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par Tamás ERDÉLYI

Abstract. We prove that there are absolute constants \( c_1 > 0 \) and \( c_2 > 0 \) such that for every

\[ \{a_0, a_1, \ldots, a_n\} \subset [1, M], \quad 1 \leq M \leq \exp(c_1 n^{1/4}), \]

there are

\[ b_0, b_1, \ldots, b_n \in \{-1, 0, 1\} \]

such that

\[ P(z) = \sum_{j=0}^{n} b_j a_j z^j \]

has at least \( c_2 n^{1/4} \) distinct sign changes in \((0,1)\). This improves and extends earlier results of Bloch and Pólya.

1. Introduction

Let \( F_n \) denote the set of polynomials of degree at most \( n \) with coefficients from \( \{-1, 0, 1\} \). Let \( L_n \) denote the set of polynomials of degree \( n \) with coefficients from \( \{-1, 1\} \). In [6] the authors write

“The study of the location of zeros of these classes of polynomials begins with Bloch and Pólya [2]. They prove that the average number of real zeros
of a polynomial from $\mathcal{F}_n$ is at most $c\sqrt{n}$. They also prove that a polynomial from $\mathcal{F}_n$ cannot have more than

$$\frac{cn \log \log n}{\log n}$$

real zeros. This quite weak result appears to be the first on this subject. Schur [13] and by different methods Szegő [15] and Erdős and Turán [8] improve this to $c\sqrt{n} \log n$ (see also [4]). (Their results are more general, but in this specialization not sharp.)

Our Theorem [4.1] gives the right upper bound of $c\sqrt{n}$ for the number of real zeros of polynomials from a much larger class, namely for all polynomials of the form

$$p(x) = \sum_{j=0}^{n} a_j x^j, \quad |a_j| \leq 1, \quad |a_0| = |a_n| = 1, \quad a_j \in \mathbb{C}.$$}

Schur [13] claims that Schmidt gives a version of part of this theorem. However, it does not appear in the reference he gives, namely [12], and we have not been able to trace it to any other source. Also, our method is able to give $c\sqrt{n}$ as an upper bound for the number of zeros of a polynomial $p \in \mathcal{P}_n$ with $|a_0| = 1, |a_j| \leq 1$, inside any polygon with vertices in the unit circle (of course, $c$ depends on the polygon). This may be discussed in a later publication.

Bloch and Pólya [2] also prove that there are polynomials $p \in \mathcal{F}_n$ with

$$c n^{1/4} \sqrt{\log n}$$

distinct real zeros of odd multiplicity. (Schur [13] claims they do it for polynomials with coefficients only from $\{-1, 1\}$, but this appears to be incorrect.)

In a seminal paper Littlewood and Offord [11] prove that the number of real roots of a $p \in \mathcal{L}_n$, on average, lies between

$$\frac{c_1 \log n}{\log \log \log n} \quad \text{and} \quad c_2 \log^2 n$$

and it is proved by Boyd [7] that every $p \in \mathcal{L}_n$ has at most $c \log^2 n / \log \log n$ zeros at 1 (in the sense of multiplicity).

Kac [10] shows that the expected number of real roots of a polynomial of degree $n$ with random uniformly distributed coefficients is asymptotically $(2/\pi) \log n$. He writes “I have also stated that the same conclusion holds if the coefficients assume only the values 1 and −1 with equal probabilities. Upon closer examination it turns out that the proof I had in mind is inapplicable.... This situation tends to emphasize the particular interest of the
discrete case, which surprisingly enough turns out to be the most difficult.” In a recent related paper Solomyak [14] studies the random series $\sum \pm \lambda^n$.


In this paper we improve the lower bound (1.1) in the result of Bloch and Pólya to $cn^{1/4}$. Moreover we allow a much more general coefficient constraint in our main result. Our approach is quite different from that of Bloch and Pólya.

2. New result

**Theorem 2.1.** There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that for every

$$\{a_0, a_1, \ldots, a_n\} \subset [1, M], \quad 1 \leq M \leq \exp(c_1 n^{1/4}),$$

there are $b_0, b_1, \ldots, b_n \in \{-1, 0, 1\}$ such that

$$P(z) = \sum_{j=0}^{n} b_j a_j z^j$$

has at least $c_2 n^{1/4}$ distinct sign changes in $(0, 1)$.

3. Lemmas

Let $D := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Denote by $S_M$ the collection of all analytic functions $f$ on the open unit disk $D$ that satisfy

$$|f(z)| \leq \frac{M}{1 - |z|}, \quad z \in D.$$

Let $\|f\|_A := \sup_{x \in A} |f(x)|$. To prove Theorem 2.1 our first lemma is the following.

