in A$. Put $N_A = \text{Ker}[F - \mu I : G_{a,A} \rightarrow G_{a,A}]$. Then $N_A$ is a commutative group scheme, finite and flat of order $p$ over $A$. Indeed, $N_A = \text{Spec} A[T]/(T^p - \mu T)$ and the addition is given by $T \mapsto T \otimes 1 + 1 \otimes T$.

**Lemma 2.2.** Under the notation of 2.1, let $R$ be an $A$-algebra and $a \in R$. If $a^p = 0$, then

$$U \mapsto \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i$$

defines a homomorphism of group schemes

$$c : N_R = \text{Spec} R[T]/(T^p - \mu T) \rightarrow \mathbb{G}_{m,R} = \text{Spec} R[U, 1/U].$$

Furthermore, the map

$$a \mapsto \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i$$

gives rise to a bijection between $\text{Ker}[F : R \rightarrow R]$ and $\text{Hom}_{R-\text{gr}}(N_R, \mathbb{G}_{m,R})$. 
Proof. Put \( f(T) = \sum_{i=0}^{p-1} \frac{a_i}{i!} T^i \). If \( a^p = 0 \), then \( f(T) \) is invertible in \( R[T]/(T^p - \mu T) \). Moreover, we obtain a functional equation \( f(X + Y) = f(X) f(Y) \). Hence \( U \mapsto f(T) \) defines a homomorphism of group \( R \)-schemes \( c : N_R \to G_{m,R} \). Conversely, assume that \( U \mapsto f(T) = \sum_{i=0}^{p-1} a_i T^i \) defines a homomorphism of group \( R \)-schemes

\[
c : N_R = \text{Spec} \ R[T]/(T^p - \mu T) \to G_{m,R} = \text{Spec} \ R[1/U].
\]

Then we obtain (1) \( f(0) = 1 \), (2) \( f(X + Y) = f(X) f(Y) \). By (1), \( a_0 = 1 \). Furthermore, comparing the coefficients of \( X^i Y^j \) in

\[
f(X + Y) = 1 + a_1 (X + Y) + a_2 (X + Y)^2 + \cdots + a_{p-1} (X + Y)^{p-1}
\]

and

\[
f(X) f(Y) = (1 + a_1 X + a_2 X^2 + \cdots + a_{p-1} X^{p-1})
\times (1 + a_1 Y + a_2 Y^2 + \cdots + a_{p-1} Y^{p-1})
\]

for each \( i, j \), we obtain

\[
a_i a_j = \begin{cases} \binom{i+j}{i} a_{i+j} & (i+j < p) \\ 0 & (i+j \geq p) \end{cases}
\]

In particular, we have \( a_1 a_{p-1} = 0 \) and \( i a_i = a_1 a_{i-1} \) for each \( i \geq 1 \). It follows that \( a_i = \frac{a_i}{i!} \) for \( 1 \leq i < p \) and \( a^p_i = 0 \).

Notation 2.3. Let \( p \) be a prime number. We put

\[
W(X, Y) = \frac{X^p + Y^p - (X + Y)^p}{p} = - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X^{p-i} Y^i \in \mathbb{Z}[X, Y].
\]

Definition 2.4. Let \( A \) be an \( \mathbb{F}_p \)-algebra and \( \mu \in A \). Define a finite flat commutative group scheme \( G \) over \( A \) by \( G = \text{Spec} \ A[T]/(T^p) \) with

(a) the multiplication:

\[
T \mapsto T \otimes 1 + 1 \otimes T + \mu W(T \otimes 1, 1 \otimes T) = T \otimes 1 + 1 \otimes T - \mu \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} T^{p-i} \otimes T^i,
\]

(b) the unit: \( T \mapsto 0 \),

(c) the inverse: \( T \mapsto -T \).

Proposition 2.5. Let \( A \) be an \( \mathbb{F}_p \)-algebra and \( \mu \in A \). Then the Cartier dual \( N_A^\vee \) of \( N_A = \text{Ker} (F - \mu I : \mathbb{G}_{a,A} \to \mathbb{G}_{a,A}) \) is isomorphic to the group
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scheme

$$G = \text{Spec } A[T]/(T^p)$$

with the multiplication

$$\Delta : T \mapsto T \otimes 1 + 1 \otimes T + \mu W(T \otimes 1, 1 \otimes T).$$

Proof. For an $A$-algebra $R$, we have $G(R) = \{a \in R ; a^p = 0\}$. Therefore, the map $\eta : G(R) \to N^\wedge(R) = \text{Hom}_{R-\text{gr}}(N_R, \mathbb{G}_{m,R})$ defined by

$$a \mapsto \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i$$

is bijective by Lemma 2.4. Moreover, for any $a, b \in G(R)$, we have

$$\left( \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i \right) \left( \sum_{i=0}^{p-1} \frac{b^i}{i!} T^i \right) = \sum_{i=0}^{p-1} \frac{c^i}{i!} T^i$$

for some $c \in G(R)$. Comparing the coefficients of $T$, we obtain

$$c = a + b + \mu \sum_{i=1}^{p-1} \frac{1}{(p-i)!} a^{p-i} b^i = a + b - \mu \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a^{p-i} b^i$$

since $T^p = \mu T$ in $R[T]/(T^p - \mu T)$. Therefore the map $\eta : G(R) \to N^\wedge_A(R) = \text{Hom}_{R-\text{gr}}(N_R, \mathbb{G}_{m,R})$ is an isomorphism of groups.

Remark 2.6. The Cartier duality asserts that the character group $\text{Hom}_{R-\text{gr}}(G \otimes_A R, \mathbb{G}_{m,R})$ is isomorphic to $N_A(R)$ for any $A$-algebra $R$. The assertion is verified directly as follows.

Let $R$ be an $A$-algebra and $a \in R$. If $a^p = \mu a$, then

$$U \mapsto \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i$$

defines a homomorphism of group schemes

$$G \otimes_A R = \text{Spec } R[T]/(T^p) \to \mathbb{G}_{m,R} = \text{Spec } R[U, 1/U]$$

since

$$\sum_{i=1}^{p-1} \frac{a^i}{i!} \left( X + Y + \mu W(X,Y) \right)^i \equiv \left( \sum_{i=0}^{p-1} \frac{a^i}{i!} X^i \right) \left( \sum_{i=0}^{p-1} \frac{a^i}{i!} Y^i \right) \mod (X^p, Y^p).$$

Furthermore,

$$a \mapsto \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i$$

gives rise to a bijection

$$\xi : N_A(R) = \{a \in R ; a^p = \mu a\} \sim \text{Hom}_{R-\text{gr}}(G \otimes_A R, \mathbb{G}_{m,R}).$$
In fact, assume that $U \mapsto f(T) = \sum_{i=0}^{p-1} a_i T^i$ defines a homomorphism of group $R$-schemes

$$G \otimes_A R = \text{Spec } R[T]/(T^p) \to \mathbb{G}_{m,R} = \text{Spec } R[U, 1/U].$$

Then we obtain (1) $f(0) = 1$ and (2) $f(X + Y + \mu W(X, Y)) = f(X)f(Y)$.

By (1), $a_0 = 1$. Furthermore, comparing the coefficients of $X^i Y^j$ in

$$f(X + Y + \mu W(X, Y)) = 1 + a_1 \{X + Y + \mu W(X, Y)\}
+ a_2 \{X + Y + \mu W(X, Y)\}^2 + \cdots
+ a_{p-1} \{X + Y + \mu W(X, Y)\}^{p-1}$$

and

$$f(X)f(Y) = (1 + a_1 X + a_2 X^2 + \cdots + a_{p-1} X^{p-1})
\times (1 + a_1 Y + a_2 Y^2 + \cdots + a_{p-1} Y^{p-1})$$

for each $i, j$, we obtain

$$a_i a_j = \begin{cases} \binom{i+j}{i} a_{i+j} & (i+j < p) \\ -(i+j-p+1) \frac{1}{p} (i+j-p+1) \binom{i+j}{i} a_{i+j-p+1} & (i+j \geq p) \end{cases}$$

In particular, we have $a_1 a_{p-1} = \mu a_1$ and $i a_i = a_1 a_{i-1}$ for each $i \geq 1$. It follows that $a_i = \frac{a_i^p}{i!}$ for $1 \leq i < p$ and $a_1^p = \mu a_1$. Hence $\xi$ is surjective. It is readily seen that $\xi$ is injective.

Moreover, for any $a, b \in N_A(R)$, we have

$$\left( \sum_{i=0}^{p-1} a_i T^i \right) \left( \sum_{i=0}^{p-1} b_i T^i \right) = \sum_{i=0}^{p-1} c_i T^i$$

for some $c \in N_A(R)$. Comparing the coefficients of $T$, we obtain $c = a + b$ since $T^p = 0$ in $R[T]/(T^p)$. Therefore the map $\xi : N_A(R) \to G^\vee(R) = \text{Hom}_{R_{gr}}(G \otimes_A R, \mathbb{G}_{m,R})$ is an isomorphism of groups.

