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Asymptotic representations for Fibonacci reciprocal sums and Euler's formulas for zeta values

par Carsten ELSNER, Shun SHIMOMURA et Iekata SHIOKAWA

Résumé. Nous présentons les représentations asymptotiques pour certaines sommes des réciproques des nombres de Fibonacci et des nombres de Lucas quand un paramètre tend vers une valeur critique. Comme cas limite de nos résultats, nous obtenons les formules d'Euler pour les valeurs des fonctions de zeta.

Abstract. We present asymptotic representations for certain reciprocal sums of Fibonacci numbers and of Lucas numbers as a parameter tends to a critical value. As limiting cases of our results, we obtain Euler's formulas for values of zeta functions.

1. Introduction

Let $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ be the Fibonacci numbers and the Lucas numbers, defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),$$
$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0).$$

For various reciprocal sums of these numbers, transcendence and algebraic relations were studied. Duverney, Ke. Nishioka, Ku. Nishioka, and the last named author [3] (see also [2]) proved the transcendence of the numbers $\sum_{n=1}^{\infty} F_n^{-2s}$, $\sum_{n=1}^{\infty} L_n^{-2s}$, $\sum_{n=1}^{\infty} F_{2n-1}^{-s}$, $\sum_{n=1}^{\infty} L_{2n}^{-s}$ $(s \in \mathbb{N})$ by using Nesterenko's theorem ([10]) on Ramanujan functions. In [4], for $\zeta_{\mathrm{F}}(s) := \sum_{n=1}^{\infty} F_n^{-s}$, we proved that the values $\zeta_{\mathrm{F}}(2)$, $\zeta_{\mathrm{F}}(4)$, $\zeta_{\mathrm{F}}(6)$ are algebraically independent, and that for any integer $s \geq 4$

$$\zeta_{\mathrm{F}}(2s) - r_s \zeta_{\mathrm{F}}(4) \in \mathbb{Q}(\zeta_{\mathrm{F}}(2), \zeta_{\mathrm{F}}(6))$$

with some $r_s \in \mathbb{Q}$; for example

$$\zeta_{\mathrm{F}}(8) - \frac{15}{14}\zeta_{\mathrm{F}}(4) = \frac{1}{378(4u+5)^2}\Big(256u^6 - 3456u^5 + 2880u^4 + 1792u^3v - 11100u^3 + 20160u^2v - 10125u^2 + 7560uv + 3136v^2 - 1050v\Big)$$

with $u := \zeta_{\mathrm{F}}(2)$, and $v := \zeta_{\mathrm{F}}(6)$.

Assume that $\alpha,\ \beta \in \mathbb{C}$ satisfy

$$\alpha\beta = -1, \quad |\beta| < 1. \tag{1}$$

Let $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ be sequences defined by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}\,, \quad V_n = \alpha^n + \beta^n \quad (n \geq 0), \tag{2}$$

which are generalized Fibonacci and Lucas numbers, respectively. Indeed, if $\beta = (1-\sqrt{5})/2$, then $U_n = F_n$, $V_n = L_n$. For $s \in \mathbb{N}$ consider the reciprocal sums

$$h_{2s} := (\alpha - \beta)^{-2s} \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}}\,, \tag{3}$$

$$g_{2s-1}^{*} := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{V_{2n-1}^{2s-1}}\,, \tag{4}$$

which are holomorphic for $|\beta| < 1$. For $\{h_{2s}\}_{s\in\mathbb{N}}$ (respectively, $\{g_{2s-1}^{*}\}_{s\in\mathbb{N}}$) algebraic relations were discussed in [6] (respectively, [5]). These sums may also be regarded as functions of the modulus k of Jacobian elliptic functions (see Sections 4 and 5, also [4], [5], [6]).

In this paper we present asymptotic representations for these sums as $\beta \to -1+0$ (or $k \to 1-0$). Degenerate cases of our expressions coincide with Euler's formulas for $\zeta(2s) = \sum_{n=1}^{\infty} n^{-2s}$ and $L(2s-1) = \sum_{n=1}^{\infty} (-1)^{n+1}(2n-1)^{-(2s-1)}$, respectively (see Section 3).

