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Abstract. Let $p$ be a rational prime and $K$ a complete discrete valuation field with residue field $k$ of positive characteristic $p$. When $k$ is finite, generalizing the theory of Deligne [1], we construct in [10] and [11] a theory of local $\varepsilon_0$-constants for representations, over a complete local ring with an algebraically closed residue field of characteristic $\neq p$, of the Weil group $W_K$ of $K$. In this paper, we generalize the results in [10] and [11] to the case where $k$ is an arbitrary perfect field.

1. Introduction

Let $K$ be a complete discrete valuation field whose residue field $k$ is of characteristic $p$. When $k$ is a finite field, the author defines in [10] local $\varepsilon_0$-constants $\varepsilon_{0,R}(V,\psi)$ for a triple $(R,(\rho,V),\psi)$ where $R$ is a strict $p'$-coefficient ring (see Section 2 for the definition), $(\rho,V)$ is an object in $\text{Rep}(W_K,R)$, and $\psi : K \to R^\times$ is a non-trivial continuous additive character. In [10] the author proved several properties including the formula for induced representations. In the present paper, we generalize the results of two papers [10] and [11] to the case where $k$ is an arbitrary perfect field of characteristic $p$. More precisely, we define an object $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ in $\text{Rep}(W_k,\widetilde{\psi})$ of rank one (where $W_k$ is a dense subgroup of the absolute Galois group of $k$ defined in 3.1), called the local $\varepsilon_0$-character, for any triple $(R,(\rho,V),\widetilde{\psi})$ where $R$ is a strict $p'$-coefficient ring, $(\rho,V)$ an object in $\text{Rep}(W_K,R)$ and $\widetilde{\psi}$ is a non-trivial invertible additive character sheaf on $K$. When $k$ is finite of order $q$, this $\widetilde{\varepsilon}_{0,R}(V,\widetilde{\psi})$ and the local $\varepsilon_0$-constants

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\( \varepsilon_{0,R}(V, \psi) \) are related by
\[
\text{Tr}(\text{Fr}_q : \varepsilon_{0,R}(V, \tilde{\psi})) = (-1)^{\text{rank} V + \text{sw}(V)} \varepsilon_{0,R}(V, \psi),
\]
where \( \tilde{\psi} \) is the invertible character sheaf associated to \( \psi \).

We generalize the properties of local \( \varepsilon_0 \)-constants stated in [10] to those of local \( \tilde{\varepsilon}_0 \)-characters by using the specialization argument. We also prove the product formula which describes the determinant of the étale cohomology of a \( R_0 \)-sheaf on a curve over a perfect field \( k \) as a tensor product of local \( \tilde{\varepsilon}_0 \)-characters.

2. Notation

Let \( \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \) denote the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively.

Let \( \mathbb{Z}_{>0} \) (resp. \( \mathbb{Z}_{\geq 0} \)) be the ordered set of positive (resp. non-negative) integers. We also define \( \mathbb{Q}_{>0}, \mathbb{Q}_{\geq 0}, \mathbb{R}_{>0} \) and \( \mathbb{R}_{\geq 0} \) in a similar way. For \( \alpha \in \mathbb{R} \), let \( \lfloor \alpha \rfloor \) (resp. \( \lceil \alpha \rceil \)) denote the maximum integer not larger than \( \alpha \) (resp. the minimum integer not smaller than \( \alpha \)).

For a prime number \( \ell \), let \( \mathbb{F}_\ell \) denote the finite field of \( \ell \) elements, \( \mathbb{F}_\ell^n \) the unique extension of \( \mathbb{F}_\ell \) of degree \( n \) for \( n \in \mathbb{Z}_{>0} \), \( \mathbb{F}_\ell \) the algebraic closure of \( \mathbb{F}_\ell \), \( \mathbb{Z}_\ell = W(\mathbb{F}_\ell) \) (resp. \( W(\mathbb{F}_\ell) \)) the ring of Witt vectors of \( \mathbb{F}_\ell \) (resp. \( \mathbb{F}_\ell \)), \( \mathbb{Q}_\ell = \text{Frac}(\mathbb{Z}_\ell) \) the field of fractions of \( \mathbb{Z}_\ell \). Let \( \varphi : W(\mathbb{F}_\ell) \to W(\mathbb{F}_\ell) \) be the Frobenius automorphism of \( W(\mathbb{F}_\ell) \).

For a ring \( R \), let \( R^\times \) denote the group of units in \( R \). For a positive integer \( n \in \mathbb{Z}_{>0} \), let \( \mu_n(R) \) denote the group of \( n \)-th roots of unity in \( R \), \( \mu_{n,\infty}(R) \) denotes the union \( \cup_i \mu_{n_i}(R) \).