**Theorem 2.7.** Let $A$ be an $\mathbb{F}_p$-algebra and $\mu \in A$. If $\mu = \lambda^{p-1}$ for some $\lambda \in A$, then $G$ is isomorphic to

$$\text{Ker}[F : G_A^{(\lambda)} \to G_A^{(\lambda^p)}] = \text{Spec } A[X]/(X^p)$$

with the multiplication

$$\Delta : X \mapsto X \otimes 1 + 1 \otimes X + \lambda X \otimes X.$$

Here $F$ denotes the absolute Frobenius map.
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**Proof.** Define a homomorphism of $A$-algebra $\tilde{\eta} : A[X]/(X^p) \to A[T]/(T^p)$ by

$$X \mapsto \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{i!} T^i.$$

Then $\tilde{\eta}$ is an isomorphism. Indeed, the inverse of $\tilde{\eta}$ is given by

$$T \mapsto \sum_{i=1}^{p-1} \frac{(-\lambda)^{i-1}}{i} X^i.$$

Hereafter we show that $\tilde{\eta}$ is a Hopf homomorphism. It is sufficient to verify that

$$\sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} \left\{ X + Y + \lambda^{p-1} W(X,Y) \right\}^k = \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} X^k + \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} Y^k + \lambda \left( \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} X^k \right) \left( \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} Y^k \right)$$

in $A[X,Y]/(X^p, Y^p)$. At first note that

$$\sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} X^k + \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} Y^k + \lambda \left( \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} X^k \right) \left( \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} Y^k \right)$$

$$= \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} X^k + \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} Y^k + \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} \left\{ \sum_{l=1}^{k-1} \frac{1}{l!(k-l)!} X^{k-l} Y^l \right\}$$

$$= \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} (X + Y)^k + \sum_{k=1}^{p-1} \lambda^{k+p-2} \left( \sum_{l=k}^{p-1} \frac{1}{(k+p-1-l)!} (X + Y)^{k+p-1-l} Y^l \right).$$

We have

$$\sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} \left\{ X + Y + \lambda^{p-1} W(X,Y) \right\}^k$$

$$= \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} \left\{ (X + Y)^k + k \lambda^{p-1} (X + Y)^{k-1} W(X,Y) \right\}$$

$$= \sum_{k=1}^{p-1} \frac{\lambda^{k-1}}{k!} (X + Y)^k + \sum_{k=1}^{p-1} \frac{\lambda^{k+p-2}}{(k-1)!} (X + Y)^{k-1} W(X,Y) \right\}$$
since $X^p = Y^p = 0$. Note now that we have a congruence relation
\[
(X + Y)^{k-1}W(X, Y) = (X + Y)^{k-1} \frac{X^p + Y^p - (X + Y)^p}{p}
\]
\[
\equiv -\frac{1}{p} (X + Y)^{k+p-1}
\]
\[
\equiv -\sum_{l=k}^{p-1} \frac{1}{l!} \binom{k + p - 1}{l} X^{k+p-1-l}Y^l \mod (X^p, Y^p)
\]
in $\mathbb{Q}[X, Y]$, and therefore
\[
(X + Y)^{k-1}W(X, Y) \equiv -\sum_{l=k}^{p-1} \frac{1}{l!} \binom{k + p - 1}{l} X^{k+p-1-l}Y^l \mod (X^p, Y^p)
\]
in $\mathbb{Z}[X, Y]$. Moreover, we have
\[
-\frac{1}{p} \binom{k + p - 1}{l}
\]
\[
\equiv -\frac{1}{p!} (k + p - 1)(k + p - 2) \cdots \times (k + p - k + 1)(k + p - k)(k + p - k - 1) \cdots (k + p - l)
\]
\[
\equiv -\frac{(k - 1)!(p - 1)(p - 2) \cdots (k + p - l)}{l!}
\]
\[
\equiv -\frac{(k - 1)!(p - 1)!}{(k + p - l - 1)!l!}
\]
\[
\equiv \frac{(k - 1)!}{(k + p - l - 1)!l!} \mod p
\]
in $\mathbb{Z}_{(p)}$. Hence the result.

**Remark 2.8.** We obtain an exact sequence of group $A$-schemes
\[
0 \longrightarrow G \xrightarrow{\eta} G^{(\lambda)}_A \xrightarrow{F} G^{(\lambda p)}_A \longrightarrow 0.
\]
When $\lambda = 0$, the sequence (1) is nothing but the radicial sequence
\[
0 \longrightarrow \alpha_{p,A} \longrightarrow \mathbb{G}_{m,A} \xrightarrow{F} \mathbb{G}_{m,A} \longrightarrow 0.
\]

On the other hand, if $\lambda$ is invertible $A$, we have a commutative diagram of group $A$-schemes with exact rows
\[
0 \longrightarrow G \xrightarrow{\eta} G^{(\lambda)}_A \xrightarrow{F} G^{(\lambda p)}_A \longrightarrow 0
\]
\[
0 \longrightarrow \alpha_{p,A} \longrightarrow \mathbb{G}_{m,A} \xrightarrow{F} \mathbb{G}_{m,A} \longrightarrow 0.
\]
Therefore, the exact sequence (1) gives a deformation of the Kummer sequence to the radicial sequence.

**Corollary 2.9.** Let $R$ be an $A$-algebra. If $R$ is a local ring or $\lambda$ is nilpotent, then $H^1(R, G)$ is isomorphic to $\text{Coker}[F : G^A(R) \rightarrow G^{(\lambda^p)}(R)]$.

**Proof.** From the exact sequence of group schemes over $R$

$$0 \rightarrow G \rightarrow G^A(\lambda) \xrightarrow{F} G^{(\lambda^p)}(\lambda) \rightarrow 0,$$

we obtain a long exact sequence

$$G(\lambda) \xrightarrow{F} G^{(\lambda^p)}(\lambda) \rightarrow H^1(R, G) \rightarrow H^1(R, G^A(\lambda)) \xrightarrow{F} H^1(R, G^{(\lambda^p)}).$$

We know that $H^1(R, G^A(\lambda)) = 0$ under the assumption ([5], Cor 1.3), which implies the assertion.

The above assertion is restated as follows:

**Corollary 2.10.** Let $R$ be an $A$-algebra and $S$ an $R$-algebra. Assume that $\text{Spec} \, S$ has a structure of $G$-torsor over $\text{Spec} \, R$. If $R$ is a local ring or $\lambda$ is nilpotent, then there exists a morphism $\text{Spec} \, R \rightarrow G^{(\lambda^p)}$ such that the square

$$\begin{array}{ccc}
\text{Spec} \, R & \longrightarrow & G^{(\lambda^p)} \\
\downarrow & & \downarrow F \\
\text{Spec} \, S & \longrightarrow & G^{(\lambda^p)}
\end{array}$$

is cartesian. More precisely, $S$ is isomorphic to $R[X]/(X^p - a)$ for some $a \in R$ with $1 + \lambda^p a \in R^\times$, and the action of $G$ on $\text{Spec} \, S$ over $R$ is defined by

$$R[X]/(X^p - a) \rightarrow R[T]/(T^p) \otimes_R R[X]/(X^p - a)$$

$$X \mapsto \sum_{i=1}^{p-1} \frac{\lambda^i}{i!} T^i \otimes 1 + \sum_{i=0}^{p-1} \frac{\lambda^i}{i!} T^i \otimes X.$$

Hereafter we study a quadratic twist of the exact sequence (1).

**Notation 2.11.** Let $A$ be a ring and $\lambda \in A$. Put $B = A[\sqrt{\lambda}] = A[t]/(t^2 - \lambda)$ and $\tilde{B} = A[\sqrt{\lambda^p}] = A[t]/(t^2 - \lambda^p)$. As is done in 1.4, we define group schemes $G_{B/A}$ and $G_{\tilde{B}/A}$ over $A$ by

$$G_{B/A} = \text{Spec} \, A[X, Y]/(X^2 - \lambda Y^2 - Y)$$

with the multiplication:

$$X \mapsto X \otimes 1 + 1 \otimes X + 2\lambda X \otimes Y + 2\lambda Y \otimes X, \quad Y \mapsto Y \otimes 1 + 1 \otimes Y + 2\lambda Y \otimes Y + 2X \otimes X.$$
with the multiplication:

\[
X \mapsto X \otimes 1 + 1 \otimes X + 2\lambda^p X \otimes Y + 2\lambda^p Y \otimes X, \\
Y \mapsto Y \otimes 1 + 1 \otimes Y + 2\lambda^p Y \otimes Y + 2X \otimes X.
\]

Furthermore a homomorphism of group \( A \)-schemes

\[
F : G_{B/A} = \text{Spec } A[X, Y]/(X^2 - \lambda Y^2 - Y) \\
\to G_{\tilde{B}/A} = \text{Spec } A[X, Y]/(X^2 - \lambda^p Y^2 - Y)
\]

is defined by

\[
X \mapsto X^p, \\
Y \mapsto Y^p.
\]

It is readily seen that \( F : G_{B/A} \to G_{\tilde{B}/A} \) is finite flat.

**Theorem 2.12.** Let \( p \) be a prime number > 2, \( A \) an \( \mathbb{F}_p \)-algebra and \( \mu \in A \). If \( \mu = \lambda(p-1)/2 \) for some \( \lambda \in A \), then \( G \) is isomorphic to

\[
\text{Ker}[F : G_{B/A} \to G_{\tilde{B}/A}] = \text{Spec } A[X, Y]/(X^2 - \lambda^p Y^2 - Y, X^p, Y^p)
\]

with the multiplication

\[
\Delta : X \mapsto X \otimes 1 + 1 \otimes X + 2\lambda X \otimes Y + 2\lambda Y \otimes X, \\
Y \mapsto Y \otimes 1 + 1 \otimes Y + 2\lambda Y \otimes Y + 2X \otimes X.
\]

Here \( F \) denotes the absolute Frobenius map.

**Proof.** We verify that

\[
X \mapsto \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i - 1)!} T^{2i-1}, \\
Y \mapsto \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} T^{2i}
\]

defines a homomorphism of group schemes

\[
\xi : G = \text{Spec } A[X]/(T^p) \to G_{B/A} = \text{Spec } A[X, Y]/(X^2 - \lambda Y^2 - Y)
\]

Noting

\[
\frac{\sinh \sqrt{AT}}{\sqrt{A}} = \sum_{i=1}^{\infty} \frac{A^{i-1}}{(2i - 1)!} T^{2i-1}, \\
\frac{\cosh \sqrt{AT} - 1}{A} = \sum_{i=1}^{\infty} \frac{A^{i-1}}{(2i)!} T^{2i},
\]

and

\[
\left( \frac{1}{2} \frac{\sinh \sqrt{AT}}{\sqrt{A}} \right)^2 - A \left( \frac{1}{2} \frac{\cosh \sqrt{AT} - 1}{A} \right)^2 - \frac{1}{2} \frac{\cosh \sqrt{AT} - 1}{A} = 0
\]

we obtain an identity in \( \mathbb{Q}[A][[T]] \)

\[
\left\{ \frac{1}{2} \sum_{i=1}^{\infty} \frac{A^{i-1}}{(2i - 1)!} T^{2i-1} \right\}^2 - A \left\{ \frac{1}{2} \sum_{i=1}^{\infty} \frac{A^{i-1}}{(2i)!} T^{2i} \right\}^2 - \frac{1}{2} \sum_{i=1}^{\infty} \frac{A^{i-1}}{(2i)!} T^{2i} = 0,
\]
and therefore an identity in $\mathbb{Q}[A][T]/(T^p)$

$$\left\{ \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} T^{2i-1} \right\}^2 - A\left\{ \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} T^{2i} \right\}^2 - \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} T^{2i} = 0,$$

which reads as an identity in $\mathbb{F}_p[A][T]/(T^p)$

$$\left\{ \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} T^{2i-1} \right\}^2 - A\left\{ \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} T^{2i} \right\}^2 - \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} T^{2i} = 0$$

since

$$\sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} T^{2i-1}, \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} T^{2i} \in \mathbb{Z}_p[A][T].$$

