

JOURNAL

de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 21, n° 3 (2009), p. 771-786.

<http://jtnb.cedram.org/item?id=JTNB_2009__21_3_771_0>

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The optimality of the Bounded Height Conjecture

par EVELINA VIADA

RÉSUMÉ. Nous démontrons que la “conjecture de hauteur bornée” est optimale dans le sens suivant. Soit V une variété irréductible dans une puissance d’une courbe elliptique. Si les sous-variétés anormales de V recouvrent tout V , alors chaque ouvert de V a une hauteur non bornée. Nous donnons aussi quelques exemples

ABSTRACT. In this article we show that the Bounded Height Conjecture is optimal in the sense that, if V is an irreducible subvariety with empty deprived set in a power of an elliptic curve, then every open subset of V does not have bounded height. The Bounded Height Conjecture is known to hold. We also present some examples and remarks.

1. Introduction

This work concerns principally the optimality of the Bounded Height Conjecture, stated by Bombieri, Masser and Zannier [2] and proven by Habegger [3]. In section 2, we clarify the assumption on the varieties, understanding such a hypothesis geometrically and from different points of view. We give some examples, to make sure that certain situations can occur. In section 3, we prove the optimality of the Bounded Height Conjecture. In the final section we present some further remarks and possible open questions.

Denote by A an abelian variety over $\overline{\mathbb{Q}}$ of dimension g . Consider on $A(\overline{\mathbb{Q}})$ a canonical height function. Denote by $\|\cdot\|$ the induced semi-norm. For $\varepsilon \geq 0$, we define

$$\mathcal{O}_\varepsilon = \{\xi \in A(\overline{\mathbb{Q}}) : \|\xi\| \leq \varepsilon\}.$$

Consider a proper irreducible algebraic subvariety V of dimension d embedded in A and defined over $\overline{\mathbb{Q}}$. We say that:

- V is transverse, if V is not contained in any translate of a proper algebraic subgroup of A .
- V is weak-transverse, if V is not contained in any proper algebraic subgroup of A .

Manuscrit reçu le 21 décembre 2007, révisé le 19 décembre 2008.

Mots clefs. Height, Elliptic curves, Subvarieties.

Classification math. 11G50, 14H52, 14K12.

Given an integer r with $1 \leq r \leq g$ and a subset F of $A(\overline{\mathbb{Q}})$, we define the set

$$S_r(V, F) = V(\overline{\mathbb{Q}}) \cap \bigcup_{\text{cod}B \geq r} B + F$$

where B varies over all abelian subvarieties of A of codimension at least r and

$$B + F = \{b + f : b \in B, f \in F\}.$$

Note that

$$S_{r+1}(V, F) \subset S_r(V, F).$$

We denote the set $S_r(V, A_{\text{Tor}})$ simply by $S_r(V)$, where A_{Tor} is the torsion of A . For a subset $V' \subset V$, we denote

$$S_r(V', F) = V' \cap S_r(V, F).$$

It is natural to ask: ‘For which sets F and integers r , does the set $S_r(V, F)$ have bounded height or is Zariski non-dense in V ?’

Sets of this kind, for $r = g$, appear in the literature in the context of the Mordell-Lang, of the Manin-Mumford and of the Bogomolov Conjectures. More recently Bombieri, Masser and Zannier [1] have proven that:

For a transverse curve C in a torus,

- (1) The set $S_1(C)$ has bounded height,
- (2) The set $S_2(C)$ is finite.

They investigate intersections with the union of all algebraic subgroups of a given codimension. This opens a vast number of conjectures for subvarieties of semi-abelian varieties. Most naively, one could risk the following:

For V a transverse subvariety of A ,

- (1) $S_d(V)$ has bounded height,
- (2) $S_{d+1}(V)$ is Zariski non-dense in V .

We will show that (1) is a too optimistic guess.

Several problems rise for varieties. A proper Zariski closed subset of a curve has bounded height. In general, a proper Zariski closed subset of a variety does not have bounded height, however it is still a ‘small’ set. So one shall say, that outside an anomalous Zariski closed subset of V , the points we consider have bounded height. Bombieri, Masser and Zannier introduced the anomalous set. Hardest is to show that it is Zariski closed.

Definition 1.1 ([2] Definition 1.1 and 1.2). *An irreducible subvariety X of V is anomalous if it has positive dimension and lies in a coset H of A satisfying*

$$\dim H \leq g - \dim V + \dim X - 1.$$

The deprived set V^{oa} is what remains of V after removing all anomalous subvarieties.

In [2, Theorem 1.4], they prove the following theorem, for a subvariety V of a torus. Rémond [5] proves it for a subvariety of an abelian variety.

Theorem 1.1. *The deprived set V^{oa} is a Zariski open of V .*

Then, they state the following conjecture for tori and $\varepsilon = 0$.

Conjecture 1.1 (Bounded Height Conjecture). *Let V be an irreducible variety in A of dimension d . Then, there exists $\varepsilon > 0$ such that $S_d(V^{oa}, \mathcal{O}_\varepsilon)$ has bounded height.*

We remark that in all known effective proofs, the bound for the height of $S_d(V^{oa})$ is independent of the field of definition of V . Then, a set F of bounded height does not harm. For transverse curves in a torus [1] and in a product of elliptic curves [7], Conjecture 1.1 is effectively proven. P. Habegger [3] deals with subvarieties of an abelian variety A defined over the algebraic numbers. He shows

Theorem 1.2 (Habegger [3]). *Conjecture 1.1 holds.*

In the first instance we analyze several geometric properties which are different for varieties, but they all collapse to the transversal condition for curves.