2. Statement of results

Let $B_{2\nu}$ and $E_{2\nu}$ ($\nu = 0, 1, 2, \ldots$) denote, respectively, the Bernoulli and the Euler numbers:

$$\begin{aligned} &B_0 = 1, \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \ldots; \\ &E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \ldots, \end{aligned}$$

and let s be a positive integer. For $s \geq 2$, let $\sigma_1(s), \ldots, \sigma_{s-1}(s)$ be the elementary symmetric functions of the $s-1$ numbers $-1^2, -2^2, \ldots, -(s-1)^2$ defined by

$$\sigma_i(s) = (-1)^i \sum_{1\leq r_1 < \ldots < r_i \leq s-1} r_1^2 \cdots r_i^2 \quad (1 \leq i \leq s-1),$$

and for $s \geq 0$, set $\sigma_0(s) = 1$. For $s \geq 1$ let $\tau_1(s), \ldots, \tau_s(s)$ be the elementary symmetric functions of the s numbers $-1^2, -3^2, \ldots, -(2s-1)^2$ given by

$$\tau_i(s) = (-1)^i \sum_{1\leq r_1 < \cdots < r_i \leq s} (2r_1 - 1)^2 \cdots (2r_i - 1)^2 \qquad (1 \leq i \leq s),$$

and for $s \geq 0$ set $\tau_0(s) = 1$. Our results are given as follows:

Theorem 1. *For $-1 < \beta < -1 + \delta_0$, we have*

$$(\alpha^2 - \beta^2)^{2s} h_{2s} = \Phi_s(\eta)\big(1 + O(e^{-\pi^2/(2\eta)})\big),$$
$$\eta := -\log(-\beta) = (1+\beta)(1 + O(1+\beta)).$$

Here δ_0 is a sufficiently small positive number, and $\Phi_s(X)$ is an entire function written in the form

$$\Phi_s(X) := \frac{2^s}{(2s-1)!}\Big(X^{-2}(\cosh(4X) - 1)\Big)^s \sum_{p=1}^{s} \Lambda_p(X)$$

with

$$\Lambda_p(X) := \begin{cases} \dfrac{\sigma_{s-1}(s)}{96} X^{2s-2}(\pi^2 - 12X + 4X^2), & \text{if } p = 1; \\ \dfrac{(-1)^{p-1}}{2^{2p+2}p}\sigma_{s-p}(s) B_{2p} X^{2s-2p}(\pi^{2p} - (-1)^p 2^{2p} X^{2p}), & \text{if } p \geq 2. \end{cases}$$

Several coefficients of the series expansion

$$\Phi_s(X) = \sum_{j=0}^{\infty} \lambda_j^{(s)} X^j, \qquad \lambda_j^{(s)} \in \mathbb{Q}[\pi]$$

are given by

$$\lambda_0^{(s)} = \frac{2^{2s-1}(-1)^{s-1}B_{2s}}{(2s)!}\pi^{2s},$$

and (i) *for $s = 1, 2$,*

$$\lambda_1^{(1)} = -2, \quad \lambda_2^{(1)} = \frac{2}{9}\pi^2 + \frac{2}{3}, \qquad \lambda_3^{(1)} = -\frac{8}{3}, \quad \lambda_4^{(1)} = \frac{16}{135}\pi^2 + \frac{8}{9},$$
$$\lambda_1^{(2)} = 0, \qquad \lambda_2^{(2)} = \frac{4}{135}\pi^4 - \frac{4}{9}\pi^2, \quad \lambda_3^{(2)} = \frac{16}{3},$$

(ii) *for $s \geq 3$,*

$$\lambda_1^{(s)} = 0, \quad \lambda_2^{(s)} = \frac{2^{2s-1}(-1)^{s-1}}{3 \cdot (2s-1)!}\pi^{2s-2}\big(2B_{2s}\pi^2 + s(2s-1)B_{2s-2}\big), \quad \lambda_3^{(s)} = 0.$$

Theorem 2. *For $-1 < \beta < -1 + \delta_0$, we have*

$$(\alpha + \beta)^{2s-1} g^*_{2s-1} = \Psi_s(\eta)\big(1 + O(e^{-\pi^2/(2\eta)})\big).$$

Here δ_0 and η are as in Theorem 1, and $\Psi_s(X)$ is an entire function written in the form

$$\Psi_s(X) := \frac{(X^{-1}\sinh X)^{2s-1}}{(2s-2)!} \times \sum_{p=0}^{s-1} (-1)^p 2^{-2p-2} \pi^{2p+1} \tau_{s-1-p}(s-1) E_{2p} X^{2s-2p-2}.$$