Specializing $A$ to $\lambda$, we obtain

$$\left\{ \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i-1)!} T^{2i-1} \right\}^2 - \lambda\left\{ \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} T^{2i} \right\}^2 - \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} T^{2i} = 0$$

in $A[T]/(T^p)$, which implies that

$$X \mapsto \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i-1)!} T^{2i-1}, \ Y \mapsto \frac{p-1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} T^{2i}$$

defines a homomorphism of $A$-algebras

$$\tilde{\xi} : A[X,Y]/(X^2 - \lambda Y^2 - Y, X^p, Y^p) \to A[T]/(T^p).$$

Furthermore, as is remarked in 1.4,

$$X \mapsto 2(X + \sqrt{\lambda}Y)$$
gives rise to an isomorphism of group scheme over $B = A[\sqrt{\lambda}]$

$$\sigma : G_{B/A} \otimes_A B = \text{Spec } B[X,Y]/(X^2 - \lambda Y^2 - Y) \cong G_B^{(\sqrt{\lambda})} = \text{Spec } B[X, \frac{1}{1 + \sqrt{\lambda}X}].$$

On the other hand,

$$X \mapsto \sum_{i=1}^{p-1} \frac{\sqrt{\lambda}^{i-1}}{i!} T^i$$
gives an isomorphism of group scheme over $B$

$$\eta_B : G \otimes_A B = \text{Spec } B[T]/(T^p) \cong \text{Ker}[F : G_B^{(\sqrt{\lambda})} \to G_B^{(\sqrt{\lambda}^p)}] = \text{Spec } B[X]/(X^p).$$
Moreover, we have $\sigma \circ \eta_B = \xi_B$ since
\[
2\left\{ \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda_i - 1}{(2i - 1)!} T^{2i-1} + \sqrt{\lambda} \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda_i - 1}{(2i)!} T^{2i} \right\} = \sum_{i=1}^{p-1} \sqrt{\lambda} \frac{1}{i!} T^i.
\]
Hence, $\xi_B$ is an homomorphism of group schemes over $B$. It follows that $\xi$ is an homomorphism of group schemes over $A$ since $B = A[\sqrt{\lambda}]$ is faithfully flat over $A$.

**Remark 2.13.** We obtain an exact sequence of group schemes
\[
0 \rightarrow G \xrightarrow{\xi} G_{B/A} \xrightarrow{F} G_{\tilde{B}/A} \rightarrow 0.
\]
In the proof of 2.12, we obtained an isomorphism of exact sequences:
\[
0 \rightarrow G \xrightarrow{\xi_B} G_{B/A} \xrightarrow{F} G_{\tilde{B}/A} \rightarrow 0,
\]
\[
0 \rightarrow G \xrightarrow{\eta_B} G^{(\sqrt{\lambda})} \xrightarrow{F} G^{(\sqrt{\lambda^p})} \rightarrow 0.
\]
That is to say, the sequence (2) is a quadratic twist of (1).

**Corollary 2.14.** Let $R$ be an $A$-algebra. If $R$ is a local ring or $\lambda$ is nilpotent, then $H^1(R, G)$ is isomorphic to $\text{Coker}[F : G_{B/A}(R) \rightarrow G_{\tilde{B}/A}(R)]$.

**Proof.** From the exact sequence of group schemes over $R$
\[
0 \rightarrow G \rightarrow G_{B/A} \xrightarrow{F} G_{\tilde{B}/A} \rightarrow 0,
\]
we obtain a long exact sequence
\[
G_{B/A}(R) \xrightarrow{F} G_{\tilde{B}/A}(R) \rightarrow H^1(R, G) \rightarrow H^1(R, G_{B/A}) \xrightarrow{F} H^1(R, G_{\tilde{B}/A}).
\]
We know that $H^1(R, G_{B/A})$ is annihilated by 2 under the assumption ([9], Prop 4.3.) and that $H^1(R, G)$ is annihilated by $p$, which imply the assertion.

The above assertion is restated as follows:

**Corollary 2.15.** Let $R$ be an $A$-algebra and $S$ an $R$-algebra. Assume that $\text{Spec } S$ has a structure of $G$-torsor over $\text{Spec } R$. If $R$ is a local ring or $\lambda$ is nilpotent, then there exists a morphism $\text{Spec } R \rightarrow G_{\tilde{B}/A}$ such that the square
\[
\text{Spec } S \xrightarrow{G_{B/A}} \xrightarrow{F} \text{Spec } R \xrightarrow{G_{\tilde{B}/A}}
\]
is cartesian. More precisely, $S$ is isomorphic to
\[
R[X, Y]/(X^p - a, Y^p - b, X^2 - \lambda Y^2 - Y)
\]
Degeneration of the Kummer sequence in characteristic $p > 0$

for some $a, b \in R$ with $a^2 - \lambda p b^2 - b = 0$, and the action of $G$ on $\text{Spec } S$ over $R$ is defined by

\[ R[X, Y]/(X^p - a, Y^p - b, X^2 - \lambda Y^2 - Y) \]

\[ \to R[T]/(T^p) \otimes_R R[X, Y]/(X^p - a, Y^p - b, X^2 - \lambda Y^2 - Y) : \]

\[ X \mapsto \frac{p-1}{2} \lambda^{-1} \frac{i}{(2i-1)!} T^{2i-1} \otimes 1 + \sum_{i=1}^{p-1} \frac{\lambda^i}{(2i-1)!} T^{2i-1} \otimes Y + \sum_{i=0}^{p-1} \frac{\lambda^i}{(2i)!} T^{2i} \otimes X, \]

\[ Y \mapsto \frac{p-1}{2} \lambda^{-1} \frac{i}{(2i)!} T^{2i} \otimes 1 + \sum_{i=0}^{p-1} \frac{\lambda^i}{(2i)!} T^{2i} \otimes Y + \sum_{i=1}^{p-1} \frac{\lambda^{-1}}{(2i-1)!} T^{2i-1} \otimes X. \]

**Remark 2.16.** The Artin-Hasse exponential series $E_p(T) \in \mathbb{Z}_{(p)}[[T]]$ is defined by

\[ E_p(T) = \exp \left( \sum_{r=0}^{\infty} \frac{T^p}{p^r} \right). \]

For an $\mathbb{Z}_{(p)}$-algebra $R$ and $a = (a_k)_{k \geq 0} \in R^N$, we define $E_p(a; T) \in A[[T]]$ by

\[ E_p(a; T) = \prod_{k=0}^{\infty} E_p(a_k T^p^k). \]

It is known that

\[ E_p(a + b; T) = E_p(a; T) E_p(b; T) \]

where $+$ denotes the addition of Witt vectors.

Let $\hat{W}$ denote the formal completion of the additive group scheme of Witt vectors. Then, if $R$ is an $\mathbb{F}_p$-algebra, we have

\[ \hat{W}(R) = \left\{ (a_0, a_1, a_2, \ldots) \in W(R) : a_i \text{ is nilpotent for all } i \text{ and } a_i = 0 \right\} \text{ for all but a finite number of } i \].

Moreover,

\[ a = (a_k)_{k \geq 0} \mapsto E_p(a; T) = \prod_{k=0}^{\infty} \left( \sum_{i=0}^{p-1} \frac{a_i}{i!} T^p^k \right) \]

gives rise to an isomorphism

\[ \eta : F \hat{W}(R) = \text{Ker}[F : \hat{W}(R) \to \hat{W}(R)] \isom \text{Hom}_{R-\text{gr}}(\mathbb{G}_{a,R}, \mathbb{G}_{m,R}) \]

(cf. [1, Ch II, Sec 2, 2.7]). Under this identification,

\[ F - \mu I : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R} \]

induces

\[ V - [\mu]I : F \hat{W}(R) \to F \hat{W}(R). \]
In fact, if $a \in F \hat{W}(R)$, we have
\[
E_p(a; T^p - \mu T) = \prod_{k=0}^{\infty} \left( \sum_{i=0}^{p-1} \frac{a_k^i}{i!} (T^p - \mu^p)^i \right)
\]
\[
= \prod_{k=0}^{\infty} \left( \sum_{i=0}^{p-1} \frac{a_k^i}{i!} (T^{p+1} - \mu^p T^p)^i \right).
\]

Now, by the functional equation of the exponential series, we obtain
\[
\sum_{i=0}^{p-1} \frac{a_k^i}{i!} (T^{p+1} - \mu^p T^p)^i = \left( \sum_{i=0}^{p-1} \frac{a_k^i}{i!} (T^{p+1})^i \right) \left( \sum_{i=0}^{p-1} \frac{(\mu^p a_k^i)^i}{i!} (T^p)^i \right)^{-1}
\]
for each $k$ since we have $a_k^p = 0$. Therefore, we have gotten
\[
E_p(a; T^p - \mu T) = \prod_{k=0}^{\infty} \left( \sum_{i=0}^{p-1} \frac{a_k^i}{i!} (T^{p+1})^i \right) \left( \sum_{i=0}^{p-1} \frac{(\mu^p a_k^i)^i}{i!} (T^p)^i \right)^{-1}
\]
\[
= E_p(Va; T)E_p([\mu]a; T)^{-1}
\]
\[
= E_p((V - [\mu])a; T).
\]

Moreover, we obtain a commutative diagram with exact rows:
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}(G_{a,R}, G_{m,R}) & \stackrel{(\varepsilon - [\mu]I)^*}{\longrightarrow} & \text{Hom}(G_{a,R}, G_{m,R}) & \longrightarrow & \text{Hom}(N, G_{m,R}) & \longrightarrow & 0 \\
0 & \longrightarrow & F \hat{W}(R) & \stackrel{V - [\mu]I}{\longrightarrow} & F \hat{W}(R) & \longrightarrow & G(R) & \longrightarrow & 0.
\end{array}
\]

**Remark 2.17.** Put $A_p = \mathbb{Z}[\zeta, 1/p(p + 1)] \cap \mathbb{Z}_p$, where $\zeta$ is a primitive $(p - 1)$-th root of unity in the ring of $p$-adic integers. For any scheme $S$ over $A_p$. In [11] Tate and Oort defined a commutative group scheme $G_{a,b}^L$ over $S$, where $L$ is an invertible $\mathcal{O}_S$-module and $a \in \Gamma(S, L^{\otimes (p-1)})$, $b \in \Gamma(S, L^{\otimes (1-p)})$ with $a \otimes b = p$. The group scheme $G_{a,b}^L$ is finite flat of order $p$ over $S$, and the Cartier dual $(G_{a,b}^L)^\vee$ is isomorphic to $G_{b,a}^{L^\vee}$. If $A$ is an $\mathbb{F}_p$-algebra, $S = \text{Spec} A$ and $L = \mathcal{O}_S$, then we have $ab = 0$ and
\[
G_{a,b}^L = \text{Spec} A[T]/(T^p - aT)
\]
with the multiplication
\[
\Delta : T \mapsto T \otimes 1 + 1 \otimes T + bW(T \otimes 1, 1 \otimes T).
\]
In particular, we have $N = G_{\mu,0}^A$ and $G = G_{0,\mu}^A$. 

