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The integral logarithm in Iwasawa theory : an exercise

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The integral logarithm in Iwasawa theory: 
an exercise

par JÜRGEN RITTER et ALFRED WEISS

Résumé. Soient $l$ un nombre premier impair et $H$ un groupe fini abélien. Nous décrivons le groupe d’unités de $\Lambda_H$ (la complétion du localisé de $\mathbb{Z}_l[[T]][H]$ en $l$) ainsi que le noyau et le conoyau du logarithme intégral $L : \Lambda^\times_H \to \Lambda^\times$, qui apparaît dans la théorie d’Iwasawa non-commutative.

Abstract. Let $l$ be an odd prime number and $H$ a finite abelian $l$-group. We describe the unit group of $\Lambda^\times_H$ (the completion of the localization at $l$ of $\mathbb{Z}_l[[T]][H]$) as well as the kernel and cokernel of the integral logarithm $L : \Lambda^\times_H \to \Lambda^\times$, which appears in non-commutative Iwasawa theory.

1. Introduction

Let $\Lambda = \mathbb{Z}_l[[T]]$ denote the ring of power series in one variable over the $l$-adic integers $\mathbb{Z}_l$, where $l$ is an odd prime number. We localize $\Lambda$ at the prime ideal $l \cdot \Lambda$ to arrive at $\Lambda_*$ and then form the completion

$$\Lambda_\Lambda = \lim_{\leftarrow n} \Lambda_*/l^n\Lambda_* .$$

The integral logarithm $L : \Lambda^\times_\Lambda \to \Lambda_\Lambda$ is defined by

$$L(e) = \frac{1}{l} \log \frac{e^l}{\psi(e)} ,$$

where $\psi : \Lambda_\Lambda \to \Lambda_\Lambda$ is the $\mathbb{Z}_l$-algebra homomorphism induced by $\psi(T) = (1 + T)^l - 1$ and with ‘log’ defined by the usual power series.

In this paper, the unit group $\Lambda^\times_\Lambda$ as well as ker($L$) and coker ($L$) are studied – more precisely, we study the analogous objects when $\Lambda_\Lambda$ is replaced by the group ring $\Lambda_H$ of a finite abelian $l$-group $H$.

The interest in doing so comes from recent work in Iwasawa theory in which refined ‘main conjectures’ are formulated in terms of the $K$-theory of completed group algebras $\mathbb{Z}_l[[G]]$ with $G$ an $l$-adic Lie group (see [5],[9]). For $l$-adic Lie groups of dimension 1, use of the integral logarithm $L$ has reduced the ‘main conjecture’ to questions of the existence of special elements (“pseudomeasures”) in $K_1(\mathbb{Z}_l[[G]]_*)$, by Theorem A of [10], and,

more recently ([11],[12]), to still unproved logarithmic congruences between Iwasawa $L$-functions. Moreover, $L$ has been indispensable for the proof of the ‘main conjecture’ in the few special cases ([6],[13]) which have been settled so far.

The integral logarithm $L$ when applied to $K_1(\mathbb{Z}_l[[T]]_\Lambda)$ takes its values only in $\mathbb{Z}_l[[T]]^{\times}_\Lambda$, which is why we need to consider completions. As for finite $G$ (see [8] and [4]), $L$ can be used to obtain structural information about $K_1$; in particular, $\text{coker}(L)$ can be detected on the abelianization $G^{ab}$ of $G$ (by [10], Theorem 8). Then

\[ G^{ab} = H \times \Gamma, \text{ with } \Gamma \simeq \mathbb{Z}_l \text{ and } H \text{ as before,} \]

\[ K_1(\mathbb{Z}_l[[G^{ab}]]_\Lambda) = \mathbb{Z}_l[[G^{ab}]]^{\times}_\Lambda \] (see [1], 40.31 and 40.32 (ii)),

\[ \mathbb{Z}_l[[G^{ab}]], \mathbb{Z}_l[[G^{ab}]]^{\times}_\Lambda \text{ are } \Lambda[H] \text{ and } \Lambda^{\times}_\Lambda[H], \text{ respectively,} \]

and $\psi$ is induced by the map $g \mapsto g^l$ on $G^{ab}$.

For these reasons it seems worthwhile to present a rather complete understanding of $L$ in the abelian situation, which is the purpose of our exercise.

The content of the paper is as follows. In section 2 we consider $\Lambda$ and define an integral exponential $E$ on $T^2\Lambda$ which is inverse to $L$ (on $1 + T^2\Lambda$).

As a consequence, we obtain the decomposition

\[ \Lambda^{\times} = \mathbb{Z}_l^{\times} \times (1 + T)\mathbb{Z}_l \times E(T^2\Lambda) \]

for the unit group $\Lambda^{\times}$ of $\Lambda$ (which reminds us of [2], Theorem 1). Applying $L$ to the decomposition yields a generalization of the Oliver congruences [8], Theorem 6.6.