Several coefficients of the series expansion

$$\Psi_s(X) = \sum_{j=0}^{\infty} \mu_j^{(s)} X^{2j}, \qquad \mu_j^{(s)} \in \mathbb{Q}[\pi]$$

are given by

$$\mu_0^{(s)} = \frac{(-1)^{s-1} E_{2s-2}}{2^{2s}(2s-2)!} \pi^{2s-1},$$

and (i) *for* $s = 1$,

$$\mu_m^{(1)} = \frac{\pi}{4 \cdot (2m+1)!} \quad (m \geq 1),$$

(ii) *for* $s \geq 2$,

$$\mu_1^{(s)} = \frac{(-1)^{s-1}(2s-1)}{2^{2s+1} \cdot 3 \cdot (2s-2)!} \pi^{2s-3} \big(E_{2s-2}\pi^2 + 8(s-1)(2s-3)E_{2s-4} \big).$$

Remark 1. The asymptotic formulas above are also valid as $\beta \to -1$ through the sector $|\arg(\beta+1)| < \pi/2 - \theta_0$, where $\theta_0 > 0$ is an arbitrary small number (then $\eta \to 0$, $|\arg \eta| < \pi/2 - \theta_0'$ for some $\theta_0' > 0$) (see Remark 4 and Section 5).

Remark 2. The quantity β may be regarded as a function of the modulus k of Jacobian elliptic functions, which satisfies $\beta \to -1+0$ as $k \to 1-0$ (see Section 4). Moreover, $\beta \to i$ and $\alpha \to i$ as $k \to \infty$ through a suitable sector. In this case, for

$$(\alpha - \beta)^{2s} h_{2s} = \sum_{n=1}^{\infty} \frac{1}{U_{2n}^{2s}} \quad \text{and} \quad (\alpha - \beta)^{2s-1} f_{2s-1} = \sum_{n=1}^{\infty} \frac{1}{U_{2n-1}^{2s-1}},$$

we obtain similar asymptotic formulas as $\beta \to i$.

3. Euler's formulas for zeta functions

For $-1 < \beta < -1 + \delta_0$ and for $n \geq 1$, observing that

$$\begin{aligned} \Big| \frac{\alpha^{2n} - \beta^{2n}}{\alpha^2 - \beta^2} \Big| &= \Big| \sum_{\nu=0}^{n-1} \alpha^{2n-2-2\nu} \beta^{2\nu} \Big| \\ &= \Big| \sum_{\nu=0}^{n-1} \beta^{4\nu-2n+2} \Big| = \Big| \frac{1}{2} \sum_{\nu=0}^{n-1} \big(\beta^{4\nu-2n+2} + \beta^{-(4\nu-2n+2)} \big) \Big| \geq n, \end{aligned}$$

we have for $s \in \mathbb{N}$

$$\lim_{\beta \to -1+0} (\alpha^2 - \beta^2)^{2s} h_{2s} = \sum_{n=1}^{\infty} \frac{1}{n^{2s}}.$$

Therefore, letting β tend to $-1+0$ in Theorem 1, we obtain

$$\zeta(2s) = \frac{2^{2s-1}(-1)^{s-1}B_{2s}}{(2s)!}\pi^{2s} \quad (s \in \mathbb{N}).$$

For each $s \in \mathbb{N}$ a similar argument concerning Theorem 2 leads us

$$\lim_{\beta\to-1+0}(\alpha+\beta)^{2s-1}g^*_{2s-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^{2s-1}} = \frac{(-1)^{s-1}E_{2s-2}}{2^{2s}(2s-2)!}\pi^{2s-1}.$$

4. Preliminaries

Consider the complete elliptic integrals of the first and the second kinds with the modulus k

(5)

$$K = K(k) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad E = E(k) := \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt$$

for $k^2 \in \mathbb{C} \setminus (\{0\} \cup [1, +\infty))$, where the branch of each integrand is chosen so that it tends to 1 as $t \to 0$. Put

(6) $$\beta = \beta(k) = -e^{-\pi K'/(2K)}, \quad \alpha = \alpha(k) = -1/\beta(k),$$

where

(7) $$K' := K(k'), \quad k' := \sqrt{1-k^2},$$

the branch of k' being chosen so that $k' > 0$ for $0 < k < 1$. These integrals are expressible as follows (cf. [1, 17. 3. 9 and 17. 3. 10]):