Property (S^n). *We say that V satisfies Property (S^n) if, for all morphisms $\phi : A \rightarrow A$ such that $\dim \phi(A) \geq d + n$,*

$$\dim \phi(V) = d.$$

We simply say Property (S) for (S^0).

In some sense Property (S) is natural. Property (S^n) implies Property (S^{n+1}) and also implies transversality. For curves, transverse implies Property (S). The fact that an abelian variety needs not to have algebraic subgroups of arbitrary dimension between 0 and g , makes delicate to compare Property (S) with the set V^{oa} . For instance, consider $A = B \times B'$ for B and B' simple abelian varieties of positive dimension. If V is a subvariety of B' satisfying property (S) in B' and $\dim V > \dim B$, then $\{0\} \times V$ satisfies property (S) in A . On the other hand $V^{oa} = \emptyset$, because one can take $H = \{0\} \times B'$ and $X = V$ in Definition 1.1. This example also shows that Conjecture 1.1 is not optimal for such A . On the contrary, if we work in a power of an elliptic curve E , property (S) is equivalent to the assumption $V^{oa} \neq \emptyset$ (see lemma 3.2). Then, in E^g , one can easily reformulate the Bounded Height Conjecture in terms of Property (S), avoiding the notion of deprived set.

Conjecture 1.2 (Bounded Height Conjecture). *Let V be an algebraic subvariety of E^g defined over $\overline{\mathbb{Q}}$. Suppose that V satisfies Property (S).*

Then, there exists $\varepsilon > 0$ and a non-empty open subset V^e of V such that $S_d(V^e, \mathcal{O}_\varepsilon)$ has bounded height.

One could hope to relax the assumption of Property (S) on the variety. Could it be sufficient to assume, as we do for curves, that V is transverse? What about a product of varieties which do satisfy Property (S)? In section 3, we prove that Theorem 1.2 is optimal for subvarieties of a power of an elliptic curve E^g .

Theorem 1.3. *Let V be a subvariety of E^g of dimension d . Suppose that V does not satisfy Property (S) (or equivalently that $V^{oa} = \emptyset$). Then, for every non-empty Zariski open subset U of V the set $S_d(U)$ does not have bounded height.*

The proof is constructive. A fundamental point is to associate to a non-torsion point of $E(\mathbb{Q})$ a Zariski dense subgroup of E^n .

A natural rising question is to investigate the height for larger codimension of the algebraic subgroup. Let Γ be a subgroup of $A(\overline{\mathbb{Q}})$ of finite rank. We denote $\Gamma_\varepsilon = \Gamma + \mathcal{O}_\varepsilon$.

Conjecture 1.3. *Let V be an irreducible algebraic subvariety of A of dimension d , defined over $\overline{\mathbb{Q}}$. Then there exists $\varepsilon > 0$ and a non-empty Zariski open subset V^e of V such that:*

- (1) *If V is weak-transverse, $S_{d+1}(V^e, \mathcal{O}_\varepsilon)$ has bounded height.*
- (2) *If V is transverse, $S_{d+1}(V^e, \Gamma_\varepsilon)$ has bounded height.*

In some cases Conjecture 1.3 is proven. For $\Gamma \neq 0$ or V weak-transverse but not transverse, the method used for the proofs is based on a Vojta inequality. This method is not effective. It gives optimal results for curves (see [6] Theorem 1.5 and [8] Theorem 1.2). On the contrary, for varieties of dimension at least two a hypothesis stronger than transversality is needed. Part (1) of the following theorem is proven by Rémond [4] Theorem 1.2 and [5]. Whereas, Part (2) is proven by the author [9] Theorem 1.5.

Theorem 1.4. *Let V be an irreducible subvariety of E^g of dimension d , defined over \mathbb{Q} . Let p be a point in $E^s(\mathbb{Q})$ not lying in any proper algebraic subgroup of E^s . Assume that V satisfies*

$$(1.1) \quad \dim(V + B) = \min(\dim V + \dim B, g)$$

for all abelian subvarieties B of E^g . Then there exists a non-empty Zariski open subset V^e of V and $\varepsilon > 0$ such that:

- (1) $S_{d+1}(V^e, \Gamma_\varepsilon)$ has bounded height,
- (2) $S_{d+1}(V^e \times p, \mathcal{O}_\varepsilon)$ has bounded height.

In Lemma 4.1, we will see that the assumption (1.1) is equivalent to Property (S). Finally we give some examples of varieties satisfying Property (S) and of varieties which do not satisfy Property (S) but for which Conjecture 1.3 holds.

To conclude we remark that, if one knows that, for $r \geq d + 1$ and V transverse, the set $S_r(V^e, \Gamma_\varepsilon)$ has bounded height, then [9] Theorem 1.1 implies that $S_r(V^e, \Gamma_\varepsilon)$ is not Zariski dense in V . If Γ has trivial rank, it is sufficient to assume V weak-transverse. This makes results on heights particularly interesting.

2. Preliminaries

Let E be an elliptic curve defined over a number field. All statement in the introduction become trivially verified for a zero-dimensional variety. In the following we avoid this case. Let V be an irreducible algebraic subvariety of E^g of dimension $0 < d < g$ defined over $\overline{\mathbb{Q}}$.

We fix on $E(\overline{\mathbb{Q}})$ the canonical Néron-Tate height function. We denote by $\|\cdot\|$ the induced semi-norm on $E(\overline{\mathbb{Q}})$. For $x = (x_1, \dots, x_g) \in E^g(\overline{\mathbb{Q}})$, we denote

$$\|x\| = \max_i \|x_i\|.$$

For $\varepsilon \geq 0$, we define

$$\mathcal{O}_\varepsilon = \{\xi \in E^g(\overline{\mathbb{Q}}) : \|\xi\| \leq \varepsilon\}.$$

The height of a non-empty set $S \subset E^g(\overline{\mathbb{Q}})$ is the supremum of the heights of its elements. The degree of S is the degree (possibly ∞) of the field of definition of the points of S .