The third section centers around $\Lambda^{\times}$ and two important subgroups

\[ \Xi = \{ \sum_{k=-\infty}^{\infty} x_k T^k \in \Lambda^{\times} : x_k = 0 \text{ for } l | k \} \text{ and } \Xi_2 = \{ \sum_{k \geq 2} x_k T^k \in \Xi \} . \]

In terms of these we exhibit natural decompositions of $\Lambda^{\times}$ and $\Lambda^{\times}_\Lambda$, which leads immediately to $\ker(L)$ and $\text{im}(L)$, $\text{coker}(L)$.

Section 4 is still concerned with $\Lambda^{\times}$: we determine the kernel and cokernel of its endomorphism $1 - \psi$.

This will be used in the last section, §5, where we extend most of the results to the group ring $\Lambda[H]$ of a finite abelian $l$-group $H$ over $\Lambda^{\times}$ and determine $\ker(L)$ and $\text{coker}(L)$ here.

2. The integral exponential $E$ and $\Lambda^{\times}$

Recall that $\Lambda$ is the ring $\mathbb{Z}_l[[T]]$ of formal power series $\sum_{k \geq 0} y_k T^k$ with coefficients $y_k \in \mathbb{Z}_l$, and that the integral logarithm is defined on the units $e \in \Lambda^{\times}$ of $\Lambda$ by

\[ L(e) = \frac{1}{l} \log \frac{e^l}{\psi(e)} \quad \text{where} \quad \psi(T) = (1 + T)^l - 1 . \]
Moreover, note that $1 + T^2 \Lambda$ is a subgroup of $\Lambda^\times$ since $T \in \text{rad}(\Lambda) = (l, T)$.

We now turn to the integral exponential $E$ on $T^2 \Lambda$: This is the formal power series, with coefficients in $\mathbb{Q}_l$, defined by
\[
E(y) = \exp \left( \sum_{i \geq 0} \frac{\psi^i(y)}{l^i} \right) \in \mathbb{Q}_l[[T]] \quad \text{for each } y \in T^2 \Lambda.
\]

Observe that $E$ and $\psi$ commute.

**Lemma 2.1.** $E(y) \in 1 + T^2 \Lambda$, and $E$ and $L$ are inverse to each other

\[
T^2 \Lambda \overset{E}{=} T^2 L = 1 + T^2 \Lambda.
\]

**Proof.** The proof is an adaptation of that of the Dwork-Dieudonné lemma (see [7], 14 §2): if $f(T) \in 1 + T^2 \mathbb{Q}_l[[T]]$ satisfies $f(T) \in 1 + lT^2\mathbb{Z}_l[[T]]$, then $f(T) \in 1 + T^2\mathbb{Z}_l[[T]]$.

First, $\psi^i(T) \equiv l^iT \text{ mod } T^2\mathbb{Z}_l[[T]]$ implies $\psi^i(T^k) \equiv l^iT^k \text{ mod } T^{k+1}\mathbb{Z}_l[[T]]$, and thus, if $y \in y_kT^k + T^{k+1}\mathbb{Z}_l[[T]]$ with $y_k \in \mathbb{Z}_l$, then
\[
\psi^i(y)/l^i \in y_k l^{(k-1)i}T^k + T^{k+1}\mathbb{Q}_l[[T]]; \\
\sum_{i \geq 0} \frac{\psi^i(y)}{l^i} \in \left( \sum_{i \geq 0} y_k l^{(k-1)i}T^k \right) + T^{k+1}\mathbb{Q}_l[[T]] \\
= y_k(1-l^{k-1})^{-1}T^k + T^{k+1}\mathbb{Q}_l[[T]],
\]
so $E(y) \in 1 + y_k(1-l^{k-1})^{-1}T^k + T^{k+1}\mathbb{Q}_l[[T]]$.

Second,
\[
E(y)/E(\psi(y)) = \exp \left( l \sum_{i \geq 0} \frac{\psi^i(y)}{l^i} - \sum_{i \geq 0} \frac{\psi^{i+1}(y)}{l^i} \right) \\
= \exp(l^i) \in 1 + lT^2\mathbb{Z}_l[[T]],
\]
which brings us in a position to employ the Dwork-Dieudonné argument to obtain $E(y) \in 1 + T^2\mathbb{Z}_l[[T]] = 1 + T^2 \Lambda$ and, in particular, $E(y) \in 1 + y_k(1-l^{k-1})^{-1}T^k + T^{k+1}\mathbb{Z}_l[[T]]$.

Moreover, given $b_k \in \mathbb{Z}_l$ and setting $a_k = (1-l^{k-1})b_k$, then $E(a_kT^k) \in 1 + b_kT^k + T^{k+1}\mathbb{Z}_l[[T]]$ for all $k$, which implies that $E(T^2 \Lambda) = 1 + T^2 \Lambda$.