(8) $$K = \frac{\pi}{2}F(1/2, 1/2, 1, k^2),$$

(9) $$K' = \frac{\pi}{2}F(1/2, 1/2, 1, 1-k^2),$$

(10) $$E = \frac{\pi}{2}F(-1/2, 1/2, 1, k^2),$$

with the hypergeometric function

$$F(a,b,c,x) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n\, n!}x^n, \quad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1).$$

Lemma 3. *As $k \to 1-0$, we have*

(11) $$K = l(k)(1 + 2\log 2 \cdot l(k)^{-1})(1 + O(1-k^2)),$$

(12) $$K' = \frac{\pi}{2}(1 + O(1-k^2)),$$

(13) $$E = 1 + O((1-k^2)l(k)),$$

with

$$l(k) := \log(1/k') = \log(1/\sqrt{1-k^2}).$$

Furthermore $l(k) \to +\infty$ *and* $K \to +\infty$ *as* $k \to 1-0$.

Proof. Formula (12) follows immediately from (9). Recall the connection formula for $F(1/2, 1/2, 1, z)$ around $z = 0$ and $z = 1$:

$$F(1/2,1/2,1,z) = \frac{\Gamma(1)}{\Gamma(1/2)^2}\sum_{n=0}^{\infty}\frac{(1/2)_n^2}{(n!)^2}\Big(2\psi(n+1) - 2\psi(n+1/2) - \log(1-z)\Big)(1-z)^n,$$

$$\psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} = -\gamma + \sum_{m=0}^{\infty}\Big(\frac{1}{m+1} - \frac{1}{m+t}\Big)$$

for $|1-z| < 1$, $|\arg(1-z)| < \pi$ (cf. [1, 15. 3. 10]). Putting $z = k^2$, we derive for $|1-k^2| < 1$, $|\arg(1-k^2)| < \pi$,

$$\begin{aligned} K &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(1/2)_n^2}{(n!)^2} \\ &\quad \times \Big(2\sum_{m=0}^{\infty}\Big(\frac{1}{m+n+1/2} - \frac{1}{m+n+1}\Big) - \log(1-k^2)\Big)(1-k^2)^n \\ &= 2\log 2 - \frac{1}{2}\log(1-k^2) + O\big((1-k^2)\log(1-k^2)\big) \\ &= l(k) + 2\log 2 + O((1-k^2)l(k)), \end{aligned}$$

which implies (11), where the branch of $\log(1-k^2)$ is chosen so that $\operatorname{Im}\log(1-k^2) = 0$ for $0 < k < 1$, namely for $\arg k = \arg(1-k) = 0$. Using the formula

$$\begin{aligned} F(-1/2,1/2,1,z) &= \frac{\Gamma(1)^2}{\Gamma(1/2)\Gamma(3/2)} + \frac{\Gamma(1)}{\Gamma(-1/2)\Gamma(1/2)}\sum_{n=0}^{\infty}\frac{(1/2)_n(3/2)_n}{n!(n+1)!}(1-z)^{n+1} \\ &\quad \times \Big(\log(1-z) - \psi(n+1) - \psi(n+2) + \psi(n+1/2) + \psi(n+3/2)\Big) \end{aligned}$$

for $|1-z| < 1$, $|\arg(1-z)| < \pi$ (cf. [1, 15. 3. 11]), we obtain (13). □

From (11) it follows that

$$1 - k^2 = e^{-2l(k)} = O(e^{-2K}) \tag{14}$$

as $k \to 1-0$. Hence, by (12), for $1 - \delta_1 < k < 1$,

$$\beta(k) = -\exp\Big(-\frac{\pi^2}{4K}(1 + O(e^{-2K}))\Big), \tag{15}$$

and $|\beta(k)| < 1$, provided that $\delta_1 > 0$ is sufficiently small. This implies that $\beta(k) \to -1+0$ as $k \to 1-0$.