3. Relations with the Grothendieck resolution

Throughout the section, $A$ denotes an $\mathbb{F}_p$-algebra.

3.1. First we recall a resolution of a finite flat commutative group scheme by smooth affine commutative group schemes, constructed by Grothendieck (cf. [3. Sec 6]). Let $S$ be a scheme and $F$ an affine commutative $S$-group scheme such that $\mathcal{O}_F$ is a locally free $\mathcal{O}_S$-module of finite rank. Then the functor $\text{Hom}_{S-\text{gr}}(F, \mathbb{G}_{m,S})$ is represented by a commutative group scheme $F^\vee$, called the Cartier dual of $F$. The $\mathcal{O}_S$-module $\mathcal{O}_{F^\vee}$ is also locally free of finite rank. The Cartier duality asserts that $\text{Hom}_{S-\text{gr}}(F^\vee, \mathbb{G}_{m,S})$ is isomorphic to $F$.

Furthermore the functor $\text{Hom}_{S-\text{sch}}(F^\vee, \mathbb{G}_{m,S})$ is nothing but the Weil restriction $\prod_{F^\vee/S} \mathbb{G}_{m,F^\vee}$, which is representable since $\mathcal{O}_{F^\vee}$ is a locally free $\mathcal{O}_S$-module of finite rank (cf. [1, Ch.I, Sec.1,6.6]). Then we obtain an exact sequence of commutative group schemes:

$$0 \to F \xrightarrow{i} \prod_{F^\vee/S} \mathbb{G}_{m,F^\vee} \to \left( \prod_{F^\vee/S} \mathbb{G}_{m,F^\vee} \right)/F \to 0.$$ 

The Weil restriction $\prod_{F^\vee/S} \mathbb{G}_{m,F^\vee}$ is smooth over $S$ since $\mathbb{G}_{m,F^\vee}$ is smooth over $F^\vee$, and therefore the quotient $\left( \prod_{F^\vee/S} \mathbb{G}_{m,F^\vee} \right)/F$ is also smooth over $S$.

The canonical map

$$H^1(S, \prod_{F^\vee/S} \mathbb{G}_{m,F^\vee}) \to H^1(F^\vee, \mathbb{G}_{m,F^\vee}) = \text{Pic}(F^\vee)$$

is an isomorphism since $F^\vee$ is finite over $S$ and $\mathbb{G}_{m,F^\vee}$ is smooth over $F^\vee$. Let $X$ be an $F$-torsor over $S$. Then the inclusion $F \to \prod_{F^\vee/S} \mathbb{G}_{m,F^\vee}$ defines a class $[X]$ in $\text{Pic}(F^\vee)$.

First we treat the sequence: (0) $0 \to N_A \to \mathbb{G}_{m,A} \xrightarrow{F-\mu_1} \mathbb{G}_{m,A} \to 0$.

3.2. Let $A$ be an $\mathbb{F}_p$-algebra, and $B = A[T]/(T^p)$. Then $\prod_{B/A} \mathbb{G}_{m,B}$ is represented by

$$\text{Spec } A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{T_0}]$$
with the multiplication
\[ T_k \mapsto \sum_{i=0}^{k} T_{k-1} \otimes T_i \ (0 \leq k \leq p - 1) \]

with the unit
\[ T_0 \mapsto 1, \ T_k \mapsto 0 \ (1 \leq k \leq p - 1) \]

In fact, let \( R \) be an \( A \)-algebra. The multiplication of \( R[T]/(T^p) = R \otimes_A A[T]/(T^p) \) is given by
\[
\left( \sum_{i=0}^{p-1} a_i T^i \right) \left( \sum_{i=0}^{p-1} b_i T^i \right) = \sum_{k=1}^{p-1} \left( \sum_{i=0}^{p-1} a_{k-i} b_i \right) T^k.
\]

It is now sufficient to note that \( \sum_{k=0}^{p-1} a_k T^k \) is invertible in \( R[T]/(T^p) \) if and only if \( a_0 \) is invertible in \( R \).

**Theorem 3.3.** Let \( A \) be an \( \mathbb{F}_p \)-algebra and \( \mu \in A \). Put \( N_A = \text{Ker}[F - \mu I : \mathbb{G}_{a,A} \to \mathbb{G}_{a,A}] \) and \( B = A[T]/(T^p) \). Then:

1. A homomorphism of group schemes
\[
\tilde{\chi} : \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec} \ A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{T_0}] \to \mathbb{G}_{a,A} = \text{Spec} \ A[T]
\]
is defined by
\[ T \mapsto \frac{T_1}{T_0}. \]

Moreover, the diagram of group schemes
\[
\begin{array}{ccc}
N_A & \xrightarrow{i} & \prod_{B/A} \mathbb{G}_{m,B} \\
\| & & \downarrow \tilde{\chi} \\
N_A & \xrightarrow{\xi} & \mathbb{G}_{a,A}
\end{array}
\]
is commutative.

2. A homomorphism of group schemes
\[
\tilde{\sigma} : \mathbb{G}_{a,A} = \text{Spec} \ A[T] \to \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec} \ A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{T_0}]
\]
is defined by
\[ T_0 \mapsto 1, \ T_k \mapsto \frac{1}{k!} T^k \ (1 \leq k \leq p - 1). \]
Moreover, the diagram of group schemes

\[ N_A \xrightarrow{\xi} \mathbb{G}_{a,A} \]

\[ \parallel \quad \downarrow \tilde{\sigma} \]

\[ N_A \xrightarrow{i} \prod_{B/A} \mathbb{G}_{m,B} \]

is commutative, and \( \tilde{\sigma} \) is a section of \( \tilde{\chi} \).

**Proof.** The addition of \( \mathbb{G}_{a,A} \) is given by

\[ T \mapsto T \otimes 1 + 1 \otimes T. \]

On the other hand, we have

\[ \frac{T_1}{T_0} \mapsto \frac{T_1 \otimes T_0 + T_0 \otimes T_1}{T_0 \otimes T_0} = \frac{T_1}{T_0} \otimes 1 + 1 \otimes \frac{T_1}{T_0} \]

by the definition of multiplication of \( \prod_{B/A} \mathbb{G}_{m,B} \). Therefore \( \tilde{\chi} \) is a homomorphism of group. Furthermore, comparing

\[ \frac{1}{k!} T^k \mapsto \frac{1}{k!} (T \otimes 1 + 1 \otimes T)^k = \sum_{i=1}^{k} \frac{1}{(k-i)!} T^{k-i} \otimes \frac{1}{i!} T^i \]

and

\[ T_k \mapsto \sum_{i=0}^{k} T_{k-i} \otimes T_i, \]

we find that \( \tilde{\sigma} \) is group homomorphism.

We obtain the commutativity of the two squares, noting that

\[ i : N_A \to \prod_{B/A} \mathbb{G}_{m,B} \]

is defined by

\[ A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{T_0}] \to A[T]/(T^p - \mu T) \]

\[ T_0 \mapsto 1, \]

\[ T_k \mapsto \frac{1}{k!} T^k \quad (1 \leq k \leq p - 1). \]

Next we examine the exact sequence: (1) \[ 0 \to G \xrightarrow{\eta} \mathcal{G}_{A}^{(\lambda)} \xrightarrow{F} \mathcal{G}_{A}^{(\lambda^p)} \to 0. \]
Notation 3.4. Let $A$ be an $\mathbb{F}_p$-algebra and $\mu \in A$. Put

$$\Delta(\mu; T_0, T_1, \ldots, T_{p-1}) = \begin{vmatrix} T_0 & 0 & 0 & \ldots & 0 & O \\ T_1 & T_0 + \mu T_{p-1} & \mu T_{p-2} & \ldots & \mu T_2 & \mu T_1 \\ T_2 & T_1 & T_0 + \mu T_{p-1} & \ldots & \mu T_3 & \mu T_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ T_{p-2} & T_{p-3} & T_{p-4} & \ldots & T_0 + \mu T_{p-1} & \mu T_{p-2} \\ T_{p-1} & T_{p-2} & T_{p-3} & \ldots & T_1 & T_0 + \mu T_{p-1} \end{vmatrix}.$$ 

Proposition 3.5. Let $A$ be an $\mathbb{F}_p$-algebra and $\mu \in A$. Then $f(T) = \sum_{k=0}^{p-1} a_k T^k$ is invertible in $A[T]/(T^p - \mu T)$ if and only if $\Delta(\mu; a_0, a_1, \ldots, a_{p-1})$ is invertible in $A$.

Proof. The $A$-module $A[T]/(T^p - \mu T)$ has a basis $\{1, T, T^2, \ldots, T^{p-1}\}$. Moreover, we have

$$(1 \ T \ T^2 \ \ldots \ T^{p-1})(a_0 + a_1 T + a_2 T^2 + \cdots + a_{p-1} T^{p-1}) = (1 \ T \ T^2 \ \ldots \ T^{p-1})$$

$$\times \begin{pmatrix} a_0 & 0 & 0 & \ldots & 0 & 0 \\ a_1 & a_0 + \mu a_{p-1} & \mu a_{p-2} & \ldots & \mu a_2 & \mu a_1 \\ a_2 & a_1 & a_0 + \mu a_{p-1} & \ldots & \mu a_3 & \mu a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{p-2} & a_{p-3} & a_{p-4} & \ldots & a_0 + \mu a_{p-1} & \mu a_{p-2} \\ a_{p-1} & a_{p-2} & a_{p-3} & \ldots & a_1 & a_0 + \mu a_{p-1} \end{pmatrix}.$$ 

Hence the result.