The ring of endomorphism $\text{End}(E)$ is isomorphic either to \mathbb{Z} (if E does not have C.M.) or to an order in an imaginary quadratic field (if E has C.M.). We consider on $\text{End}(E)$ the hermitian scalar product $\langle \cdot, \cdot \rangle$ induced by \mathbb{C} and denote by $|\cdot|$ the associated norm. Note that the metric does not depend on the embedding of $\text{End}(E)$ in \mathbb{C} . We denote by $M_{r,g}(\text{End}(E))$ the module of $r \times g$ matrices with entries in $\text{End}(E)$. For $F = (f_{ij}) \in M_{r,g}(\text{End}(E))$, we define

$$|F| = \max_{ij} |f_{ij}|.$$

We identify a morphism $\phi : E^g \rightarrow E^r$ with a matrix in $M_{r,g}(\text{End}(E))$.

Let B be an algebraic subgroup of E^g of codimension r . Then $B \subset \ker \phi_B$ for a surjective morphism $\phi_B : E^g \rightarrow E^r$. Conversely, we denote by B_ϕ the kernel of a surjective morphism $\phi : E^g \rightarrow E^r$. Then B_ϕ is an algebraic subgroup of E^g of codimension r .

If $\phi : E^g \rightarrow E^{g'}$ is a surjective morphism, we can complement ϕ and define an isogeny $f : E^g \rightarrow E^g$ such that $f(\ker \phi) = 0 \times E^{g-g'}$ and $\pi_1 f = \phi$, where $\pi_1 : E^g \rightarrow E^{g'}$ is the natural projection on the first g' coordinate.

More precisely; recall that every abelian subvariety of E^g of dimension n is isogenous to E^n . Then $\ker \phi$ is isogenous to $E^{g-g'}$, let i be such an isogeny. Let $(\ker \phi)^\perp$ be an orthogonal complement of $\ker \phi$ in E^g . Then $E^{g'}$ is isogenous to $(\ker \phi)^\perp$. Let $j : E^{g'} \rightarrow (\ker \phi)^\perp$ be such an isogeny. Define the isogeny

$$f : E^g \rightarrow E^g$$

$$x \rightarrow (\phi(x), i(x - j(\phi(x)))) .$$

This f has the wished property.

Let us state a classical

Lemma 2.1. *For every algebraic subvariety X of E^g of dimension d there exists a projection on d coordinates such that the restriction to X is dominant.*

Proof. Let d_0 be the maximal integer such that the restriction of $\pi_0 : E^g \rightarrow E^{d_0}$ to X is surjective. If $d_0 \geq d$, nothing has to be shown. Suppose that $d_0 < d$. Without loss of generality, suppose that π_0 projects on the first d_0 coordinates. For $d_0 < i \leq g$, we define $\pi_i : E^g \rightarrow E^{d_0+1}$ to be the projection $\pi_i(x_1, \dots, x_g) \rightarrow (x_1, \dots, x_{d_0}, x_i)$. Let $i_\pi : E^{d_0+1} \rightarrow E^g$ be the immersion such that $\pi_i \cdot i_\pi = id_{E^{d_0+1}}$.

We denote by $X_i = i_\pi \cdot \pi_i(X) \subset E^g$. By maximality of d_0 we see that $\dim X_i = d_0$. Furthermore X is the fiber product of X_i over $\pi_0(X) = \pi_0(X_i)$. Then $d = \dim X = \dim(X_{d_0+1} \times_{\pi_0(X)} \cdots \times_{\pi_0(X)} X_g) = d_0$, which contradicts $d > d_0$. □

We show an easy application.

Lemma 2.2. *If V does not satisfy Property (S) then there exists a surjective morphism $\phi : E^g \rightarrow E^d$ such that $0 < \dim \phi(V) < d$.*

Proof. If V does not satisfy Property (S), then there exists a surjective morphism $\phi : E^g \rightarrow E^d$ such that $\dim \phi(V) < d$. If $\dim \phi(V) > 0$, nothing has to be shown. If $\dim \phi(V) = 0$, Lemma 2.1 gives a morphism $r : E^g \rightarrow E$ such that the restriction to X is surjective. Replace the first row of ϕ by r . □

3. The Bounded Height Conjecture and its optimality

In the following we first show that the set $S_d(V)$ is dense in V . We then ask if Property (S) is necessary to show that $S_d(V)$ has bounded height. We give here a positive answer. Meanwhile we try to understand the geometric aspect of Property (S).

An easy example of a variety which does not satisfy Property (S) is a split variety $V_1 \times V_2 \times \cdots \times V_n$ with the $V_i \subset E^{g_i}$. It is natural to ask if

only this kind of product varieties do not satisfy Property (S). This is not the case, as Lemma 3.1 and the examples below show.

Definition 3.1. Let $V \subset E^g$ be a variety of dimension d .

- (1) V is split if there exists an isogeny $\phi : E^g \rightarrow E^g$ such that

$$\phi(V) = V_1 \times V_2$$

with $V_i \subset E^{g_i}$ and $g_i \neq 0$, for $i = 1, 2$.

We say that V is non-split if the above property is not verified.

- (2) V is n -generically split if there exists an isogeny $\phi : E^g \rightarrow E^g$ such that $\phi(V)$ is contained in a proper split variety $W = W_1 \times W_2$ with $W_i \subset E^{g_i}$ and

$$\dim W_1 < \min(d, g_1 - n).$$

We say that V is n -generically non-split if it is not n -generically split.