**Lemma 3.1.** There is an absolute constants $c_3 > 0$ such that

$$\|f\|_{[\alpha, \beta]} \geq \exp \left( \frac{-c_3(1 + \log M)}{\beta - \alpha} \right)$$

for every $f \in S_M$ and $0 < \alpha < \beta \leq 1$ with $|f(0)| \geq 1$ and for every $M \geq 1$.

This follows from the lemma below by a linear scaling:
Lemma 3.2. There are absolute constants \( c_4 > 0 \) and \( c_5 > 0 \) such that

\[
|f(0)|^{c_5/a} \leq \exp \left( \frac{c_4(1 + \log M)}{a} \right) \|f\|_{[1-a,1]}
\]

for every \( f \in S_M \) and \( a \in (0,1] \).

To prove Lemma 3.2 we need some corollaries of the following well known result.

**Hadamard three circles theorem.** Let \( 0 < r_1 < r_2 \). Suppose \( f \) is regular in

\[
\{ z \in \mathbb{C} : r_1 \leq |z| \leq r_2 \}.
\]

For \( r \in [r_1,r_2] \), let

\[
M(r) := \max_{|z|=r} |f(z)|.
\]

Then

\[
M(r)^{\log(r_2/r_1)} \leq M(r_1)^{\log(r_2/r)} M(r_2)^{\log(r/r_1)}.
\]

**Corollary 3.1.** Let \( a \in (0,1] \). Suppose \( f \) is regular inside and on the ellipse \( E_a \) with foci at \( 1 - a \) and \( 1 - a + \frac{1}{4}a \) and with major axis

\[
\left[ 1 - a - \frac{9a}{64}, 1 - a + \frac{25a}{64} \right].
\]

Let \( \tilde{E}_a \) be the ellipse with foci at \( 1 - a \) and \( 1 - a + \frac{1}{4}a \) and with major axis

\[
\left[ 1 - a - \frac{a}{32}, 1 - a + \frac{9a}{32} \right].
\]

Then

\[
\max_{z \in E_a} |f(z)| \leq \left( \max_{z \in [1-a,1-a+\frac{1}{4}a]} |f(z)| \right)^{1/2} \left( \max_{z \in E_a} |f(z)| \right)^{1/2}.
\]

**Proof.** This follows from the Hadamard three circles theorem with the substitution

\[
w = \frac{a}{8} \left( \frac{z + z^{-1}}{2} \right) + \left( 1 - a + \frac{a}{8} \right).
\]

The Hadamard three circles theorem is applied with \( r_1 := 1, r := 2 \), and \( r_2 := 4 \). \( \square \)

**Corollary 3.2.** For every \( f \in S_M \) and \( a \in (0,1] \) we have

\[
\max_{z \in \tilde{E}_a} |f(z)| \leq \left( \frac{64M}{39a} \right)^{1/2} \left( \max_{z \in [1-a,1]} |f(z)| \right)^{1/2}.
\]
Proof of Lemma 3.2. Let \( f \in S_M \) and \( h(z) = \frac{1}{2}(1-a)(z + z^2) \). Observe that \( h(0) = 0 \), and there are absolute constants \( c_6 > 0 \) and \( c_7 > 0 \) such that
\[
|h(e^{it})| \leq 1 - c_6 t^2, \quad -\pi \leq t \leq \pi,
\]
and for \( t \in [-c_7 a, c_7 a] \), \( h(e^{it}) \) lies inside the ellipse \( \tilde{E}_a \). Now let \( m := \lceil \pi/(c_7 a) \rceil + 1 \). Let \( \xi := \exp(2\pi i / (2m)) \) be the first \( 2m \)-th root of unity, and let
\[
g(z) = \prod_{j=0}^{2m-1} f(h(\xi^j z)).
\]
Using the Maximum Principle and the properties of \( h \), we obtain
\[
|f(0)|^{2m} = |g(0)| \leq \max_{|z|=1} |g(z)| \leq \left( \max_{z \in \tilde{E}_a} |f(z)| \right)^{2m-1} \prod_{k=1}^m \left( \frac{M}{c_6(k/m)^2} \right)^2 \left( M^{2m-2} \exp(c_8(m-1)) \frac{m^{m-1}}{(m-1)!} \right)^4 \\
< \left( \max_{z \in \tilde{E}_a} |f(z)| \right)^{2m} (Me)^{c_9(m-1)}
\]
with absolute constants \( c_8 \) and \( c_9 \), and the result follows from Corollary 3.2.