Corollary 3.6. Let $A$ be an $\mathbb{F}_p$-algebra, $\mu \in A$ and $B = A[T]/(T^p - \mu T)$. Then $\prod_{B/A} \mathbb{G}_{m,B}$ is represented by

$$\text{Spec } A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\mu; T_0, T_1, \ldots, T_{p-1})}]$$

with

(a) the multiplication : $T_0 \mapsto T_0 \otimes T_0$, $T_k \mapsto \sum_{i=0}^{k} T_i \otimes T_{k-i} + \mu \sum_{i=k}^{p-1} T_i \otimes T_{k+p-i-1}$ $(1 \leq k \leq p-1)$,

(b) the unit : $T_0 \mapsto 1$, $T_k \mapsto 0$ $(1 \leq k \leq p-1)$.
Proof. Let $R$ be an $A$-algebra. The multiplication of $R[T]/(T^p - \mu T) = R \otimes_A A[T]/(T^p - \mu T)$ is given by

\[
\left( \sum_{i=0}^{p-1} a_i T^i \right) \left( \sum_{i=0}^{p-1} b_i T^i \right) = a_0 b_0 + \sum_{k=1}^{p-1} \left( \sum_{i=0}^{k} a_i b_{k-i} + \mu \sum_{i=k}^{p-1} a_i b_{k+p-i-1} \right) T^k.
\]

Lemma 3.7. Let $A$ be an $\mathbb{F}_p$-algebra and $\lambda \in A$. Then we have

\[
\Delta(\lambda^{p-1}; a_0, a_1, \ldots, a_{p-1}) = a_0 \prod_{k=1}^{p-1} \left\{ \sum_{l=0}^{p-1} (k \lambda)^l a_l \right\}
\]

for $a_0, a_1, \ldots, a_{p-1} \in A$.

Proof. We define $e_k(T) \in A[A, A^{-1}][T]$ $(0 \leq k < p)$ by

\[
e_0(T) = 1 - A^{-p+1}T^{p-1},
\]

\[
e_k(T) = 1 - A^{-p+1}(T - k\lambda)^{p-1} = -\sum_{l=1}^{p-1} (k \lambda)^{-l} T^l \hspace{1cm} (0 < k < p).
\]

Then we obtain

\[
e_k(j\lambda) = \begin{cases} 
1 & (j = k) \\
0 & (j \neq k).
\end{cases}
\]

Therefore $\{e_0(T), e_1(T), e_2(T), \ldots, e_{p-1}(T)\}$ is a basis over $A[A, A^{-1}]$ of $A[A, A^{-1}][T]/(T^p - A^{p-1}T)$. Moreover, we have

\[
1 = e_0(T) + e_1(T) + e_2(T) + \cdots + e_{p-1}(T),
\]

\[
T = A e_1(T) + 2A e_2(T) + \cdots + (p-1)A e_{p-1}(T),
\]

\[
T^2 = (A)^2 e_1(T) + (2A)^2 e_2(T) + \cdots + ((p-1)A)^2 e_{p-1}(T),
\]

\[
\vdots
\]

\[
T^{p-1} = A^{p-1} e_1(T) + (2A)^{p-1} e_2(T) + \cdots + ((p-1)A)^{p-1} e_{p-1}(T).
\]
Hence we obtain
\[
(e_0(T) e_1(T) e_2(T) \ldots e_{p-1}(T))(a_0 + a_1 T + a_2 T^2 + \ldots + a_{p-1} T^{p-1}) =
(e_0(T) e_1(T) e_2(T) \ldots e_{p-1}(T))
\]
\[
\begin{pmatrix}
    a_0 & 0 & 0 & \ldots & 0 \\
    0 & \sum_{k=0}^{p-1} \Lambda^k a_k & 0 & \ldots & 0 \\
    0 & 0 & \sum_{k=0}^{p-1} (2\Lambda)^k a_k & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & \sum_{k=0}^{p-1} ((p-1)\Lambda)^k a_k
\end{pmatrix}
\]

Therefore we obtain an identity in $A[\Lambda]
\[
\Delta(\Lambda^{p-1}; a_0, a_1, \ldots, a_{p-1}) = a_0 \prod_{k=1}^{p-1} \left\{ \left( \sum_{l=0}^{k-1} (k\Lambda)^l a_l \right) \right\}.
\]

Furthermore, we obtain the required result, specializing $\Lambda$ to $\lambda$.

Combining the above assertion with Proposition 3.5, we obtain the following:

**Corollary 3.8.** Let $A$ be an $\mathbb{F}_p$-algebra, $\lambda \in A$ and $f(T) \in A[T]/(T^p - \lambda^{p-1} T)$. Then $f(T)$ is invertible if and only if $f(j\lambda) \in A^\times$ for $0 \leq j < p$.

**Notation 3.9.** Let $A$ be an $\mathbb{F}_p$-algebra and $\lambda, a \in A$. We put
\[
F_p(\lambda, a; T) = 1 + aT + \frac{a^2}{2!} T(T - \lambda) + \frac{a^3}{3!} T(T - \lambda)(T - 2\lambda) + \ldots
\]
\[
\ldots + \frac{a^{p-1}}{(p-1)!} T(T - \lambda) \cdots (T - (p-2)\lambda).
\]

**Notation 3.10.** Recall now the definition of the Stirling number $S_{k,l}$ of first kind:
\[
T(T-1) \ldots (T - (k-1)) = \sum_{l=1}^{k} S_{k,l} T^l
\]

For example, we have
\[
S_{1,1} = 1, \quad S_{2,1} = -1, \quad S_{2,2} = 1, \\
S_{3,1} = 2, \quad S_{3,2} = -3, \quad S_{3,3} = 1, \\
S_{4,1} = -6, \quad S_{4,2} = 11, \quad S_{4,3} = -6, \quad S_{4,4} = 1, \\
S_{5,1} = 24, \quad S_{5,2} = -50, \quad S_{5,3} = 35, \quad S_{5,4} = -10, \quad S_{5,5} = 1.
\]
Lemma 3.11. Let $A$ be an $\mathbb{F}_p$-algebra and $\lambda, a \in A$. Then $F_p(\lambda, a; T)$ is invertible in $A[T]/(T^p - \lambda^p - 1)$ if and only if $1 + \lambda a$ is invertible in $A$.

Proof. By Corollary 3.8, we obtain the result since $F_p(\lambda, a; j\lambda) = (1 + \lambda a)^j$ for $1 \leq j < p$.

Theorem 3.12. Let $A$ be an $\mathbb{F}_p$-algebra, $\lambda \in A$, $N_A = \text{Ker}[F - \lambda^{p-1}I : \mathbb{G}_{a,A} \rightarrow \mathbb{G}_{a,A}]$ and $G = N_A^\vee$. Then:

(1) A homomorphism of group schemes

$$\tilde{\chi} : \prod_{N_A/A} \mathbb{G}_{m,N_A} = \text{Spec } A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\lambda^{p-1}; T_0, T_1, \ldots, T_{p-1})}]$$

$$\quad \rightarrow \mathcal{G}_A^{(\lambda)} = \text{Spec } A[X, \frac{1}{1 + \lambda X}]$$

is defined by

$$X \mapsto \sum_{l=1}^{p-1} \lambda^{l-1} T_l / T_0.$$  

Moreover, the diagram of group schemes

$$
\begin{array}{c}
G \rightarrow^i \prod_{N_A/A} \mathbb{G}_{m,N_A} \\
\downarrow_{\tilde{\chi}} \\
G \rightarrow^\eta \mathcal{G}_A^{(\lambda)}
\end{array}
$$

is commutative.

(2) A homomorphism of group schemes

$$\tilde{\sigma} : \mathcal{G}_A^{(\lambda)} = \text{Spec } A[X, \frac{1}{1 + \lambda X}] \rightarrow \prod_{N_A/A} \mathbb{G}_{m,N_A} = \text{Spec } A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\lambda^{p-1}; T_0, T_1, \ldots, T_{p-1})}].$$

is defined by

$$T_0 \mapsto 1, \quad T_l \mapsto \sum_{k=l}^{p-1} \frac{S_k}{k!} \lambda^{k-l} X^k \quad (1 \leq l \leq p - 1).$$

Moreover, the diagram of group schemes

$$
\begin{array}{c}
G \rightarrow^\eta \mathcal{G}_A^{(\lambda)} \\
\downarrow_{\tilde{\sigma}} \\
G \rightarrow^i \prod_{N_A/A} \mathbb{G}_{m,N_A}
\end{array}
$$
is commutative, and \( \tilde{\sigma} \) is a section of \( \tilde{\chi} \).

**Proof.** (1) At first we consider the case where \( A = \mathbb{F}_p[A] \) and \( \lambda = \Lambda \). Let \( R \) be an \( A \)-algebra and \( f(T) = \sum_{k=0}^{p-1} a_k T_k \in \prod_{N_A/A} \mathbb{G}_{m,N_A}(R) = (R[T]/(T^p - A^{p-1}T))^\times \). Then we obtain

\[
\tilde{\chi}(f(T)) = \sum_{k=0}^{p-1} a_k \Lambda^{k-1} = \frac{1}{A} \left\{ \frac{f(A)}{f(0)} - 1 \right\}
\]

by the definition of \( \tilde{\chi} \). Moreover, we obtain

\[
\frac{1}{A} \left\{ \frac{f(A)}{f(0)} - 1 \right\} + \frac{1}{A} \left\{ \frac{g(A)}{g(0)} - 1 \right\} + \Lambda \frac{1}{A} \left\{ \frac{f(A)}{f(0)} - 1 \right\} \frac{1}{A} \left\{ \frac{g(A)}{g(0)} - 1 \right\} = \frac{1}{A} \left\{ \frac{f(A)g(A)}{f(0)g(0)} - 1 \right\}
\]

for \( f(T), g(T) \in (R[T]/(T^p - A^{p-1}T))^\times \), which means that \( \tilde{\chi} \) is a group homomorphism. In the general case, we see that \( \tilde{\chi} \) is a group homomorphism, specializing \( \Lambda \) to \( \lambda \).