Remark 3. The connection formulas in the proof of Lemma 3 are also valid, as long as we consider the analytic continuation of $\log(1-k^2)$ on the universal covering of the punctured disk $0 < |k-1| < 1$. Hence as $k \to 1$ (without any condition on $\arg(1-k^2)$) we have $\operatorname{Re} K(k) \to +\infty$ and $\beta(k) \to -1$, $|\arg(\beta(k)+1)| < \pi/2 - \theta_0$, where θ_0 is as in Remark 1. Conversely, for every path C_β tending to $\beta = -1$ through the sector $|\arg(\beta+1)| < \pi/2$, there exists a curve C_k tending to $k = 1$ such that $\beta(C_k) = C_\beta$.

Lemma 4. *There exists a small positive number δ_1 such that, for $1-\delta_1 < k < 1$, the quantities $\beta = \beta(k)$ and $\alpha = \alpha(k) = -1/\beta(k)$ satisfy*

$$\alpha(k)^2 - \beta(k)^2 = \left(e^{\pi^2/(2K)} - e^{-\pi^2/(2K)}\right)(1 + O(e^{-2K})), \tag{16}$$

$$\alpha(k) + \beta(k) = \left(e^{\pi^2/(4K)} - e^{-\pi^2/(4K)}\right)(1 + O(e^{-2K})). \tag{17}$$

Proof. Using (6), (12) and (14), we have

$$\begin{aligned}
\alpha(k)^2 - \beta(k)^2 &= e^{\pi K'/K} - e^{-\pi K'/K} \\
&= e^{-\pi K'/K}(e^{2\pi K'/K} - 1) \\
&= \exp\Big(-\frac{\pi^2}{2K}(1 + O(e^{-2K}))\Big)\Big(\exp\Big(\frac{\pi^2}{K}(1 + O(e^{-2K}))\Big) - 1\Big) \\
&= e^{\frac{-\pi^2}{2K}}(1 + O(K^{-1}e^{-2K}))\Big(e^{\frac{\pi^2}{K}}(1 + O(K^{-1}e^{-2K})) - 1\Big) \\
&= e^{-\pi^2/(2K)}(1 + O(e^{-2K}))\Big((e^{\pi^2/K} - 1) + O(K^{-1}e^{-2K})\Big) \\
&= e^{-\pi^2/(2K)}(e^{\pi^2/K} - 1)(1 + O(e^{-2K}))
\end{aligned}$$

as $k \to 1-0$ (and $K \to +\infty$), which implies (16). Estimate (17) is obtained by a similar argument. □

We note the following lemmas concerning the Jacobian elliptic functions

$$\operatorname{ns}(k,z) = 1/\operatorname{sn}(k,z), \quad \operatorname{cn}(k,z) = \sqrt{1 - \operatorname{sn}^2(k,z)}$$

with $w = \operatorname{sn}(k,z)$ defined by

$$z = \int_0^w \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}\,.$$

Lemma 5. *Around $z = 0$,*

$$\operatorname{ns}^2(k,z) = z^{-2} + \sum_{j=0}^{\infty} c_j(k) z^{2j},$$

where $c_j(k) \in \mathbb{Q}[k^2]$. In particular,

$$\operatorname{ns}^2(1,z) = \coth^2 z = z^{-2} + \sum_{j=0}^{\infty} c_j(1) z^{2j},$$

where

$$c_0(1) = \frac{2}{3}, \quad c_j(1) = -\frac{2^{2j+2}(2j+1)B_{2j+2}}{(2j+2)!} \quad (j \geq 1).$$

Proof. The function $u = \mathrm{ns}^2(k,z)$ satisfies the differential equations

$$(u')^2 = 4u(u-1)(u-k^2) \quad \text{and} \quad u'' = 6u^2 - 4(k^2+1)u + 2k^2.$$

Substituting the expansion $z^{-2} + \sum_{j=0}^{\infty} c_j(k) z^{2j}$ of $\mathrm{ns}^2(k,z)$ into these equations, we can determine the coefficients $c_j(k)$ (see [4, Lemma 2] and the recursive relation in it). The relation $\mathrm{ns}^2(1,z) = \coth^2 z$ immediately follows from the fact that $u = \mathrm{ns}^2(1,z)$ satisfies the equation $(u')^2 = 4u(u-1)^2$ admitting the general solution $\coth^2(z - z_0)$. □

The function $u = k\,\mathrm{cn}(k,z)$ satisfies

$$(u')^2 = (k^2 - u^2)(u^2 + 1 - k^2), \quad u(0) = k.$$

If $k = 1$, then $(u')^2 = u^2(1-u^2)$ admits the general solution $\mathrm{sech}\,(z - z_0)$. From this fact we obtain the following (see also [6, Lemma 2]):