We simply say generically split for 0-generically split.

Clearly generically non-split implies non-split. Note that non-split implies transverse. Indeed if V is not transverse, then there exists an isogeny $\phi : E^g \rightarrow E^g$ such that $\phi(V) \subset p \times E^r$. Set $V_1 = p$ and $V_2 = \pi(\phi(V))$, where π is the projection on the last r coordinates.

The following lemma clarifies the equivalence between Property (S^n) and the n -generically non-split property.

Lemma 3.1. A subvariety $V \subset E^g$ satisfies Property (S^n) if and only if V is n -generically non-split.

Proof. First suppose that V does not satisfy Property (S^n). Then, there exists $\phi_1 : E^g \rightarrow E^{d+n}$ such that $V_1 = \phi_1(V)$ has dimension $d_1 < d$. Let $f = \begin{pmatrix} \phi_1 \\ \phi_1^\perp \end{pmatrix}$. Then

$$f(V) \subset V_1 \times E^{g-d-n}.$$

Furthermore $\dim V_1 < d = \min(d, d+n-n)$. Thus V is n -generically split.

Secondly suppose V is n -generically non-split. Then, up to an isogeny, V is contained in $W = W_1 \times W_2$ with $W_i \subset E^{g_i}$ and $\dim W_1 = d_1 < \min(d, g_1 - n)$. Consider the projection V_1 of V on the first $d+n$ coordinates.

If $g_1 \geq d+n$, then V_1 is contained in the projection of W_1 . As dimensions cannot increase by projection, $\dim V_1 \leq \dim W_1 < d$.

If $g_1 < d+n$, then we have $V_1 \subset W_1 \times E^{d+n-g_1}$. Thus $\dim V_1 \leq d_1 + d + n - g_1 < d$ because $d_1 < g_1 - n$. So V does not satisfy Property (S^n). \square

It is then natural to give an example of a non-split variety which is generically split, or equivalently which does not satisfy Property (S).

Example. Let us show at once that for a hypersurface, the notion of non-split and generically non-split coincide.

Let V be a non-split hypersurface in E^{d+1} . If V were generically split, then, for an isogeny ϕ , $\phi(V)$ would be contained in a proper split variety $W_1 \times W_2$. For a dimensional argument $\phi(V) = W_1 \times W_2$, contradicting the non-split assumption.

Example. In some sense, to give an example of a non-split but generically-split variety it is necessary to consider varieties of large codimension.

In \mathbb{G}_m^n it is easier to write equations. Consider the surface V in \mathbb{G}_m^4 parameterized by u and v , and given by the set of points $(u, u^5 + 1, 5u^4v + u, v + u^5 + 1)$. This is simply the envelope variety V of the irreducible plane curve $C = (u, u^5 + 1)$. The envelope is constructed as follows. To a point $p \in C$ we associate the tangent line t_p in p . Then $V = \cup_{p \in C}(p, t_p)$. The set V is an algebraic surface; let z_1, z_2, z_3, z_4 be the variables, then V is the zero set of

$$\begin{cases} z_3 = 5z_1^4(z_4 - z_2) + z_1, \\ z_2 = z_1^5 + 1. \end{cases}$$

Note that V has trivial stabilizer. Indeed if $(\xi_1, \dots, \xi_4) \in \text{stab } V$, then for any u, v there exist a, b such that

$$\begin{cases} a = \xi_1 u \\ a^5 + 1 = \xi_2 (u^5 + 1) \\ 5a^4 b + a = \xi_3 (5u^4 v + u) \\ b + a^5 + 1 = \xi_4 (v + u^5 + 1) \end{cases}$$

This implies that $\xi_i = 1$.

The projection of V on the first two coordinates is exactly the curve C defined by $z_2 = z_1^5 + 1$. Thus V does not satisfy Property (S), however it is non-split. If, on the contrary, V were split, then, for an isogeny ϕ , $\phi(V) = V_1 \times V_2$. Since V is transverse, V_i have positive dimension and $V_i \subset \mathbb{G}_m^{n_i}$ for $n_1 + n_2 = 4$. Since V has trivial stabilizer, $n_i > 1$. Thus $n_i = 2$. Then, the non-singular integral 4×4 -matrix $\phi = (\varphi_{ij})$ is such that the image of

$$(u, v) \rightarrow \begin{pmatrix} f_1(u, v) \\ f_2(u, v) \end{pmatrix} = \begin{pmatrix} u^{\varphi_{11}}(u^5 + 1)^{\varphi_{12}}(5u^4v + u)^{\varphi_{13}}(v + u^5 + 1)^{\varphi_{14}} \\ u^{\varphi_{21}}(u^5 + 1)^{\varphi_{22}}(5u^4v + u)^{\varphi_{23}}(v + u^5 + 1)^{\varphi_{24}} \end{pmatrix}$$

is a curve. This means that

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial u} \end{pmatrix} = 0$$

as a rational function in u and v . This determinant factorises as a product of polynomials which can not be identically zero times $P(u, v)$. Here $P(u, v)$ is a polynomial in u and v with seven coefficients which are quadratic

functions of φ_{ij} . A tedious, but elementary computation shows that if these 7 coefficients are all zero, then all the minors of $(\varphi_{ij})_{1 \leq i \leq 2, 1 \leq j \leq 4}$ are zero, except possibly the minor $(\varphi_{ij})_{1 \leq i \leq 2, 1 \leq j \leq 2}$. The same argument for the functions $f_3(u, v), f_4(u, v)$ defined by the last two rows of ϕ shows that for the matrix $(\varphi_{ij})_{3 \leq i \leq 4, 1 \leq j \leq 4}$ only the minor $(\varphi_{ij})_{3 \leq i \leq 4, 1 \leq j \leq 2}$ could be non-zero. This contradicts $\det \phi \neq 0$.