We finish the proof of the lemma by showing $\mathbb{L}E(y) = y$, and $\mathbb{L}(1+y) = 1+y$ whenever $y \in T^2 \Lambda$, so $1+y = \mathbb{E}(\tilde{y})$:
\[
\mathbb{L}E(y) = \frac{1}{l} \log \frac{E(y)}{E(\psi(y))} \overset{(*)}{=} \frac{1}{l} \log \exp(l^i) = y, \\
\mathbb{L}(1+y) = \mathbb{L}(E(\tilde{y})) = \mathbb{E}(\tilde{y}) = y.
\]

$\square$
Corollary 2.1.

\[ L(1 + l\Lambda) = (l - \psi)\Lambda, \]
\[ \exp(ly) = (1 + T)^{ly} E((l - \psi)y) \]

if \( y \equiv y_1 T \mod T^2\Lambda \) (and \( y_1 \in \mathbb{Z}_l \)).

Proof. Since \( \exp(l\Lambda) = 1 + l\Lambda \), for the first assertion it suffices to compute

\[ L(\exp(ly)) = \frac{1}{l} \log(\exp(ly)^{l-\psi}) = \frac{1}{l} \log \exp(l(l - \psi)y) = (l - \psi)y. \]

The second assertion holds for \( y = T \) (so \( y_1 = 1 \)):

\( (1 + T)^{-l} \exp(lT) \in 1 + T^2\mathbb{Z}_l[[T]] \),

hence \( (1 + T)^{-l} \exp(lT) = E(z) \) for some \( z \in T^2\mathbb{Z}_l[[T]] \).

Apply \( L \) and get \( (l - \psi)(T) = z \) from the last but one displayed formula and as \( L(1 + T) = \frac{1}{l} \log (1+(1+T)^{l-1} - 1) = 0 \).

Next, take \( y \in T^2\mathbb{Z}_l[[T]] \), so \( \exp(ly) \in 1 + T^2\mathbb{Z}_l[[T]] \) and again \( \exp(ly) = E((l - \psi)y) \).

The two special cases can be combined on writing \( y = y_1 T + (y - y_1 T) \). □

Denote by \( \mu_{l-1} \) the group of roots of unity in \( \mathbb{Z}_l^\times \).

Corollary 2.2.

\[ \Lambda^\times = \mathbb{Z}_l^\times \times (1 + T)^{\mathbb{Z}_l} \times E(T^2\Lambda), \]
\[ \ker(L) = \mu_{l-1} \times (1 + T)^{\mathbb{Z}_l}, \]
\[ \text{im}(L) = \mathbb{Z}_l \oplus T^2\Lambda \]

Proof. The first coefficient \( e_0 \) of \( e = \sum_{k \geq 0} e_k T^k \in \Lambda^\times \) is a unit in \( \mathbb{Z}_l \).

Replacing \( e \) by \( e_0^{-1} \cdot e = 1 + e_1' T + \cdots \) and then multiplying by \( (1 + T)^{-e_1'} \) gives the new unit \( 1 + \tilde{e}_2 T^2 + \cdots \in 1 + T^2\Lambda \). Thus, by Lemma 2.1, \( \Lambda^\times = \mathbb{Z}_l^\times \cdot (1 + T)^{\mathbb{Z}_l} \cdot E(T^2\Lambda) \), and the product is obviously direct. Since, on \( \mathbb{Z}_l^\times \), \( L(\zeta) = 0 \) precisely for \( \zeta \in \mu_{l-1} \), and since \( L(1 + T) = 0 \), we also get the claimed description of the kernel and image of \( L \). □

3. The integral logarithm \( L \) on \( \Lambda_\Lambda \)

We recall that \( \Lambda_* \) denotes the localization of \( \Lambda \) at the prime ideal \( l\Lambda \) and that \( \Lambda_\Lambda = \lim_{n} \Lambda_*/l^n\Lambda_* \). In particular, \( \Lambda_* \) and \( \Lambda_\Lambda \) have the same residue field \( \mathbb{F}_l((T)) \) (which carries the natural \( T \)-valuation \( v_T \)). It follows that

\[ \Lambda_\Lambda = \{ x = \sum_{k \in \mathbb{Z}} x_k T^k : x_k \in \mathbb{Z}_l, \lim_{k \to -\infty} x_k = 0 \}. \]
Such large rings are basic objects in the theory of higher dimensional local fields $[3]$; the map $\psi$ on $\Lambda_\Lambda$ is extra structure which remembers the group $\Gamma$.

In what follows we frequently use the decomposition

$$\Lambda_\Lambda = \Lambda_\Lambda^- \oplus \mathbb{Z}_l \oplus \Lambda_\Lambda^+,$$

where $\Lambda_\Lambda^\pm = \begin{cases} \{ x \in \Lambda_\Lambda : x_k = 0 \text{ for } k \leq 0 \} \\ \{ x \in \Lambda_\Lambda : x_k = 0 \text{ for } k \geq 0 \} \end{cases}$.

Note that the three summands are subrings which are preserved by $\psi$. As a consequence, we see that $\Lambda_\Lambda \cap (l - \psi)\Lambda_\Lambda = (l - \psi)\Lambda_\Lambda$.