Lemma 6. *Around* $z = 0$*,*

$$k\,\mathrm{cn}(k,z) = k + \sum_{j=1}^{\infty} c_j^*(k) z^{2j},$$

where $c_j^*(k) \in \mathbb{Q}[k]$*. In particular,*

$$\mathrm{cn}(1,z) = \mathrm{sech}\, z = 1 + \sum_{j=1}^{\infty} c_j^*(1) z^{2j}, \qquad c_j^*(1) = \frac{E_{2j}}{(2j)!}.$$

5. Proofs of results

5.1. Proof of Theorem 1. Recall that

$$(18) \qquad (\alpha^2 - \beta^2)^{2s} h_{2s} = \Big(\frac{\alpha^2 - \beta^2}{2}\Big)^{2s} \sum_{\nu=1}^{\infty} \mathrm{cosech}^{2s}(\nu\pi K'/K)$$
$$= \frac{(\alpha^2-\beta^2)^{2s}}{(2s-1)!} \sum_{j=0}^{s-1} \sigma_{s-j-1}(s) A_{2j+1}(\beta^2),$$

where

$$A_{2j+1}(q) := \sum_{n=1}^{\infty} \frac{n^{2j+1} q^{2n}}{1 - q^{2n}}$$

(cf. [12, Table 1(i)], [4]). The series $A_{2j+1}(\beta(k)^2)$ are generated by

$$\Big(\frac{2K}{\pi}\Big)^2 \mathrm{ns}^2\Big(\frac{2Kx}{\pi}\Big) = \frac{4K(K-E)}{\pi^2} + \mathrm{cosec}^2 x - 8 \sum_{j=0}^{\infty} (-1)^j A_{2j+1} \frac{(2x)^{2j}}{(2j)!}$$

(cf. [12, Tables 1(i)], [9], [11, p. 535]), namely

$$A_1 = \frac{1}{24}\Big(1 - \Big(\frac{2K}{\pi}\Big)^2\Big(\frac{3E}{K} - 2 + k^2\Big)\Big), \tag{19}$$

$$\Big(\frac{2K}{\pi}\Big)^{2j+2} c_j(k) = a_j - (-1)^j \frac{2^{2j+3}}{(2j)!} A_{2j+1} \quad (j \geq 1). \tag{20}$$

Here $c_j(k)$ are the coefficients given in Lemma 5, and a_j are the coefficients of the expansion

$$\operatorname{cosec}^2 z = z^{-2} + \sum_{j=0}^{\infty} a_j z^{2j},$$

namely, $a_0 = 1/3$, $a_j = (-1)^{j-1} c_j(1)$ $(j \geq 1)$.

Recall that $K \to +\infty$ and that $\beta(k) \to -1+0$ as $k \to 1-0$ (cf. (15) and Lemma 3). By (16) we have

$$\begin{aligned} (\alpha^2 - \beta^2)^{2s} &= (e^{\pi^2/K} + e^{-\pi^2/K} - 2)^s (1 + O(e^{-2K})) \\ &= \pi^{4s} K^{-2s} \Big(2\pi^{-4} K^2 (\cosh(\pi^2/K) - 1)\Big)^s (1 + O(e^{-2K})) \end{aligned} \tag{21}$$

as $k \to 1-0$. Note that

$$2\pi^{-4} K^2 (\cosh(\pi^2/K) - 1) = \sum_{m=1}^{\infty} \frac{2\pi^{4(m-1)}}{(2m)!} K^{-2(m-1)} = 1 + O(K^{-2}).$$

By (19), (14) and Lemma 3, the quantity $A_1(\beta^2)$ can be written as

$$\begin{aligned} A_1 &= \frac{1}{24}\Big(1 - \frac{4K^2}{\pi^2}\Big(\frac{3}{K}(1 + O(l(k)e^{-2K})) - 1 + O(e^{-2K})\Big)\Big) \\ &= \frac{1}{24}\Big(1 - \frac{4K^2}{\pi^2}(-1 + 3K^{-1} + O(e^{-2K}))\Big) \\ &= \frac{K^2}{6\pi^2}\Big(1 - 3K^{-1} + \frac{\pi^2}{4}K^{-2}\Big)(1 + O(e^{-2K})). \end{aligned}$$