Let \( R \) be an \( A \)-algebra. By definition,

(a) \( i : G(R) \to \left( \prod_{N_A/A} \mathbb{G}_{m,N_A}(R) = (R[T]/(T^p - A^{p-1}T))^\times \right) \) is given by

\[
a \mapsto \sum_{i=1}^{p-1} \frac{a^i}{i!} T^i;
\]

(b) \( \tilde{\chi} : \left( \prod_{N_A/A} \mathbb{G}_{m,N_A}(R) = (R[T]/(T^p - A^{p-1}T))^\times \right) \to \mathcal{G}_A^{(\lambda)}(R) \) is given by

\[
\sum_{i=0}^{p-1} a_i T^i \mapsto \sum_{i=0}^{p-1} \lambda^{i-1} a_i / a_0;
\]

(c) \( \eta : G(R) \to \mathcal{G}_A^{(\lambda)}(R) \) is given by

\[
a \mapsto \sum_{i=1}^{p-1} \lambda^{i-1} a^i.
\]

These imply the commutativity of the first square.
(2) Let \( R \) be an \( A \)-algebra. Then, by definition, we have
\[
F_p(\lambda, a; T) = 1 + aT + \frac{a^{p-1}}{2!}T(T - \lambda) + \cdots
+ \frac{a^{p-1}}{(p-1)!}T(T - \lambda) \cdots (T - (p-2)\lambda)
= 1 + \sum_{l=1}^{p-1} \left( \sum_{k=l}^{p-1} \frac{S_{k,l}}{k!} \lambda^{k-l} a^k \right) T^l
\]
for \( a \in R \). If \( a \in G_A^{(\lambda)}(R) \), then \( 1 + \lambda a \) is invertible in \( R \), and therefore \( F_p(\lambda, a; T) \) is invertible in \( R[[T]]/(T^p - \lambda^{p-1}T) \).

At first we consider the case where \( A = \mathbb{F}_p[A] \) and \( \lambda = A \). We define a ring homomorphism
\[
\varphi : R[A][T]/(T^p - A^{p-1}T) \to R[A]^p
\]
by
\[
f(T) \mapsto (f(0), f(A), f(2A), \ldots, f(p-1)A)).
\]
Then,
\[
\varphi \otimes_{R[A]} R[A, A^{-1}] : R[A, A^{-1}][T]/(T^p - A^{p-1}T) \to R[A, A^{-1}]^p
\]
is an isomorphism of \( R[A, A^{-1}] \)-algebra since we have
\[
T^p - A^{p-1}T = T(T - A)(T - 2A) \cdots (T - (p-1)A).
\]
Therefore the map \( \varphi : R[A][T]/(T^p - A^{p-1}T) \to R[A]^p \) is injective. Now we have
\[
\varphi(F_p(A, a : T)) = (1, 1 + Aa, (1 + Aa)^2, \ldots, (1 + Aa)^{p-1}).
\]
Moreover, we have an identity in \( R[A][T]/(T^p - A^{p-1}T) \)
\[
F_p(A, a; T)F_p(A, b; T) = F_p(A, a + b + Aab; T)
\]
since \( (1 + Aa)(1 + Ab) = 1 + A(a + b + Aab) \). Therefore \( \tilde{\sigma} \) is a group homomorphism.

By the definition of \( \tilde{\chi} \), we have also
\[
\tilde{\chi}(\tilde{\sigma}(a)) = \tilde{\chi}(F_p(A, a; T)) = a
\]
for \( f(T) \in (R[T]/(T^p - A^{p-1}T))^\times \). It follows that \( \tilde{\sigma} \) is a section of \( \tilde{\chi} \).

Now we verify the commutativity of the second square. As is known, we have an identity in \( \mathbb{Q}[[U]] \)
\[
\sum_{k=l}^{\infty} \frac{S_{k,l}}{k!} U^k = \frac{1}{l!} \{ \log(1 + U) \}_l
\]
for each \( l \geq 1 \) (cf.\cite[1.1.11]{8}). Then we obtain in \( \mathbb{Q}[A, T][[U]] \)

\[
\sum_{k=l}^{p-1} S_{k,l} \frac{A^{k-l}U^k}{k!} \equiv \frac{1}{l!}\left\{ \frac{\log(1 + AU)}{A} \right\}^l \mod U^p
\]

for \( 1 \leq l \leq p - 1 \), and therefore,

\[
1 + \sum_{l=1}^{p-1} \left( \sum_{k=l}^{p-1} \frac{S_{k,l} A^{k-l}U^k}{k!} \right) T^l \equiv 1 + \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ \frac{T \log(1 + AU)}{A} \right\}^l \equiv \exp \left[ \frac{T}{A} \log(1 + AU) \right] \mod U^p.
\]

Furthermore we obtain

\[
1 + \sum_{l=1}^{p-1} \left( \sum_{k=l}^{p-1} \frac{S_{k,l} A^{k-l}U^k}{k!} \left( \sum_{i=1}^{p-1} \frac{A^{i-1}U^i}{i!} \right) \right) T^l \equiv \exp TU \mod U^p,
\]

noting that

\[
\sum_{i=1}^{p-1} \frac{A^{i-1}U^i}{i!} \equiv \exp AU - 1 \mod U^p.
\]

At last we have gotten an identity in \( \mathbb{Z}_p[[A, U, T]]/(U^p) \)

\[
1 + \sum_{l=1}^{p-1} \left( \sum_{k=l}^{p-1} \frac{S_{k,l} A^{k-l}U^k}{k!} \left( \sum_{i=1}^{p-1} \frac{A^{i-1}U^i}{i!} \right) \right) T^l = \sum_{i=0}^{p-1} \frac{U^i}{i!} T^i,
\]

which reads as an identity in \( \mathbb{F}_p[[A, U, T]]/(U^p) \). This implies the commutativity of the second square.

In the general case, we obtain the required results, specializing \( A \) to \( \lambda \).

**Corollary 3.13.** Let \( S \) be an \( A \)-scheme and \( X \) a \( G \)-torsor over \( S \). Then the class \([X]\) belongs to \( \ker[H^1(S, G) \to H^1(S, G_A^{(\lambda)})] \) if and only if \([X]\) is trivial in \( \text{Pic}(S \times_A N_A) \).

**Proof.** By Theorem 3.12.(1), we obtain a commutative diagram of cohomology groups

\[
\begin{array}{ccc}
H^1(S, G) & \xrightarrow{i} & H^1(S, \prod_{N_A/A} \mathbb{G}_{m,N_A}) \\
\| & & \| \\
H^1(S, G) & \xrightarrow{\eta} & H^1(S, G_A^{(\lambda)})
\end{array}
\]

Hence we obtain an implication

\([X]\) is trivial in \( \text{Pic}(S \times_A N_A) \) \( \Rightarrow \) \([X]\) \( \in \ker[H^1(S, G) \to H^1(S, G_A^{(\lambda)})] \).
Degeneration of the Kummer sequence in characteristic $p > 0$

On the other hand, by Theorem 3.12. (2), we obtain a commutative diagram of cohomology groups

\[
\begin{array}{ccc}
H^1(S, G) & \xrightarrow{\eta} & H^1(S, G^{(\lambda)}) \\
\| & & \downarrow \tilde{\sigma} \\
H^1(S, G) & \xrightarrow{i} & H^1(S, \prod_{N_A/A} \mathbb{G}_{m,N_A})
\end{array}
\]

Hence we obtain an implication

\[[X] \in \text{Ker}[H^1(S, G) \to H^1(S, G^{(\lambda)})] \Rightarrow [X] \text{ is trivial in } \text{Pic}(S \times_A N)\).

**Remark 3.14.** Let $A$ be an $\mathbb{F}_p$-algebra, $\lambda \in A$ and $B = A[T]/(T^p - \lambda^{p-1}T)$. We define a homomorphism of group schemes

\[\varepsilon : \prod_{B/A} \mathbb{G}_{m,B} \to \mathbb{G}_{m,A}\]

by

\[U \mapsto T_0.\]

We define also a homomorphism of group schemes

\[\chi_k : \prod_{B/A} \mathbb{G}_{m,B} \to \mathbb{G}_{m,A}\]

by

\[U \mapsto \sum_{l=0}^{p-1} k^l \lambda^l T_l\]

for $0 < k < p$.

If $\lambda$ is invertible in $A$,

\[(\varepsilon, \chi_1, \ldots, \chi_{p-1}) : \prod_{B/A} \mathbb{G}_{m,B} \to (\mathbb{G}_{m,A})^{p-1}\]

is an isomorphism. The inverse of $(\varepsilon, \chi_1, \ldots, \chi_{p-1})$ is given by

\[T_0 \mapsto U_0, \ T_l \mapsto -\lambda^{-l} \sum_{k=1}^{p-1} k^{-l} U_k \ (1 \leq l \leq p - 2), \ T_{p-1} = -\lambda^{-p+1} \sum_{k=0}^{p-1} U_k.\]

Furthermore the homomorphism

\[\sigma_0 : \mathbb{G}_{m,A} \to \prod_{B/A} \mathbb{G}_{m,B}\]

defined by

\[T_0 \mapsto U, \ T_l \mapsto 0 \ (1 \leq l \leq p - 2), \ T_{p-1} \mapsto \lambda^{-p+1}(1 - U)\]
is a section of $\varepsilon$. For $1 \leq k \leq p - 1$, the homomorphism
\[ \sigma_k : \mathbb{G}_{m,A} \to \prod_{B/A} \mathbb{G}_{m,B} \]
defined by
\[ T_0 \mapsto 1, \ T_l \mapsto (k\lambda)^{-1}(1 - U) \ (1 \leq l \leq p - 1) \]
is section of $\chi_k$.

The composition $\alpha(\lambda) \circ \widetilde{\chi}$ coincides with the homomorphism $\chi_1/\varepsilon$. Moreover, if $\lambda$ is invertible in $A$, then the homomorphism $\widetilde{\sigma}$ coincides with the composition $(\sigma_1 \sigma_2^2 \cdots \sigma_{p-1}^{p-1}) \circ \alpha(\lambda)$.