**Definition.**

$$\Xi = \{ x = \sum_{k \in \mathbb{Z}} x_k T^k \in \Lambda_\Lambda : x_k = 0 \text{ when } l \text{ divides } k \},$$

$$\Xi_s = \{ x = \sum_{k \geq s} x_k T^k \in \Xi \}, \text{ where } s \in \mathbb{Z}.$$

**Lemma 3.1.**

1. $l - \psi$ is injective on $\Lambda_\Lambda$ and has image $\mathbb{L}(1 + l\Lambda_\Lambda)$,
2. $\Lambda_\Lambda = \Xi \oplus (l - \psi)\Lambda_\Lambda$

**Proof.** For the first assertion we make use of the commuting diagram

$$\begin{array}{ccc}
\Lambda_\Lambda & \xrightarrow{l} & \Lambda_\Lambda \\
\downarrow{l - \psi} & & \downarrow{l - \psi}
\end{array}$$

with exact rows and with $\overline{\psi(T)} = \overline{T}^l$, so $\overline{\psi(x)} = \overline{x}^l$ for $x \in \mathbb{F}_l((T))$. In particular, $-\overline{\psi}$ is injective and hence the snake lemma implies $\ker(l - \psi) = l \cdot \ker(l - \psi)$ from which $\ker(l - \psi) = 0$ follows by $\bigcap_{n \geq 0} l^n \Lambda_\Lambda = 0$.

Regarding the image, we observe that $\exp(l\Lambda_\Lambda) = 1 + l\Lambda_\Lambda$ and recall $\mathbb{L}(\exp(ty)) = (l - \psi)y$ from the proof of Corollary 2.1 (but now with $y \in \Lambda_\Lambda$).

For the second assertion we make use of the commuting diagram

$$\begin{array}{ccc}
\Xi & \xrightarrow{l} & \Xi \\
\downarrow & & \downarrow
\end{array}$$

$$\begin{array}{ccc}
\Lambda_\Lambda / (l - \psi)\Lambda_\Lambda & \xrightarrow{l} & \Lambda_\Lambda / (l - \psi)\Lambda_\Lambda \\
\downarrow & & \downarrow
\end{array}$$

with natural vertical maps (which we denote by $\sim$). Its bottom row is the sequence of cokernels of diagram (1) and thus exact. Its right vertical map is an isomorphism, by the definition of $\Xi$ and by $\overline{\psi(x)} = \overline{x}^l$. Consequently, the other vertical map, $\Xi \to \Lambda_\Lambda / (l - \psi)\Lambda_\Lambda$, is injective, by the snake lemma and $\bigcap_{n \geq 0} l^n \Lambda_\Lambda = 0$. To finish the proof of the lemma we are left with showing the surjectivity of $\Xi \to \Lambda_\Lambda / (l - \psi)\Lambda_\Lambda$. Starting, in (2), with $\bar{x} \in \Lambda_\Lambda / (l - \psi)\Lambda_\Lambda$ (the middle term in the bottom row) we find elements $y_0 \in \Xi$...
and \( \tilde{x}_1 \in \Lambda_\Lambda/(l - \psi)\Lambda_\Lambda \) such that \( \tilde{x} - \tilde{y}_0 = l\tilde{x}_1 \). Continuing, we get \( \tilde{x} = \tilde{y}_0 + l\tilde{y}_1 + l^2\tilde{y}_2 + \cdots \), with \( y_0 + l y_1 + l^2 y_2 + \cdots \in \Xi \).

\[ \square \]

**Corollary 3.1.** \( T^2\Lambda = \Xi_2 \oplus (l - \psi)T\Lambda \)

**Proof.** Since \( \Xi \cap \Lambda_\Lambda^+ = \Xi_1 \), Lemma 3.1 gives \( \Lambda_\Lambda^+ = \Xi_1 \oplus (l - \psi)\Lambda_\Lambda^+ \), i.e., \( T\Lambda = \Xi_1 \oplus (l - \psi)T\Lambda \). We intersect with \( T^2\Lambda \) and obtain the corollary from \( (l - \psi)T\Lambda \subset T^2\Lambda \) and \( \Xi_1 \cap T^2\Lambda = \Xi_2 \).

\[ \square \]

**Proposition 3.1.** \( \Lambda_\Lambda^\times = T^2 \times \mu_{l-1} \times (1 + T)^{z_l} \times \mathbb{E}(\Xi_2) \times (1 + l\Lambda_\Lambda) \)

**Proof.** Given \( e = \sum_{k \in \mathbb{Z}} e_k T^k \in \Lambda_\Lambda^\times \), we will modify \( e \) by factors in \( T^2 \), \( \mu_{l-1} \times (1 + l\Lambda_\Lambda) \) and \( (1 + T)^{z_l} \) to arrive at a new unit \( \mathbb{E}(y) \) for some \( y \in \Xi_2 \). This confirms the claimed product decomposition of \( \Lambda_\Lambda^\times \) but not yet that it is a direct product.

1. Going modulo \( l \), let \( e = \sum_{k \geq k_0} e_k T^k \in \mathbb{F}_l((T)) \) have coefficient \( e_{k_0} \neq 0 \). Multiplying \( e \) by \( T^{-k_0} \in T^2 \) gives a new unit with zero coefficient not divisible by \( l \) but all coefficients with negative index divisible by \( l \); we denote it again by \( e \).