Combining this with (21), we have for $s \geq 1$

$$\begin{aligned} (\alpha^2 - \beta^2)^{2s} A_1 = \frac{\pi^{4s-2}}{6} K^{-2(s-1)} &\Big(2\pi^{-4} K^2 (\cosh(\pi^2/K) - 1)\Big)^s \\ &\times \Big(1 - 3K^{-1} + \frac{\pi^2}{4}K^{-2}\Big)(1 + O(e^{-2K})) \end{aligned} \tag{22}$$

as $k \to 1-0$. For $s \geq 2$ and for $2 \leq p \leq s$, using (20), (21), and Lemma 5 with the facts $a_{p-1} = (-1)^p c_{p-1}(1)$, $c_{p-1}(k) = c_{p-1}(1) + O(e^{-2K})$, we have

$$\begin{aligned}(\alpha^2-\beta^2)^{2s} A_{2p-1} &= \frac{(-1)^p(2p-2)!}{2}\pi^{4s-2p}c_{p-1}(1)K^{-2(s-p)} \\ &\quad \times \Big(2\pi^{-4}K^2(\cosh(\pi^2/K)-1)\Big)^s \\ &\quad \times (1-(-1)^p 2^{-2p}\pi^{2p}K^{-2p})(1+O(e^{-2K})) \\ &= \frac{(-1)^{p-1}}{p}2^{2p-2}\pi^{4s-2p}B_{2p}K^{-2(s-p)} \\ &\quad \times \Big(2\pi^{-4}K^2(\cosh(\pi^2/K)-1)\Big)^s \\ &\quad \times (1-(-1)^p 2^{-2p}\pi^{2p}K^{-2p})(1+O(e^{-2K})) \end{aligned} \tag{23}$$

as $k \to 1-0$. By (15),

$$K^{-1} = -\frac{4}{\pi^2}\log(-\beta)\,(1+O(e^{-2K})),$$
$$K = \frac{\pi^2}{4\eta}(1+O(e^{-2K})) = \frac{\pi^2}{4\eta} + O(Ke^{-2K}) = \frac{\pi^2}{4\eta} + O(1),$$

and hence

$$K^{-1} = \frac{4}{\pi^2}\eta(1+O(e^{-\pi^2/(2\eta)})), \quad e^{-2K} = O(e^{-\pi^2/(2\eta)}). \tag{24}$$

For $s \geq 1$ we derive from (18), (22) and (23) that

$$\begin{aligned}(\alpha^2-\beta^2)^{2s}h_{2s} &= \frac{(\alpha^2-\beta^2)^{2s}}{(2s-1)!}\sum_{p=1}^{s}\sigma_{s-p}(s)A_{2p-1} \\ &= \frac{\Big(2\pi^{-4}K^2(\cosh(\pi^2/K)-1)\Big)^s}{(2s-1)!}\sum_{p=1}^{s}\sigma_{s-p}(s)F_{s,p}(K)\end{aligned} \tag{25}$$

with

$$F_{s,1}(K) := \frac{\pi^{4s-2}}{6}K^{-2(s-1)}\Big(1-3K^{-1}+\frac{\pi^2}{4}K^{-2}\Big)(1+O(e^{-2K})),$$

and

$$\begin{aligned}F_{s,p}(K) := &\frac{(-1)^{p-1}}{p}2^{2p-2}\pi^{4s-2p}B_{2p}K^{-2(s-p)} \\ &\times (1-(-1)^p 2^{-2p}\pi^{2p}K^{-2p})(1+O(e^{-2K}))\end{aligned}$$

for $p \geq 2$. Now we note the following: for a function $\phi(z)$ holomorphic for $|z| < r_0$ satisfying $\phi(0) \neq 0$,

$$\phi(z+\Delta z) = \phi(z) + \int_0^{\Delta z}\phi'(z+t)dt = \phi(z) + O(\Delta z) = \phi(z)(1+O(\Delta z))$$

as $\Delta z \to 0$, uniformly for $|z| < r_0'$, provided that r_0' is sufficiently small. Substituting (24) into (25), and using this fact with $z = 4\pi^{-2}\eta$ and $\Delta z = O(\eta e^{-\pi^2/(2\eta)})$, we have the required formula. Using

$$2X^{-2}(\cosh(4X)-1) = \sum_{m=0}^{\infty} \frac{2^{4m+5}}{(2m+2)!} X^{2m}$$

and

$$\sigma_1(s) = -\sum_{r=1}^{s-1} r^2 = -\frac{1}{6}s(s-1)(2s-1),$$

we obtain the coefficients mentioned in the theorem.