We conclude the section, examining the sequence:
\[ 0 \to G \xrightarrow{\xi} G_{B/A} \xrightarrow{F} G_{B/A} \to 0. \]

**Lemma 3.15.** Let $p$ be a prime number $> 2$, $A$ an $\mathbb{F}_p$-algebra and $\lambda \in A$. Put $N_A = \text{Ker}[F - \lambda^{\frac{p-1}{2}} I : \mathbb{G}_{a,A} \to \mathbb{G}_{a,A}]$ and $B = A[T]/(T^2 - \lambda)$. Then a homomorphism of group schemes
\[ \pi : \prod_{N_A/A} \mathbb{G}_{m,N_A} = \text{Spec} A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\lambda^{(p-1)/2}; T_0, T_1, \ldots, T_{p-1})}] \]
\[ \to \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec} A[U, V, \frac{1}{U^2 - \lambda V^2}] \]
is defined by
\[ U \mapsto \sum_{l=0}^{(p-1)/2} \lambda^l T_{2l}, \ V \mapsto \sum_{l=1}^{(p-1)/2} \lambda^{l-1} T_{2l-1}. \]

**Proof.** Let $R$ be an $A$-algebra. Then a homomorphism of $R$-algebra
\[ \pi : R[T]/(T^p - \lambda^{p-1} T) \to R[T]/(T^2 - \lambda) \]
is defined by $\pi(f(T)) = f(\sqrt[2]{\lambda})$ since the polynomial $T^{p-1} - \lambda^{p-1} T$ is divisible by $T^2 - \lambda$. Hence we obtain a homomorphism of multiplicative groups
\[ \pi : (R[T]/(T^p - \lambda^{p-1} T))^\times \to (R[T]/(T^2 - \lambda))^\times, \]
which is represented by a homomorphism of group $A$-schemes
\[ \pi : \prod_{N_A/A} \mathbb{G}_{m,N_A} = \text{Spec} A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\lambda^{(p-1)/2}; T_0, T_1, \ldots, T_{p-1})}] \]
\[ \to \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec} A[U, V, \frac{1}{U^2 - \lambda V^2}]. \]
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In fact, for $f(T) = \sum_{k=0}^{p-1} a_k T^k \in (R[T]/(T^p - \lambda^{p-1}))^\times$, we have

$$\pi(f(T)) = F(\sqrt{\lambda}) = \left( \sum_{k=0}^{p-1} a_{2k} \lambda^k \right) + \left( \sum_{k=0}^{p-1} a_{2k-1} \lambda^{k-1} \right) \sqrt{\lambda}.$$

**Lemma 3.16.** Let $p$ be a prime number $> 2$. Put

$$F_p(A, U; T) = 1 + UT + \frac{U^2}{2!} T(T-A) + \cdots + \frac{U^{p-1}}{(p-1)!} T(T-A) \cdots (T-(p-2)A)$$

and

$$G_p(A, X, Y; T) = F_p(\sqrt{\lambda}, 2(X + Y\sqrt{\lambda}); T)F_p(-\sqrt{\lambda}, 2(X - Y\sqrt{\lambda}); T).$$


**Proof.** The field $\mathbb{Q}(\sqrt{\lambda}, X, Y, T)$ is a quadratic extension of $\mathbb{Q}(A, X, Y, T)$, and the Galois group is generated by $\sqrt{\lambda} \mapsto -\sqrt{\lambda}$. Hence we have $G_p(A, X, Y; T) \in \mathbb{Q}(A, X, Y, T)$ since $G_p(A, X, Y; T)$ is invariant under the action $\sqrt{\lambda} \mapsto -\sqrt{\lambda}$.

We obtain the result, noting $\mathbb{Z}(p)[\sqrt{\lambda}, X, Y, T] \cap \mathbb{Q}(A, X, Y, T) = \mathbb{Z}(p)[A, X, Y, T]$.

**Notation 3.17.** For each $l \geq 1$, we define $c_{p,l}(A; X, Y) \in \mathbb{Z}(p)[A; X, Y]$ by

$$G_p(A, X, Y; T) = 1 + \sum_{l \geq 1} c_{p,l}(A; X, Y)T^l.$$  

**Example 3.18.** When $p = 3$, we have

- $c_{3,1}(A; X, Y) = 4X - 48XYA$,
- $c_{3,2}(A; X, Y) = 28X^2 - 144X^4A$
  $- 48X^2YA + 20Y^2A + 288X^2Y^2A^2 + 48Y^3A^2 - 144Y^4A^3$,
- $c_{3,3}(A; X, Y) = 48X^3 - 48XY^2A$,
- $c_{3,4}(A; X, Y) = 144X^4 - 288X^2Y^2A + 144Y^4A^2$. 
Example 3.19. When $p = 5$, we have
\[ c_{5,1}(A; X, Y) = 4X + 32X^3 - 48XY + 3072X^3Y + 96XY^2 \]
\[ - 3072XY^3, \]
\[ c_{5,2}(A; X, Y) = 28X^2 + 1328X^4 - 192X^2Y + 20Y^2 - 8960X^6 + 204Y^4 \]
\[ - 4224X^4Y + 8736X^2Y^2 - 147456X^8, \]
\[ c_{5,3}(A; X, Y) = 64X^3 + 2624X^5 - 3264X^3Y + 4096X^7 \]
\[ + 28000X^8 + 1920X^5Y + 200Y^5 - 480X^2Y^2, \]
\[ c_{5,4}(A; X, Y) = 304X^4 + 7360X^6 + 4992X^4Y + 480X^2Y^2 \]
\[ + 200704X^8 + 13520Y^3 - 6720X^4Y^2 - 802816X^6Y^2, \]
\[ c_{5,5}(A; X, Y) = 448X^5 - 5120X^7 + 15360Y^3 + 128X^3Y^2 \]
\[ + 15360X^5Y^2 + 30720X^3Y^3 - 576XY^4, \]
\[ c_{5,6}(A; X, Y) = 1600X^6 - 57344X^8 + 3072X^6Y^2 - 1728X^4Y^2 \]
\[ + 229376X^6Y^2 + 9216X^4Y^3 - 1344X^2Y^4, \]
\[ c_{5,7}(A; X, Y) = 1024X^7 + 3072X^5Y^2 - 3072X^3Y^4 - 1024XY^6 + 4096X^8 + 128X^3Y^2 + 24576X^4Y^4, \]
\[ - 16384X^2Y^6 + 1024X^5Y^2, \]
\[ c_{5,8}(A; X, Y) = 4096X^8 - 16384X^6Y^2 + 24576X^4Y^4 + 128X^3Y^2 + 16384X^2Y^6, \]
\[ + 4096X^8, \]
**Theorem 3.20.** Let $p$ be a prime number $> 2$, $A$ an $\mathbb{F}_p$-algebra and $\lambda \in A$. Put $N_A = \text{Ker}[F - \lambda^{p-1}/2 I : G_{a,A} \to G_{a,A}]$ and $G = N_A^\vee$. Then:

(1) A homomorphism of group schemes

\[ \tilde{\chi} : \prod_{N_A/A} G_{m,N_A} = \text{Spec } A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\lambda(p-1)/2; T_0, T_1, \ldots, T_{p-1})}] \rightarrow G_{B/A} = \text{Spec } A[X, Y]/(X^2 - \lambda Y^2 - Y) \]

is defined by

\[ X \mapsto \frac{\left( \sum_{l=0}^{(p-1)/2} \lambda^l T_{2l} \right)^2 - \lambda \left( \sum_{l=1}^{(p-1)/2} \lambda^{l-1} T_{2l-1} \right)^2}{\left( \sum_{l=0}^{(p-1)/2} \lambda^l T_{2l} \right)^2}, \]

\[ Y \mapsto \frac{\left( \sum_{l=1}^{(p-1)/2} \lambda^{l-1} T_{2l-1} \right)^2}{\left( \sum_{l=0}^{(p-1)/2} \lambda^l T_{2l} \right)^2 - \lambda \left( \sum_{l=1}^{(p-1)/2} \lambda^{l-1} T_{2l-1} \right)^2}. \]

Moreover, the diagram of group schemes

\[ \begin{array}{ccc}
G & \longrightarrow & \prod_{N_A/A} G_{m,N_A} \\
\downarrow & & \downarrow \tilde{\chi} \\
G & \longrightarrow & G_{B/A}
\end{array} \]

is commutative.

(2) A homomorphism of group schemes

\[ \tilde{\sigma} : G_{B/A} = \text{Spec } A[X, Y]/(X^2 - \lambda Y^2 - Y) \rightarrow \prod_{N_A/A} G_{m,N_A} = \text{Spec } A[T_0, T_1, \ldots, T_{p-1}, \frac{1}{\Delta(\lambda(p-1)/2; T_0, T_1, \ldots, T_{p-1})}] \]

is defined by

\[ T_0 \mapsto 1, \ T_l \mapsto c_{p,l}(\lambda; X, Y) + \lambda^{\frac{p-1}{2}} c_{p,l+p-1}(\lambda; X, Y) \ (1 \leq l \leq p-1). \]
Moreover, the diagram of group schemes

\[
\begin{array}{ccc}
G & \xrightarrow{\xi} & G_{B/A} \\
\text{square map} & \downarrow & \downarrow \tilde{\sigma} \\
G & \hookrightarrow & \prod_{N_A/A} \mathbb{G}_{m,N_A}
\end{array}
\]

is commutative.

**Proof.** (1) At first recall that a homomorphism of group schemes

\[r : \prod_{B/A} \mathbb{G}_{m,B} = \text{Spec } A[U,V, \frac{1}{U^2 - \lambda V^2}] \to G_{B/A} = \text{Spec } A[X,Y]/(X^2 - \lambda Y^2 - Y)\]

is defined by

\[X \mapsto \frac{UV}{U^2 - \lambda V^2}, \quad Y \mapsto \frac{V^2}{U^2 - \lambda V^2}.
\]

Then \(\tilde{\chi}\) is nothing but the composite

\[r \circ \pi : \prod_{N_A/A} \mathbb{G}_{m,N_A} \to \prod_{B/A} \mathbb{G}_{m,B} \to G_{B/A}.
\]

Now we verify the commutativity of the first square. First note that the square map on \(G = \text{Spec } A[T]/(T^p)\) is given by \(T \mapsto 2T\) since the multiplication of \(G\) is defined by

\[\Delta : T \mapsto T \otimes 1 + 1 \otimes T - \mu \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} T^{p-i} \otimes T^i.
\]

Let \(R\) be an \(A\)-algebra. Then by the definition,

(a) \(i : G(R) \to (\prod_{N_A/A} \mathbb{G}_{m,N_A})(R) = (R[T]/(T^p - \lambda \frac{p-1}{2} T))^{\times}\) is given by

\[a \mapsto \sum_{i=0}^{p-1} \frac{a^i}{i!} T^i;
\]