2. Now \( e_0 \in \mathbb{Z}_l^\times = \mu_{l-1} \times (1 + l\mathbb{Z}_l) \subset \mu_{l-1} \times (1 + l\Lambda_\Lambda) \), and multiplying \( e \) by \( e_0^{-1} \) allows us to assume that \( e = le^- + 1 + e^+ \), where \( e^- \in \Lambda_\Lambda^- \) and \( e^+ \in \Lambda_\Lambda^+ \), so \( 1 + e^+ \in \Lambda_\Lambda^\times \leq \Lambda_\Lambda^\times \) and \( e(1 + e^+)^{-1} = 1 + l(e^- + 1 + e^+)^{-1} \in 1 + l\Lambda_\Lambda^- \), i.e., \( e \equiv 1 + e^+ \mod 1 + l\Lambda_\Lambda \).

3. If \( 1 + e^+ = 1 + e_1 T + e_2 T^2 + \cdots \), then multiplying \( 1 + e^+ \) by \( (1 + T)^{-e_1} \in (1 + T)^{z_l} \) produces \( 1 + T^2 \tilde{y} \) with \( \tilde{y} \in \Lambda \) (note \( (1 + T)^{z_l} \equiv 1 + zT \mod T^2\Lambda \)). Hence, by Lemma 2.1, modulo \( T^2 \cdot \mu_{l-1} \cdot (1 + T)^{z_l} \cdot (1 + l\Lambda_\Lambda) \), the original unit \( e \) satisfies \( e \equiv \mathbb{E}(y') \) with \( y' \in T^2\Lambda \).

4. As \( \mathbb{E}(T^2\Lambda) = \mathbb{E}(\Xi_2) \times \mathbb{E}((l - \psi)T\Lambda) \) by the above corollary, multiplying \( \mathbb{E}(y') \) with \( \mathbb{E}(y) \) for a suitable \( y \in \Xi_2 \) yields an element \( \mathbb{E}((l - \psi)y'') \) with \( y'' \in y'' T + T^2\Lambda \). It follows from Corollary 2.1 to Lemma 2.1 that \( \mathbb{E}((l - \psi)y'') = (1 + T)^{-ly''} \exp(l y'') \). The first factor is in \( (1 + T)^{z_l} \) and the second in \( 1 + l\Lambda \).

We now prove that we actually have a direct product.

We have already used \( (1 + T)^z \equiv 1 + zT \mod T^2\Lambda \). Together with \( \mathbb{E}(\Xi_2) \subset 1 + T^2\Lambda \) it implies that the product \( T^2 \cdot \mu_{l-1} \cdot (1 + T)^{z_l} \cdot \mathbb{E}(\Xi_2) \) is direct. Moreover, an element in it which also lies in \( 1 + l\Lambda_\Lambda \) must equal \( \mathbb{E}(y) \) with \( y \in \Xi_2 \). Indeed, \( (1 + T)^z \equiv 1 \mod l \) gives \( z \equiv 0 \mod l \), hence \( (1 + T)^z \equiv 1 \mod l \), since modulo \( l \) we are in characteristic 1. Thus \( z = 0 \).

So assume \( \mathbb{E}(y) = 1 + l z \). Applying \( L \) gives \( y = L(1 + l z) = (l - \psi)z' \in (l - \psi)\Lambda_\Lambda \), by Corollary 2.1. As \( y \in \Xi_2 \subset T^2\Lambda \), the zero coefficient of \( z' \) vanishes and Corollary 3.1 implies \( y = 0 \). This completes the proof of the proposition.

\[ \square \]
Definition. $\xi : \Lambda_\Lambda = \Xi \oplus (l - \psi)\Lambda_\Lambda \rightarrow \Xi$ is the identity on $\Xi$ and zero on $(l - \psi)\Lambda_\Lambda$

Corollary 3.2. We have an exact sequence

$$\mu_{l-1} \times (1 + T)^{Z_l} \rightarrow \Lambda_\Lambda \overset{L}{\rightarrow} \Lambda_\Lambda \rightarrow \Xi / (Z \cdot \xi((L(T))) \oplus \Xi_2).$$

Proof. For the proof note that $\xi((L(T)))$ is in $\Lambda_\Lambda^-$ and non-zero: writing $T^i = \frac{1}{1 - lv}$ with $v = -\frac{1}{l} \sum_{i=1}^{l-1} (i) T^{-i}$ we have

$$L(T) = - \log(1 - lv) = \sum_{j \geq 1} \frac{i^j - 1}{j} v^j \in \Lambda_\Lambda^-$$

with

$$\xi((L(T))) = v = \sum_{i=1}^{l-1} \frac{(-1)^i}{i} T^{-i} \mod l.$$

Recall that $\mu_{l-1} \times (1 + T)^{Z_l} \subset \ker(\mathbb{L})$, that $\mathbb{L}$ is the identity on $\Xi_2$, and that $1 + \Lambda_\Lambda = \exp(l\Lambda_\Lambda)$.