\[\square\]

4. Proof of theorem 2.1

Proof of Theorem 2.1. Let \( L \leq \frac{1}{2} n^{1/2} \) and
\[
\mathcal{M}(P) := (P(1-n^{-1/2}), P(1-2n^{-1/2}), \ldots, P(1-Ln^{-1/2})) \in [-M\sqrt{n}, M\sqrt{n}]^L.
\]

We consider the polynomials
\[
P(z) = \sum_{j=0}^{n-1} b_j a_j z^j, \quad b_j \in \{0, 1\}.
\]

There are \( 2^n \) such polynomials. Let \( K \in \mathbb{N} \). Using the box principle we can easily deduce that \( (2K)^L < 2^n \) implies that there are two different
\[
P_1(z) = \sum_{j=0}^{n-1} b_j a_j z^j, \quad b_j \in \{0, 1\},
\]
and
\[
P_2(z) = \sum_{j=0}^{n-1} \bar{b}_j a_j z^j, \quad \bar{b}_j \in \{0, 1\},
\]
such that

\[ |P_1(1 - jn^{-1/2}) - P_2(1 - jn^{-1/2})| \leq \frac{M\sqrt{n}}{K}, \quad j = 1, 2, \ldots, L. \]

Let

\[ P_1(z) - P_2(z) = \sum_{j=m}^{n-1} \beta_j a_j z^j, \quad \beta_j \in \{-1, 0, 1\}, \quad b_m \neq 0. \]

Let \( 0 \neq Q(z) := z^{-m}(P_1(z) - P_2(z)) \). Then \( Q \) is of the form

\[ Q(z) := \sum_{j=0}^{n-1} \gamma_j a_j z^j, \quad \gamma_j \in \{-1, 0, 1\}, \quad \gamma_0 \in \{-1, 1\}, \]

and, since \( 1 - x \geq e^{-2x} \) for all \( x \in [0, 1/2] \), we have

\[ |Q(1 - jn^{-1/2})| \leq \exp\left(2Ln^{1/2}\right)\frac{M\sqrt{n}}{K}, \quad j = 1, 2, \ldots, L. \]

Also, by Lemma 3.1, there are

\[ \xi_j \in I_j := [1 - jn^{-1/2}, 1 - (j - 1)n^{-1/2}], \quad j = 1, 2, \ldots, L, \]

such that

\[ |Q(\xi_j)| \geq \exp(-c_3(1 + \log M)\sqrt{n}), \quad j = 1, 2, \ldots, L. \]

Now let \( L := \lfloor(1/16)n^{1/4}\rfloor \) and \( 2K = \exp(n^{3/4}) \). Then \((2K)^L < 2^n \) holds. Also, if \( \log M = O(n^{1/4}) \), then (4.1) implies

\[ |Q(1 - jn^{-1/2})| \leq \exp\left(-(3/4)n^{3/4}\right), \quad j = 1, 2, \ldots, L, \]

for all sufficiently large \( n \). Now observe that \( 1 \leq M \leq \exp((64c_3)^{-1}n^{1/4}) \) yields that

\[ |a_n x^n| \geq |x|^n \geq \exp(-2(1 - x)) \geq \exp(-2Ln^{1/2}) \]

\[ \geq \exp(-1(8)n^{3/4}), \quad x \in [1 - Ln^{-1/2}, 1 - (L/2)n^{-1/2}], \]

and

\[ |a_n x^n| \leq M \exp(-(L/2)n^{1/2}) \]

\[ \leq \exp(-(1/33)n^{3/4}), \quad x \in [1 - Ln^{-1/2}, 1 - (L/2)n^{-1/2}], \]

for all sufficiently large \( n \). Observe also that with \( \log M \leq (64c_3)^{-1}n^{1/4} \)

(4.2) implies

\[ |Q(\xi_j)| > \exp(-(1/63)n^{3/4}), \quad j = 1, 2, \ldots, L, \]

for all sufficiently large \( n \). Now we study the polynomials

\[ S_1(z) := Q(z) - a_n z^n \quad \text{and} \quad S_2(z) := Q(z) + a_n z^n. \]
These are of the requested special form. It follows from (4.3)–(4.6) that either $S_1$ or $S_2$ has a sign change in at least half of the intervals $I_j, j = L, L-1, \ldots \lfloor L/2 \rfloor + 2$, for all sufficiently large $n$, and the theorem is proved. □

References


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