(b) \(\tilde{\chi} : (\prod_{N_A/A} \mathbb{G}_{m,N_A})(R) = (R[T]/(T^p - \lambda \frac{p-1}{2} T))^{\times} \to G_{B/A}(R)\) is given by

\[
\sum_{i=0}^{p-1} a_i T^i \mapsto \left( \frac{\sum_{i=0}^{p-1} \lambda^i a_{2i}}{\left( \sum_{i=0}^{p-1} \lambda^{i-1} a_{2i-1} \right)^2}, \frac{\sum_{i=1}^{p-1} \lambda^{i-1} a_{2i-1}}{\left( \sum_{i=0}^{p-1} \lambda^i a_{2i} \right)^2} \right).
\]
(c) $\xi : G(R) \to G_{B/A}(R)$ is given by

$$a \mapsto \left( \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} a^{2i-1} - \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} a^{2i} \right).$$

Hence the map $\tilde{\chi} \circ i : G(R) \to G_{B/A}(R)$ is given by

$$\sum_{i=0}^{p-1} a_i T^i \mapsto \left( \left\{ \sum_{i=0}^{p-1} \frac{\lambda^i}{(2i)!} a^{2i} \right\} \left\{ \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i-1)!} a^{2i-1} \right\} - \lambda \left\{ \sum_{i=1}^{p-1} \frac{\lambda^i}{(2i)!} a^{2i} \right\} \left\{ \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i-1)!} a^{2i-1} \right\} \right)^2.$$

and the map $\xi \circ \text{square} : G(R) \to G_{B/A}(R)$ is given by

$$a \mapsto \left( \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i-1)!} (2a)^{2i-1} - \frac{1}{2} \sum_{i=1}^{p-1} \frac{\lambda^{i-1}}{(2i)!} (2a)^{2i} \right).$$

Then it is sufficient to verify that, for $a \in R$ with $a^p = 0$, we have

$$\left\{ \sum_{i=0}^{(p-1)/2} \frac{\lambda^i}{(2i)!} a^{2i} \right\} \left\{ \sum_{i=1}^{p-2} \frac{\lambda^{i-1}}{(2i-1)!} a^{2i-1} \right\} = \frac{1}{2} \sum_{i=1}^{p-2} \frac{\lambda^{i-1}}{(2i-1)!} (2a)^{2i-1}$$

and

$$\left\{ \sum_{i=0}^{p-2} \frac{\lambda^i}{(2i)!} a^{2i} \right\} - \lambda \left\{ \sum_{i=1}^{p-2} \frac{\lambda^{i-1}}{(2i-1)!} a^{2i-1} \right\} = \frac{1}{2} \sum_{i=1}^{p-2} \frac{\lambda^{i-1}}{(2i)!} (2a)^{2i}.$$
These imply that

\[
\left\{ \sum_{i=0}^{p-1} \frac{A^i}{(2i)!} U^{2i} \right\}^2 - A \left\{ \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} U^{2i-1} \right\}^2 \equiv (\cosh \sqrt{A} U)^2 - A \left( \frac{\sinh \sqrt{A} U}{\sqrt{A}} \right) \equiv 1 \mod U^p,
\]

\[
\left\{ \sum_{i=0}^{p-1} \frac{A^i}{(2i)!} U^{2i} \right\} \left\{ \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} U^{2i-1} \right\} \equiv \cosh \sqrt{A} U \frac{\sinh \sqrt{A} U}{\sqrt{A}} \equiv \frac{1}{2} \sinh 2\sqrt{A} \equiv 1 \mod U^p,
\]

\[
\left\{ \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} U^{2i-1} \right\}^2 \equiv \left( \frac{\sinh \sqrt{A} U}{\sqrt{A}} \right)^2 \equiv \frac{1}{2} \cosh 2\sqrt{A} U - 1 \equiv \frac{1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} (2U)^{2i} \mod U^p.
\]

Then we obtain identities in \( \mathbb{Z}_p(\Lambda, U)/(U^p) \):

\[
\left\{ \sum_{i=0}^{p-1} \frac{A^i}{(2i)!} U^{2i} \right\}^2 - A \left\{ \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} U^{2i-1} \right\}^2 = 1,
\]

\[
\left\{ \sum_{i=0}^{p-1} \frac{A^i}{(2i)!} U^{2i} \right\} \left\{ \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} U^{2i-1} \right\} = \frac{1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} (2U)^{2i-1},
\]

\[
\left\{ \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i-1)!} U^{2i-1} \right\}^2 = \frac{1}{2} \sum_{i=1}^{p-1} \frac{A^{i-1}}{(2i)!} (2U)^{2i}.
\]

(2) As is remarked in 1.4, an isomorphism of group schemes over \( B \)

\[
s_1 : G_{B/A} \otimes_A B = \text{Spec} \ B[X, Y]/(X^2 - \lambda Y^2 - Y) \simeq G_{B}^{(\sqrt{\lambda})} = \text{Spec} \ B[T, \frac{1}{1 + \sqrt{\lambda} T}]
\]
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is defined by

$$T \mapsto 2(X + \sqrt{\lambda}Y).$$

Then we obtain a homomorphism of group schemes over $B$

$$\sigma_1 = \tilde{\sigma}_1 \circ s_1 : G_{B/A} \otimes_A B \xrightarrow{\sim} \mathcal{G}_B^{(\sqrt{\lambda})} \to \left( \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A} \right) \otimes_A B,$$

where $\tilde{\sigma}_1 : \mathcal{G}_B^{(\sqrt{\lambda})} \to \left( \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A} \right) \otimes_A B$ is the homomorphism defined as in the statement of in Theorem 3.12. For a $B$-algebra $R$, the map $\sigma_1 : G_{B/A}(R) \to \left( \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A} \right)(R) = (R[T]/(T^p - \lambda^\frac{p-1}{2} T))^\times$ is given by

$$(a, b) \mapsto F_p(\sqrt{\lambda}, 2(a + b\sqrt{\lambda}); T).$$

Similarly an isomorphism of group schemes over $B$

$$s_2 : G_{B/A} \otimes_A B = \text{Spec } B[X,Y]/(X^2 - \lambda Y^2 - Y) \xrightarrow{\sim} \mathcal{G}_B^{(-\sqrt{\lambda})} = \text{Spec } B[T, \frac{1}{1 - \sqrt{X}T}]$$

is defined by

$$T \mapsto 2(X - \sqrt{\lambda}Y).$$

Then we obtain a homomorphism of group schemes over $B$

$$\sigma_2 = \tilde{\sigma}_2 \circ s_2 : G_{B/A} \otimes_A B \xrightarrow{\sim} \mathcal{G}_B^{(-\sqrt{\lambda})} \to \left( \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A} \right) \otimes_A B.$$

For a $B$-algebra $R$, the map $\sigma_2 : G_{B/A}(R) \to \left( \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A} \right)(R) = (R[T]/(T^p - \lambda^\frac{p-1}{2} T))^\times$ is given by

$$(a, b) \mapsto F_p(-\sqrt{\lambda}, 2(a - b\sqrt{\lambda}); T).$$

Hence, by the definition, the morphism

$$\tilde{\sigma}_B : G_{B/A} \otimes_A B \to \left( \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A} \right) \otimes_A B$$

is the product of $\sigma_1$ and $\sigma_2$. It follows that $\tilde{\sigma}_B$ is a homomorphism of group schemes over $B$. Furthermore $\tilde{\sigma} : G_{B/A} \to \prod_{N_A/\mathbb{A}} \mathbb{G}_{m,N_A}$ is a homomorphism of group schemes over $A$ since $B$ is faithfully flat over $A$.

We verify now the commutativity of the second square. Consider the composite of homomorphisms

$$G \otimes_A B \xrightarrow{\xi_B} G_{B/A} \otimes_A B \xrightarrow{s_1} \mathcal{G}_B^{(\sqrt{\lambda})} \xrightarrow{\tilde{\sigma}_1} \left( \prod_{N/\mathbb{A}} \mathbb{G}_{m,N} \right) \otimes_A B.$$
As is shown in 2.12, we have
\[ \eta_B = s_1 \circ \xi_B : G \otimes_A B \to G_{B/A} \otimes_A B \cong \mathcal{G}_B^{(\sqrt{X})}. \]
Hence, by Theorem 3.12(2), we have \( \bar{\sigma}_1 \circ \eta_B = \xi_B \), and therefore \( \sigma_1 \circ \xi_B = \bar{\sigma}_1 \circ s_1 \circ \xi_B = i_B \). Similarly we obtain \( \sigma_2 \circ \xi_B = i_B \). These imply that \( (\bar{\sigma} \circ \xi)_B = i_B \circ \text{square} \). At last we obtain the required result since \( B \) is faithfully flat over \( A \).

**Corollary 3.21.** Let \( S \) be an \( A \)-scheme and \( X \) a \( G \)-torsor over \( S \). Then the class \( [X] \) belongs to \( \text{Ker}[H^1(S, G) \to H^1(S, G_{B/A})] \) if and only if \( [X] \) is trivial in \( \text{Pic}(S \times_A N_A) \).

**Proof.** By Theorem 3.18 (1), we obtain a commutative diagram of cohomology groups
\[
\begin{array}{ccc}
H^1(S, G) & \xrightarrow{i} & H^1(S, \prod_{N_A/A} \mathbb{G}_{m,N_A}) \\
\text{square map} \downarrow & & \downarrow \tilde{\chi} \\
H^1(S, G) & \xrightarrow{\xi} & H^1(S, G_{B/A})
\end{array}
\]
Hence we obtain an implication
\( [X] \) is trivial in \( \text{Pic}(S \times_A N_A) \) \( \Rightarrow \) \( [X] \in \text{Ker}[H^1(S, G) \to H^1(S, G_{B/A})] \).

On the other hand, by Theorem 3.18 (2), we obtain a commutative diagram of cohomology groups
\[
\begin{array}{ccc}
H^1(S, G) & \xrightarrow{\xi} & H^1(S, G_{B/A}) \\
\text{square map} \downarrow & & \downarrow \tilde{\sigma} \\
H^1(S, G) & \xrightarrow{i} & H^1(S, \prod_{N_A/A} \mathbb{G}_{m,N_A})
\end{array}
\]
Hence we obtain an implication
\( [X] \in \text{Ker}[H^1(S, G) \to H^1(S, G_{B/A})] \Rightarrow [X] \) is trivial in \( \text{Pic}(S \times_A N_A) \).

**References**


Degeneration of the Kummer sequence in characteristic $p > 0$


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