Suppose now that $e = T^b \zeta(1 + T)^z \mathbb{E}(x) \exp(l y)$ is in $\ker(\mathbb{L})$ (with $b \in \mathbb{Z}$, $\zeta \in \mu_{l-1}$, $z \in \mathbb{Z}_l$, $x \in \Xi_2$, $y \in \Lambda_\Lambda$). Then $-bL(T) = x + (l - \psi)y$ implies $-b\xi((L(T))) = x$ is in $\Lambda_\Lambda^- \cap \Xi_2 = 0$, hence $b = 0 = x$ and then $y = 0$ by 1. of Lemma 3.1, as required.

Concerning coker $(\mathbb{L})$, it suffices to show that $\text{im}(\mathbb{L}) = Z \cdot \xi((L(T))) \oplus \Xi_2 \oplus (l - \psi)\Lambda_\Lambda$. By Proposition 3.1, 1. of Lemma 3.1 and $L(T) - \xi((L(T))) \in (l - \psi)\Lambda_\Lambda$ this again follows from $\xi((L(T))) \notin \Xi_2$.

This finishes the proof of the corollary. \[\square\]

Remark. When $l = 2$, more effort is needed, since $-1 \in 1 + 2\Lambda_\Lambda$ and ‘log, exp’ are no longer inverse to each other.

4. Kernel and cokernel of $1 - \psi$ on $\Lambda_\Lambda$

Lemma 4.1. There is an exact sequence

$$0 \rightarrow \mathbb{Z}_l \rightarrow \Lambda_\Lambda \overset{1 - \psi}{\rightarrow} \Lambda_\Lambda \rightarrow (\Xi / \Xi_1) \oplus \mathbb{Z}_l \rightarrow 0.$$

Proof. We start its proof from the obvious diagram below and show that $\ker(\mathbb{T} - \overline{\psi}) = \mathbb{F}_l$, the constants in $\mathbb{F}_l((\mathbb{T})) = \Lambda_\Lambda / l\Lambda_\Lambda$.

Indeed,

$$(\mathbb{T} - \overline{\psi})(\sum_{k \geq -n} z_k T^k) = 0 \iff \sum_{k \geq -n} z_k T^k = \sum_{k \geq -n} z_k T^{lk} = (\sum_{k \geq -n} z_k T^k)^l,$$
and the only $l-1$st roots of unity in the field $\mathbb{F}_l((\bar{T}))$ are the constants $\neq 0$. The above implies $\ker(1-\psi) = \mathbb{Z}_l + l \ker(1-\psi)$. By successive approximation this gives $\ker(1-\psi) = \mathbb{Z}_l$.

Turning back to the diagram, we obtain from the snake lemma the short exact sequence

$$\text{coker } (1-\psi) \hookrightarrow \text{coker } (1-\psi) \rightarrow \text{coker } (1-\bar{\psi}) .$$

We compute its right end. Because $\mathbb{F}_l((\bar{T}))$ is complete in the $\nu_T$-topology, $\sum_{n \geq 0} \bar{z}'^n$ converges for every element $\bar{z} = \sum_{k \geq 1} \bar{z}_k \bar{T}^k$, hence

$$(1-\bar{\psi})\left(\sum_{n \geq 0} \bar{z}'^n\right) = \bar{z}$$

implies that these $\bar{z}$ all belong to $\text{im } (1-\bar{\psi})$. Also, $T^j - T^{li} = (1-\bar{\psi})(T^j) \in \text{im } (1-\bar{\psi})$. Thus, $\text{coker } (1-\bar{\psi})$ is spanned by the images of $T^j$ with $j = 0$ or $j < 0$ & $l \nmid j$. These elements are actually linearly independent over $\mathbb{F}_l$.

To see this, read an equation

$$\sum_{-n \leq k < 0} \bar{z}_k T^k + z_0 = (1-\bar{\psi})(\bar{x}) = \sum_{-n \leq k < 0} \bar{x}_k (T^k - T^{lk})$$

coefficientwise from $k = -n$ to $k = 0$.

Going back to the short exact sequence displayed above, we now realize that $\Xi/\Xi_1 \oplus \mathbb{Z}_l$ maps onto $\text{coker } (1-\psi)$, since $\Lambda_\wedge$ is $l$-complete. And by the last paragraph, this surjection is, in fact, an isomorphism. □

5. Kernel and cokernel of $L$ on $\Lambda_\wedge[H]$

As in the introduction, $H$ is a finite abelian $l$-group and $\Lambda_\wedge[H]$ is its group ring over $\Lambda_\wedge$. Perhaps the description $\Lambda_\wedge[H] = \mathbb{Z}_l[[\Gamma \times H]]_\wedge$, with $\Gamma$ denoting the cyclic pro-$l$ group generated by $1+T$, gives a better understanding of the ring homomorphism $\psi$ on $\Lambda_\wedge[H]$: $\psi$ is induced by $\psi(g) = g^l$ for $g \in \Gamma \times H$. And the integral logarithm $L: \Lambda_\wedge[H]^\times \rightarrow \Lambda_\wedge[H]$, as before, takes a unit $e \in \Lambda_\wedge[H]^\times$ to $L(e) = \frac{1}{l} \log \frac{e^l}{\psi(e)}$.