5.2. Proof of Theorem 2. Note that, for $s \in \mathbb{N}$,

$$(\alpha+\beta)^{2s-1} g_{2s-1}^* = \Big(\frac{\alpha+\beta}{2}\Big)^{2s-1} \sum_{\nu=1}^{\infty} (-1)^{\nu+1} \operatorname{cosech}^{2s-1}((\nu-1/2)\pi K'/K)$$

$$= \frac{(\alpha+\beta)^{2s-1}}{2^{2s-2}(2s-2)!} \sum_{j=0}^{s-1} \tau_{s-1-j}(s-1) T_{2j}(\beta^2), \tag{26}$$

where

$$T_{2j}(q) := \sum_{n=1}^{\infty} \frac{(2n-1)^{2j} q^{n-1/2}}{1+q^{2n-1}}$$

(cf. [12, Table 1(xv)]). The series $T_{2j}(\beta(k)^2)$ are generated by

$$\Big(\frac{2K}{\pi}\Big) k \operatorname{cn}\Big(\frac{2Kx}{\pi}\Big) = 4\sum_{j=0}^{\infty} (-1)^j T_{2j} \frac{x^{2j}}{(2j)!},$$

namely

$$\Big(\frac{2K}{\pi}\Big)^{2j+1} c_j^*(k) = \frac{(-1)^j 4}{(2j)!} T_{2j} \quad (j \ge 0), \tag{27}$$

where $c_j^*(k)$ are coefficients given in Lemma 6. By (17)

$$\begin{aligned} \alpha+\beta &= \big(e^{\pi^2/(4K)} - e^{-\pi^2/(4K)}\big)(1+O(e^{-2K})) \\ &= \frac{\pi^2}{2} K^{-1} \Big(4\pi^{-2} K \sinh(\pi^2/(4K))\Big)(1+O(e^{-2K})), \end{aligned} \tag{28}$$

where

$$4\pi^{-2} K \sinh(\pi^2/(4K)) = \sum_{m=1}^{\infty} \frac{\pi^{4(m-1)}}{4^{2(m-1)}(2m-1)!} K^{-2(m-1)} = 1 + O(K^{-2}).$$

Note that $c_{j-1}^*(k) = c_{j-1}^*(1) + O(1-k) = c_{j-1}^*(1) + O(e^{-2K})$ as $k \to 1-0$, because $c_{j-1}^*(k) \in \mathbb{Q}[k]$. By (27), (28) and Lemma 6, we have, for $s \ge 1$

and for $0 \le p \le s-1$,

$$(\alpha+\beta)^{2s-1}T_{2p} = (-1)^p 2^{2p-2s}\pi^{4s-2p-3}E_{2p}K^{-2(s-p-1)} \times \Big(4\pi^{-2}K\sinh(\pi^2/(4K))\Big)^{2s-1}(1+O(e^{-2K})). \tag{29}$$

Substituting these quantities into (26), we obtain

$$\begin{aligned}(\alpha+\beta)^{2s-1}g^*_{2s-1} &= \frac{(\alpha+\beta)^{2s-1}}{2^{2s-2}(2s-2)!}\sum_{p=0}^{s-1}\tau_{s-1-p}(s-1)T_{2p} \\ &= \frac{1}{(2s-2)!}\Big(4\pi^{-2}K\sinh(\pi^2/(4K))\Big)^{2s-1} \\ &\quad \times \sum_{p=0}^{s-1}\Big((-1)^p 2^{2p-4s+2}\pi^{4s-2p-3}\tau_{s-1-p}(s-1) \\ &\qquad \times E_{2p}K^{-2(s-p-1)}(1+O(e^{-2K}))\Big).\end{aligned}$$

This combined with (24) yields the required formula. Using

$$2X^{-1}\sinh X = 2\sum_{m=0}^{\infty}\frac{X^{2m}}{(2m+1)!}$$

and

$$\tau_1(s-1) = -\sum_{r=1}^{s-1}(2r-1)^2 = -\frac{1}{3}(s-1)(2s-1)(2s-3),$$

we can check the coefficients given in the theorem.

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