We remark, that there are also non-split transverse varieties which do not satisfy Property (S^n) : One can extend this last example taking the envelope surface of a transverse curve in \mathbb{G}_m^{n+2} .

Following the work of Bombieri, Masser and Zannier [2] in a torus, Rémond [5] proposition 4.2 proves that $V^{oa} \neq \emptyset$ is equivalent condition (1.1), which in turn is equivalent to property (S) in E^g . Using the generically-split property we also directly prove one implication.

Lemma 3.2. *A variety $V \subset E^g$ does not satisfy property (S) if and only if $V^{oa} = \emptyset$*

Proof. Suppose that V has dimension d and does not satisfy property (S) . By lemma 3.1, there exists an isogeny ϕ such that $\phi(V) \subset W_1 \times W_2$ with $W_i \subset E^{g_i}$ and $\dim W_1 < \min(d, g_1)$. Then, the intersection of V with the cosets $\phi^{-1}(x \times E^{g_2})$ for $x \in W_1$ are either empty or anomalous. In addition each point of V belongs to such an intersection. So V^{oa} is empty.

The reverse implication is proven by Rémond [5] proposition 4.2. According to his definition 1.2 and the sentence just after, we have $Z_{V,an}^{(d+1)} = V \setminus V^{oa}$. Thus [5] proposition 4.2.(1) is equivalent to $V^{oa} \neq \emptyset$. By [5] Definition 1.1 and Lemma 4.1 in this article, we have that [5] proposition 4.2.(2) is equivalent to Property (S) .

□

The following lemma shows that in the Bounded Height Conjecture we can not expect the set in the consequence to be non-dense. This lemma will also be used in the proof of Theorem 1.3.

Lemma 3.3. *Let V be an irreducible subvariety of E^g of dimension $1 \leq d < g$. Then the set $S_d(V) \setminus S_g(V)$ is dense in V .*

Proof. We shall distinguish two cases with regard to whether V is or is not the translate of an abelian subvariety by a torsion point.

Suppose V is not such a translate. Then, the Manin-Mumford Conjecture, a theorem of Raynaud, ensures that the torsion $S_g(V)$ is not dense in V . Our claim is then equivalent to show that $S_d(V)$ is dense in V . Consider a surjective morphism (for example a projection) $\phi : E^g \rightarrow E^d$ such that the restriction to V is dominant. Use lemma 2.1 to ensure the

existence of such a morphism. Let E_{Tor}^d be the torsion group of E^d . The preimage on V via ϕ of E_{Tor}^d is dense in V and it is a subset of $S_d(V)$.

Suppose now that V is the translate of an abelian subvariety by a torsion point. Up to an isogeny, we can assume $V = E^d \times p$ for $p = (p_1, \dots, p_{g-d}) \in E_{\text{Tor}}^{g-d}$. Note that, by Kronecker’s Theorem, for any $x \in V$, $x + E_{\text{Tor}}^d$ is dense in V .

Since p is a torsion point,

$$\left((E(\overline{\mathbb{Q}}) \setminus E_{\text{Tor}} \times \{0\}^{d-1} \times p) + E_{\text{Tor}}^d \right) \subset S_d(V) \setminus S_g(V).$$

In addition this set is dense in V , because $E(\overline{\mathbb{Q}}) \setminus E_{\text{Tor}}$ is non-empty (even dense in E). □

We now discuss the assumption of Property (S). In general, for $V = V_1 \times V_2$ with $\dim V_1 = d_1$ and $\dim V_2 = d_2$, we have $S_{d_1}(V_1) \times S_{d_2}(V_2) \subset S_d(V)$. Could we have equality if we assume, for example, that each factor satisfies Property (S)? Similarly, does Conjecture 1.2 hold for such a product variety or for a non-split variety? The answer is negative.

To simplify the formulation of the statements we characterize the sets which break Conjecture 1.2.

Definition 3.2. *We say that a subset V^u of $V(\overline{\mathbb{Q}})$ is densely unbounded if V^u is Zariski dense in V and for every non-empty Zariski open U of V the intersection $V^u \cap U$ does not have bounded height. Equivalently V^u is densely unbounded if, for a sequence $\{N\}$ of positive reals going to infinity, the set*

$$V^u[N] = \{x \in V^u : \|x\| > N\}$$

is Zariski dense in V .

Proof of the equivalence of the definitions. Suppose that there exists a non-empty open U such that $V^u \cap U$ has height bounded by N_0 . The set $Z = V \setminus U$ is a proper closed subset of V , and therefore not dense. So $V^u[N_0 + 1] \subset Z$ can not be dense.

Suppose now that there exists an unbounded sequence $\{N\}$ such that $V^u[N_0]$ is not dense for some N_0 . Then the Zariski closure Z of $V^u[N_0]$ is a proper closed subset of V . So $U = V \setminus Z$ is a non-empty open set such that $V^u \cap U$ has height bounded by N_0 . □

Let us prove a preparatory lemma for the proof of Theorem 1.3.

Lemma 3.4. *Let z_0 be a non-torsion point in $E(\overline{\mathbb{Q}})$. Let n be a positive integer. Define $G_{z_0,n} = \langle z_0 \rangle_{\text{End}(E)}^n$. For $N \in \mathbb{N}$, the set*

$$G_{z_0,n}[N\|z_0\|] = \{p \in G_{z_0,n} : \|p\| > N\|z_0\|\}$$

is Zariski dense in E^n . As a consequence $G_{z_0,n}$ is dense in E^n .