For the discussion of its kernel and cokernel we first invoke the augmentation map $\Lambda_\wedge[H] \rightarrow \Lambda_\wedge$, $h \mapsto 1$ for $h \in H$, so that we can employ our earlier results. Let $\mathfrak{g}$ denote its kernel and note that $1 + \mathfrak{g} \subset \Lambda_\wedge[H]^\times$, as $\mathfrak{g} \subset \mathfrak{r} \overset{\text{def}}{=} \text{rad}(\Lambda_\wedge[H]) = \mathfrak{g} + l\Lambda_\wedge[H]$; moreover, for the same reason, $\Lambda_\wedge[H]^\times \rightarrow \Lambda_\wedge^\times$ is surjective.

**Proposition 5.1.** $L: \Lambda_\wedge[H]^\times \rightarrow \Lambda_\wedge[H]$ has

$$\ker(L) = \mu_{l-1} \times (1+T)_{\mathbb{Z}_l} \times H \quad (= \mu_{l-1} \times (\Gamma \times H)) ,$$

$\mu_{l-1}$ being the group of $(l-1)$-st roots of unity in $\mathbb{Z}_l$. Therefore $L$ is an isomorphism.
and coker (L) is described by the split exact sequence
\[(\Xi/\Xi_1 \oplus \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{H} \rightarrow \text{coker} (L) \rightarrow \Xi / (\mathbb{Z}_l(L(T)) \oplus \Xi_2) .\]

Proof. The proof begins with the commutative diagram
\[
\begin{array}{ccc}
1 + g & \rightarrow & \Lambda^x[H] \times \\
\Lambda & \rightarrow & \Lambda^x[H] \\
g & \rightarrow & \Lambda^x[H]
\end{array}
\]

with exact rows which are split by the same inclusion \(\Lambda^x \rightarrow \Lambda^x[H]\) of rings. Here the right square commutes because \(\psi\) and ‘log’ both commute with augmentation, and thus induces the left square since the sequences are exact.

The right vertical \(L\) fits into the exact sequence of Corollary 3.2. Similarly we will need

Lemma 5.1. There is an exact sequence
\[
1 + g \rightarrow \Lambda^x[H] \rightarrow \Lambda^x[H] \\
\rightarrow \Lambda^x[H] \\
g \rightarrow \Lambda^x[H]
\]

Proposition 5.1 follows from Lemma 5.1 and the snake lemma: for \(\mu_{l-1} \times (1 + T)_{\mathbb{Z}_l} \times H \subset \ker(L)\) maps onto the kernel of the right vertical \(L\), and the cokernel sequence splits because the natural splittings in the commutative diagram are compatible. \(\square\)

So it remains to prove Lemma 5.1, which we do next.

Proof. a) \(g/g^2 \simeq \Lambda^x \otimes_{\mathbb{Z}_l} H\) by \(h - 1 \mod g^2 \rightarrow h\)

This is a consequence of \(\Lambda^x[H] = \Lambda^x \otimes_{\mathbb{Z}_l} \mathbb{H} \otimes_{\mathbb{Z}_l} [H]\) and the natural isomorphism \(\Delta H/\Delta^2H \simeq H, h - 1 \rightarrow h\), where \(\Delta H = \langle h - 1 : h \in H \rangle_{\mathbb{Z}_l}\) is the augmentation ideal of the group ring \(\mathbb{Z}_l[H]\), so \(g = \Lambda^x \otimes_{\mathbb{Z}_l} \Delta H\).

b) If \(e = 1 + \sum_{1 \neq h \in H} e_h(h - 1) \in 1 + g \) (with \(e_h \in \Lambda^x\)), then
\[
L(e) \equiv \sum_h (e_h - \psi(e_h))(h - 1) \mod g^2 .
\]

Indeed, modulo \(l g^2\) we have
\[
e^l \equiv 1 + l \sum_h e_h(h - 1) + \sum_h e^l_h(h - 1)^l \\
\equiv 1 + l \sum_h e_h(h - 1) + \sum_h \psi(e_h)(h - 1)^l \\
\equiv 1 + l \sum_h e_h(h - 1) + \sum_h \psi(e_h)(h^l - 1) - l \sum_h \psi(e_h)(h - 1) \\
\equiv \psi(e) + l \sum_h (e_h - \psi(e_h))(h - 1) ,
\]
The same argument applies to the kernel in the top row. It follows that
\[
e^l \psi(e) \equiv 1 + \psi(e)^{-1} l \sum_h (e_h - \psi(e_h))(h - 1) \mod l\mathfrak{g}^2
\]
as \psi(e)^{-1} \in 1 + \mathfrak{g}. Now apply '1 \div l log'.

From a), b) we get the right square of the commutative diagram
\[
\begin{array}{ccc}
1 + \mathfrak{g}^2 & \rightarrow & 1 + \mathfrak{g} \\
\mathbb{L} \downarrow & & \downarrow \mathbb{L} \\
\mathfrak{g}^2 & \rightarrow & \mathfrak{g}
\end{array}
\rightarrow
\begin{array}{ccc}
\Lambda_\wedge \otimes \mathbb{Z}_l H & \rightarrow & (1 - \psi) \otimes 1 \\
\downarrow & & \downarrow \\
\bar{\Lambda}_\wedge \otimes \mathbb{Z}_l H
\end{array}
\]
with left square induced by the exactness of the rows. The map \((1 - \psi) \otimes 1\) has kernel and cokernel given by tensoring the sequence in Lemma 4.1 with \(H\); it remains exact since it is composed of two short exact sequences of torsionfree \(\mathbb{Z}_l\)-modules. So the snake lemma reduces Lemma 5.1 to proving that \(\mathbb{L} : 1 + \mathfrak{g}^2 \rightarrow \mathfrak{g}^2\) is an isomorphism.

We do this by induction on \(|H|\) and, to that end, choose an element \(h_0 \in H\) of order \(l\) and let \(H \rightarrow \bar{H} = H/\langle h_0 \rangle\) be the natural map.

Recalling that \(\mathfrak{r} = \text{rad}(\Lambda_\wedge[H]) = \mathfrak{g} + l\Lambda_\wedge[H]\), we start with the right square of the diagram
\[
\begin{array}{ccc}
1 + (h_0 - 1)\mathfrak{r} & \rightarrow & 1 + \mathfrak{g}^2 \\
\mathbb{L} \downarrow & & \downarrow \mathbb{L} \\
(h_0 - 1)\mathfrak{r} & \rightarrow & \bar{\mathfrak{g}}^2
\end{array}
\rightarrow
\begin{array}{c}
\mathfrak{g}^2 \\
\downarrow \mathbb{L} \\
\bar{\mathfrak{g}}^2
\end{array}
\]
which commutes since \(\psi, \log\) commute with \(\bar{\cdot}\). Since \(\bar{\mathbb{L}}\) is an isomorphism by the induction hypothesis, it suffices to show that the kernels in the rows are as shown and that the left \(\mathbb{L}\) is an isomorphism:

i. \(\mathfrak{g}^2 \rightarrow \bar{\mathfrak{g}}^2\) has kernel \((h_0 - 1)\mathfrak{r}\). Since \((h_0 - 1)\mathfrak{r}\) is in the kernel of \(\bar{\cdot}\) and \(l(h_0 - 1)\) is in \(\mathfrak{g}^2\), by \(l(h_0 - 1) \equiv h_0^l - 1 \mod \mathfrak{g}^2\), it remains to check
\[
(h_0 - 1)\Lambda_\wedge[H] \cap \mathfrak{g}^2 \subset (h_0 - 1)\mathfrak{r}.
\]
If \((h_0 - 1)b = (h_0 - 1) \sum_{h \in H} b_h h \in \mathfrak{g}^2\) (with \(b_h \in \Lambda_\wedge\)), then the isomorphism \(\mathfrak{g}/\mathfrak{g}^2 \simeq \Lambda_\wedge \otimes \mathbb{Z}_l H\) takes \((h_0 - 1)b\) to \(0 = \sum_{h \in H} b_h \otimes h_0 = (\sum_{b \in H} b_h) \otimes h_0\), whence \(\sum_{b \in H} b_h \in l\Lambda_\wedge\), since \(h_0\) has order \(l\). Thus, \((h_0 - 1)b \in (h_0 - 1)(\sum_{b \in H} b_h(h - 1) + l\Lambda_\wedge) \subset (h_0 - 1)\mathfrak{r}\).

The same argument applies to the kernel in the top row. It follows that \(\mathbb{L}(1 + (h_0 - 1)\mathfrak{r}) \subset (h_0 - 1)\mathfrak{r}\).

ii. \(\mathbb{L} : 1 + (h_0 - 1)\mathfrak{r} \rightarrow (h_0 - 1)\mathfrak{r}\) is an isomorphism. If \(x \in \mathfrak{r}\), then \(\psi(h_0 - 1) = 0\) implies that \(\mathbb{L}(\exp((h_0 - 1)x)) = (h_0 - 1)x\), hence \(\mathbb{L}\)
is onto, and
\[ L(1 - (h_0 - 1)x) = \log(1 - (h_0 - 1)x) \]
\[ = -(h_0 - 1)(x - x^l) + (h_0 - 1)^2 x^2 \lambda_x \]
\[ = -(h_0 - 1)x + (h_0 - 1)x^2 \lambda_x' \]
with some \( \lambda_x, \lambda_x' \in \Lambda[H] \) by (†) in [10], p.40 (with \( z \) replaced by \( h_0 \)). If this is zero, then \( (h_0 - 1)x(1 - x\lambda_x') = 0 \) with \( 1 - x\lambda_x' \in \Lambda[H] \). So \( L \) is injective.

\( \square \)

**Remark.** Admittedly, Proposition 5.1 is closer to Corollary 3.1 than to Proposition 3.1 itself, as \( \Lambda[H] \) has not been determined.

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**References**


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