Proof. Denote by $\Sigma = \langle z_0 \rangle_{\text{End}(E)}$ the submodule of E generated by z_0 . Then $G_{z_0, n} = \Sigma^n$. Recall that $\Sigma[N\|z_0\|] = \{p \in \Sigma : \|p\| > N\|z_0\|\}$. Then $(\Sigma[N\|z_0\|])^n \subset G_{z_0, n}[N\|z_0\|]$. As $\Sigma[N\|z_0\|]$ is an infinite set, it is dense in E . Then $(\Sigma[N\|z_0\|])^n$ is dense in E^n .

Note that $G_{z_0, n}$ contains $G_{z_0, n}[0]$, so it is also dense. □

We are ready to show the optimality of the Bounded Height Conjecture.

Proof of Theorem 1.3. Suppose that V does not satisfy Property (S). We are going to construct a densely unbounded set of V which is a subset of $S_d(V)$.

By Lemma 2.2, there exists a surjective morphism $\psi : E^g \rightarrow E^d$ such that $0 < \dim \psi(V) < d$. Denote $V_1 = \psi(V)$ and $d_1 = \dim V_1$. We can fix an isogeny and suppose that ψ is the projection on the first d coordinates, thus $V \subset V_1 \times E^{g-d}$. Let $x \in V$. Then $x = (x_1, x_2)$ with $x_1 \in V_1$ and $x_2 \in E^{g-d}$. Consider

$$x_1 \times W_{x_1} = V \cap (x_1 \times E^{g-d}).$$

There exists an open dense subset U_1 of V_1 such that the algebraic variety W_{x_1} is equidimensional of dimension $d_2 = d - d_1$. Let V_{x_1} be an irreducible component of W_{x_1} . By Lemma 2.1, there exists a projection $\pi_{x_1} : E^{g-d} \rightarrow E^{d_2}$ such that the restriction

$$(3.1) \quad \pi_{x_1|V_{x_1}} : V_{x_1} \rightarrow E^{d_2}$$

is dominant and even surjective and therefore its fibers are generically finite.

Consider $V_1 \subset E^d$. Since V is irreducible also V_1 is. By Lemma 3.3, applied with $V = V_1$, $d = d_1$ and $g = d$, the set $S_{d_1}(V_1) \setminus S_d(V_1)$ is Zariski dense in V_1 . Define

$$V_1^u = U_1 \cap (S_{d_1}(V_1) \setminus S_d(V_1)).$$

Then all points in V_1^u are non-torsion and V_1^u is a dense subset of V_1 . By definition of $S_{d_1}(V_1)$, if $x_1 \in V_1^u \subset S_{d_1}(V_1)$, then there exists $\phi_1 : E^d \rightarrow E^{d_1}$ of rank d_1 such that

$$(3.2) \quad \phi_1(x_1) = 0.$$

Let z_k be a coordinate of $x_1 = (z_1, \dots, z_d)$ such that $\|z_k\| = \max_i \|z_i\|$. Only the torsion has norm zero. Since x_1 is non-torsion, then $\|z_k\| > 0$.

For each point $x_1 \in V_1^u$ we will construct a subset of $x_1 \times V_{x_1}$ which is, both, densely unbounded in $x_1 \times V_{x_1}$ and a subset of $S_d(V)$.

We denote by

$$\phi_2 = (0, \dots, 0, \varphi_k, 0, \dots, 0) : E^d \rightarrow E^{d_2}$$

a morphism such that only the k -th column is non zero. For a positive integer N , we define

$$\mathcal{F}(N) := \{\phi_2 = (0, \dots, 0, \varphi_k, 0, \dots, 0) : E^d \rightarrow E^{d_2} \text{ s.t. } |\phi_2| > N\}$$

and

$$x_1 \times V_{x_1}^u(N) := \{(x_1, y) \in (x_1, V_{x_1}) \text{ s.t. } \exists \phi_2 \in \mathcal{F}(N) \text{ with } \phi_2(x_1) = \pi_{x_1}(y)\}.$$

We simply denote

$$U_{x_1} = V_{x_1}^u(1).$$

We want to show that $x_1 \times U_{x_1}$ is densely unbounded in $x_1 \times V_{x_1}$.

(a) - First we show that

$$x_1 \times V_{x_1}^u(N) \subset (x_1 \times U_{x_1})[N||z_k||].$$

For $(x_1, y) \in x_1 \times V_{x_1}^u(N)$ there exists $\phi_2 \in \mathcal{F}(N)$ such that $\phi_2(x_1) = \pi_{x_1}(y)$. Thus,

$$||y|| \geq ||\pi_{x_1}(y)|| = ||\phi_2(x_1)|| \geq |\phi_2| ||z_k|| > N ||z_k||.$$

Whence $(x_1, y) \in (x_1 \times U_{x_1})[N||z_k||]$.

(b) - We now show that $x_1 \times V_{x_1}^u(N)$ is dense in $x_1 \times V_{x_1}$. Let $(a_1 z_k, \dots, a_{d_2} z_k) \in G_{z_k, d_2}[N||z_k||]$ with $a_i \in \text{End}(E)$. Then $\max_i |a_i| > N$. Let ϕ_2 be the morphism from E^d to E^{d_2} such that the k -th column of ϕ_2 is the vector $\varphi_k = (a_1, \dots, a_{d_2})^t$ and all other entries are zeros. Then $\phi_2 \in \mathcal{F}(N)$ and $\phi_2(x_1) = (a_1 z_k, \dots, a_{d_2} z_k)$. So, we have the inclusion

$$G_{z_k, d_2}[N||z_k||] \subset \bigcup_{\phi_2 \in \mathcal{F}(N)} \phi_2(x_1).$$

By Lemma 3.4, $G_{z_k, d_2}[N||z_k||]$ is a Zariski dense subset of E^{d_2} . Thus, also the set $\bigcup_{\phi_2 \in \mathcal{F}(N)} \phi_2(x_1)$ is Zariski dense in E^{d_2} . By (3.1) the map $\pi_{x_1|V_{x_1}}$ is surjective. Then for any $\phi_2 \in \mathcal{F}(N)$ there exists $y \in V_{x_1}$ such that $\pi_{x_1}(y) = \phi_2(x_1)$. Therefore $x_1 \times V_{x_1}^u(N)$ is Zariski dense in $x_1 \times V_{x_1}$.

In view of Definition 3.2, part (a) and (b) above show that $x_1 \times U_{x_1}$ is a densely unbounded subset of $x_1 \times V_{x_1}$. In addition, by definition of $x_1 \times U_{x_1}$, for every $(x_1, y) \in x_1 \times U_{x_1}$ there exists $\phi_2 : E^d \rightarrow E^{d_2}$ such that

$$(3.3) \quad \phi_2(x_1) = \pi_{x_1}(y).$$

Consider (x_1, y) with $x_1 \in V_1^u$ and $y \in U_{x_1}$. By relations (3.2) and (3.3), the morphism

$$\phi = \begin{pmatrix} \phi_1 & 0 \\ -\phi_2 & \pi_{x_1} \end{pmatrix} : E^g \rightarrow E^d,$$

has rank equal to $\text{rk } \phi_1 + \text{rk } \pi_{x_1} = d_1 + d_2$ and

$$\phi(x_1, y) = 0.$$

So $(x_1, y) \in S_d(V)$.

Let

$$W_{x_1}^u = \bigcup V_{x_1}^u(1) = \bigcup U_{x_1}$$

for V_{x_1} varying over the irreducible components of W_{x_1} . We conclude that the set

$$\bigcup_{x_1 \in V_1^u} x_1 \times W_{x_1}^u \subset S_d(V)$$

is densely unbounded in V .

□

4. Final remarks

It is then natural to investigate the height property for the codimension of the algebraic subgroups at least $d + 1$. We expect that Conjecture 1.3 holds. Let us say at once that the (weak)-transverse hypothesis is in general necessary, however it is not clear if it is sufficient.

Theorem 1.4 is a special case of Conjecture 1.3. We show that the condition (1.1) coincides with Property (S). Compare the following lemma with [4] lemma 7.2.

Lemma 4.1. *An irreducible variety $V \subset E^g$ satisfies Property (S) if and only if $\dim(V + B) = \min(\dim V + \dim B, g)$ for all abelian subvarieties B of E^g .*

Proof. Note that E^g/B is isogenous to $E^{g-\dim B}$. Consider the natural projection $\pi_B : E^g \rightarrow E^{g-\dim B}$. Then

$$(4.1) \quad \dim \pi_B(V) = \dim(V + B) - \dim B.$$

Denote by d the dimension of V . Suppose that V satisfies Property (S), we have

- If $g - \dim B \geq d$, then $\dim \pi_B(V) = d$.
- If $g - \dim B \leq d$, then $\dim \pi_B(V) = g - \dim B$.

Use (4.1) to deduce $\dim(V + B) = \min(d + \dim B, g)$.

Suppose now that $\dim(V + B) = \min(d + \dim B, g)$ for all abelian subvarieties B of codimension d . Note that, if $\phi : E^g \rightarrow E^d$ is a surjective morphism, then the zero component of $\ker \phi$ is an abelian variety of codimension d . Relation (4.1) shows at once that V satisfies Property (S). □

We observe that, for $S_{d+1}(V)$, the natural analogue to Conjecture 1.2, is to assume Property (S^1). Property (S^1) is weaker than (S). There are even split varieties which satisfy Property (S^1). Potentially, the method used by Habegger to prove Theorem 1.2 extends to show that, for V satisfying Property (S^1), there exists a non-empty open V^e such that $S_{d+1}(V^e)$ has bounded height. However, neither such a statement nor Theorem 1.4 are

optimal: transversality is expected to be a sufficient assumption, as the following examples suggest. We give simple examples of a transverse variety V of dimension d which does not satisfy Property (S) or (S^1) but such that $S_{d+1}(V, \Gamma_\varepsilon)$ is not Zariski dense.

Example. Let V_1 be a variety in E^{d_1+n+1} of dimension d_1 . Suppose that V_1 satisfies Property (S) . If you like take a transverse curve. By Theorem 1.4 (1), for every Γ' of finite rank there exists $\varepsilon > 0$ such that $S_{d_1+1}(V_1, \Gamma'_\varepsilon)$ has bounded height. By [9] Theorem 1.1, applied to V_1 of dimension d_1 , we obtain that there exists $\varepsilon > 0$ such that:

$$(4.2) \quad S_{d_1+1}(V_1, \Gamma'_\varepsilon) \text{ is not Zariski dense in } V_1.$$

Let $V = V_1 \times E^{d_2}$ and $g = d_1 + d_2 + n + 1$, then V is transverse in E^g . Furthermore, V does not satisfy Property (S^n) . Indeed $\dim V = d = d_1 + d_2$. The projection on the first $d + n$ coordinates is $V_1 \times E^{d_2-1}$ which has dimension $d - 1$. Let $\Gamma \subset E^g$ be a subgroup of finite rank and let Γ' be its projection on the first $d_1 + n + 1$ coordinates.

By [9] Lemma 4.1 we obtain

$$S_{d+1}(V, \Gamma_\varepsilon) \subset S_{d_1+1}(V_1, \Gamma'_\varepsilon) \times E^{d_2}.$$

Then, using (4.2), $S_{d+1}(V, \Gamma_\varepsilon)$ is Zariski non-dense in V . Define $Z = S_{d+1}(V, \Gamma_\varepsilon)$. Then $S_{d+1}(V \setminus Z, \Gamma_\varepsilon)$ is empty and so it also has bounded height.

Example. Let $V = V_1 \times V_2$ with V_i a hypersurface in E^{d_i+1} satisfying Property (S) . The projection on the first $d = d_1 + d_2$ coordinates shows that V does not satisfy Property (S) . However V satisfies Property (S^1) .

We are going to show that

$$(4.3) \quad S_{d+1}(V, F) \subset (S_{d_1}(V_1, F) \times S_{d_2}(V_2, F)) \cup (S_{d_1+1}(V_1, F) \times V_2) \cup (V_1 \times S_{d_2+1}(V_2, F)).$$

Let $g = d_1 + d_2 + 2$. Let $(x, y) \in S_{d+1}(V, F)$ with $x \in V_1$ and $y \in V_2$. Then, there exist $\phi : E^g \rightarrow E^{d+1}$ of rank $d + 1$ and $(f, f') \in F$ such that

$$\phi((x, y) - (f, f')) = 0.$$

Decompose $\phi = (A|B)$ with $A : E^{d_1+1} \rightarrow E^{d+1}$ and $B : E^{d_2+1} \rightarrow E^{d+1}$. Then

$$d_2 + d_1 + 1 = \text{rk } \phi \leq \text{rk } A + \text{rk } B.$$

Note that $\text{rk } A \leq d_1 + 1$ and $\text{rk } B \leq d_2 + 1$ because of the number of columns. Then one of the following cases occurs:

- (1) $\text{rk } A = d_1$ or $\text{rk } B = d_2$,
- (2) $\text{rk } A = d_1 + 1$ and $\text{rk } B = d_2 + 1$.

(1)- If the rank of B is d_2 then, with the Gauss algorithm, one finds an invertible matrix $\Delta \in \text{Mat}_{d+1}(\text{End}(E))$ such that

$$\Delta\phi = \begin{pmatrix} \varphi_1 & 0 \\ \star & \varphi_2 \end{pmatrix},$$

with φ_1 of rank $d_1 + 1$.

If the rank of A is d_1 then one finds an invertible matrix $\Delta \in \text{Mat}_{d+1}(\text{End}(E))$ such that

$$\Delta\phi = \begin{pmatrix} \varphi_1 & \star \\ 0 & \varphi_2 \end{pmatrix},$$

with φ_2 of rank $d_2 + 1$.

Then either $x \in S_{d_1+1}(V_1, F)$ or $y \in S_{d_2+1}(V_2, F)$. So $(x, y) \in (S_{d_1+1}(V_1, F) \times V_2) \cup (V_1 \times S_{d_2+1}(V_2, F))$.

(2)- With the Gauss algorithm one can find two invertible matrices $\Delta_i \in \text{Mat}_{d+1}(\text{End}(E))$ such that

$$\begin{aligned} \Delta_1\phi &= (aI_{d+1}|l) \\ \Delta_2\phi &= (l'|bI_{d+1}) \end{aligned}$$

with $a, b \in \text{End}(E) \setminus 0$ and I_{d+1} the identity matrix. The last d_2 rows of $\Delta_1\phi$ show that $y \in S_{d_2}(V_2, F)$ and the first d_1 rows of $\Delta_2\phi$ show that $x \in S_{d_1}(V_1, F)$. Thus $(x, y) \in (S_{d_1}(V_1, F) \times S_{d_2}(V_2, F))$.

We now apply the inclusion (4.3) to the case of curves, and we deduce a non-density result for surfaces. Let $V_i = C_i$ be transverse curves in E^2 . By Theorem 1.2, there exists $\varepsilon > 0$ such that $S_1(C_i, \mathcal{O}_\varepsilon)$ has bounded height. In view of the Bogomolov Conjecture, a theorem of Ullmo, one can choose ε such that $S_2(C_i, \mathcal{O}_\varepsilon)$ is finite. Define $F = \mathcal{O}_\varepsilon$. Then, relation (4.3) implies that $S_3(C_1 \times C_2, \mathcal{O}_\varepsilon)$ has bounded height. In addition $C_1 \times C_2$ is transverse in E^4 . Using [9] Theorem 1.1, we conclude that $S_3(C_1 \times C_2, \mathcal{O}_\varepsilon)$ is Zariski non-dense.

According to Theorem 1.2 and [9] Theorem 1.1, one can do similar considerations for hypersurfaces.

These last examples give evidence that the transverse or weak-transverse hypothesis is sufficient for Conjecture 1.3. Precisely, the idea is that if U_1 is a dense subset of V_1 of bounded height, then the set $U_1 \times V_2$ is densely unbounded in $V_1 \times V_2$, (this is more or less what makes Property (S) necessary for Theorem 1.2). Instead if U_1 is Zariski closed in V_1 , then the set $U_1 \times V_2$ is still Zariski closed in $V_1 \times V_2$.

Could one extend the idea in the last examples to show that for the product of varieties satisfying Property (S) Conjecture 1.3 holds?

This is not an easy matter; even the case of $C_1 \times C_2$ for C_1 transverse in E^2 and C_2 transverse in E^3 remains open.

Acknowledgments: I kindly thank the Referee for his accurate and valuable remarks.

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