For a finite extension \( L/K \) of fields, let \( [L : K] \) denote the degree of \( L \) over \( K \). For a subgroup \( H \) of a group \( G \) of finite index, its index is denoted by \( [G : H] \).

For a finite field \( k \) of characteristic \( \neq 2 \), let \( (\_ : k) : k^\times \to \{\pm 1\} \) denote the unique surjective homomorphism.

Throughout this paper, we fix once and for all a prime number \( p \). We consider a complete discrete valuation field \( K \) whose residue field is perfect of characteristic \( p \). We say such a field \( K \) is a \( p \)-CDVF. We sometimes consider a \( p \)-CDVF whose residue field is finite. We say such a field is a \( p \)-local field.

For a \( p \)-CDVF \( K \), let \( \mathcal{O}_K \) denote its ring of integers, \( \mathfrak{m}_K \) the maximal ideal of \( \mathcal{O}_K \), \( k_K = \mathcal{O}_K/\mathfrak{m}_K \) the residue field of \( \mathcal{O}_K \), and \( v_K : K^\times \to \mathbb{Z} \) the normalized valuation. If \( K \) is a \( p \)-local field, we also denote by \( (\_ , \_)_K : K^\times \times K^\times \to \{\pm 1\} \) the Hilbert symbol, by \( W_K \) the Weil group of \( K \), by \( \text{rec} = \text{rec}_K : K^\times \to W_K^{ab} \) the reciprocity map of local class field theory, which sends a prime element of \( K \) to a lift of geometric Frobenius of \( k \). If
Let $3.1$. Ramification subgroups.

Ring $R$ is a topological ring with the topology induced from the direct product topology of $\End_R(V)$, and whose morphisms are $R$-linear maps compatible with actions of $G$.

A sequence

$$0 \to (\rho', V') \to (\rho, V) \to (\rho'', V'') \to 0$$

of morphisms in $\Rep(G, R)$ is called a short exact sequence in $\Rep(G, R)$ if $0 \to V' \to V \to V'' \to 0$ is the short exact sequence of $R$-modules.

In this paper, a noetherian local ring with residue field of characteristic $\neq p$ is called a $p'$-coefficient ring. Any $p'$-coefficient ring $(R, m_R)$ is considered as a topological ring with the $m_R$-preadic topology. A strict $p'$-coefficient ring $R$ with an algebraically closed residue field such that $(R^\times)^p = R^\times$.

3. Review of basic facts

3.1. Ramification subgroups. Let $K$ be a $p$-CDVF with a residue field $k$, and $\overline{K}$ (resp. $\overline{k}$) a separable closure of $K$ (resp. $k$). Let $k_0$ be the algebraic closure of $\mathbb{F}_p$ in $k$. If $k_0$ is finite, define the Weil group $W_k \subset \Gal(\overline{k}/k)$ of $k$ as the inverse image of $\mathbb{Z}$ under the canonical map

$$\Gal(\overline{k}/k) \to \Gal(\overline{k}/k_0) \xrightarrow{\cong} \mathbb{Z}.$$ 

If $k_0$ is infinite, we put $W_k = \Gal(\overline{k}/k)$. Define the Weil group $W_K \subset \Gal(\overline{K}/K)$ of $K$ as the inverse image of $W_k$ under the canonical map $\Gal(\overline{K}/K) \to \Gal(\overline{k}/k)$. Let $G = W_K$ denote the Weil group of $K$. Put $G^v = G \cap \Gal(\overline{K}/K)^v$ and $G^{v+} = G \cap \Gal(\overline{K}/K)^{v+}$, where $\Gal(\overline{K}/K)^v$ and $\Gal(\overline{K}/K)^{v+}$ are the upper numbering ramification subgroups (see [9, IV, §3] for definition) of $\Gal(\overline{K}/K)$. The groups $G^v$, $G^{v+}$ are called the upper numbering ramification subgroups of $G$. They have the following properties:

- $G^v$ and $G^{v+}$ are closed normal subgroups of $G$.
- $G^v \supset G^{v+} \supset G^w$ for every $v, w \in \mathbb{Q}_{\geq 0}$ with $w > v$.
- $G^{v+}$ is equal to the closure of $\bigcup_{w > v} G^w$.
- $G^0 = I_K$, the inertia subgroup of $W_K$. $G^{0+} = P_K$, the wild inertia subgroup of $W_K$. In particular, $G^w$ for $w > 0$ and $G^{w+}$ for $w \geq 0$ are pro $p$-groups.
- For $w \in \mathbb{Q}$, $w > 0$, $G^w/G^{w+}$ is an abelian group which is killed by $p$. 
3.2. Herbrand’s function $\psi_{L/K}$. Let $L/K$ be a finite separable extension of a $p$-CDVF. Let $\psi_{L/K} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be the Herbrand function (see [9, IV, §3] for definition) of $L/K$. The function $\psi_{L/K}$ has the following properties:

- $\psi_{L/K}$ is continuous, strictly increasing, piecewise linear, and convex function on $\mathbb{R}_{\geq 0}$.
- For sufficiently large $w$, $\psi_{L/K}(w)$ is linear with slope $e_{L/K}$.
- We have $\psi_{L/K}(0) = 0$.
- We have $\psi_{L/K}(\mathbb{Z}_{\geq 0}) \subset \mathbb{Z}_{\geq 0}$ and $\psi_{L/K}(\mathbb{Q}_{\geq 0}) = \mathbb{Q}_{\geq 0}$.
- Let $G = W_K$, $H = W_L$. Then for $w \in \mathbb{Q}_{\geq 0}$, we have $G^w \cap H = H^{\psi_{L/K}(w)}$ and $G^{w+} \cap H = H^{\psi_{L/K}(w)+}$.

3.3. Slope decomposition and refined slope decomposition. Let $K$ be a $p$-CDVF, $G = W_K$ the Weil group of $K$. Let $(R, \mathfrak{m}_R)$ be a $p'$-coefficient ring.

Let $V$ be an $R[G]$-module. We say that $V$ is tamely ramified or pure of slope $0$ if $G^{0+}$ acts trivially on $V$. $V$ is called totally wild if the $G^{0+}$-fixed part $V^{G^{0+}}$ is $0$. For $v \in \mathbb{Q}_{\geq 0}$, we say that $V$ is pure of slope $v$ if $V^{G^v}$ is $0$ and if $G^{v+}$ acts trivially on $V$.

Let $K^{\text{tm}}$ be the maximal tamely ramified extension of $K$ in a fixed separable closure $\overline{K}$ of $K$. Let $(\rho, V)$ be an object in $\text{Rep}(G, R)$. Since $G^{0+}$ is a pro-$p$ group, there exists a finite Galois extension $L$ of $K^{\text{tm}}$ in $\overline{K}$ such that $\rho$ factors through the quotient $W(L/K) = W_K/\text{Gal}(\overline{K}/L)$ of $W_K$.

Let $G(L/K)^v$ (resp. $G(L/K)^{v+}$) denotes the image of $G^v$ (resp. $G^{v+}$) in $W(L/K)$.

**Lemma 3.1.** There exists a finite number of rational numbers $v_1, \cdots, v_n \in \mathbb{Q}_{\geq 0}$ with $0 = v_1 < \cdots < v_n$ such that $G(L/K)^{v_i} = G(L/K)^{v_{i+1}}$ for $1 \leq i \leq n - 1$ and that $G(L/K)^{v_n} = \{1\}$.

**Proof.** There exists a finite Galois extension $L'$ of $K$ contained in $L$ such that the composite map $G(L/K)^{0+} \subset W(L/K) \to \text{Gal}(L'/K)$ is injective. Then the image of $G(L/K)^v$, $G(L/K)^{v+}$ in $\text{Gal}(L'/K)$ is equal to the upper numbering ramification subgroups $\text{Gal}(L'/K)^v$, $\text{Gal}(L'/K)^{v+}$ of $\text{Gal}(L'/K)$ respectively. Hence the lemma follows. \qed

**Corollary 3.2.** Let $(\rho, V)$ be an object in $\text{Rep}(G, R)$. Then for any $v \in \mathbb{Q}_{\geq 0}$, there exists a unique maximal sub $R[G]$-module $V^v$ of $V$ which is pure of slope $v$. $V^v = \{0\}$ except for a finite number of $v$ and we have $V = \bigoplus_{v \in \mathbb{Q}_{\geq 0}} V^v$. \qed
For $v \in \mathbb{Q}_{\geq 0}$, the object in $\text{Rep}(G, R)$ defined by $V^v$ is called the slope $v$-part of $(\rho, V)$.

$V \mapsto V^v$ define a functor from $\text{Rep}(G, R)$ to itself which preserves short exact sequences. These functors commute with base changes by $R \to R'$.

**Definition 3.3.** Let $(\rho, V)$ be an object in $\text{Rep}(G, R)$, $V = \bigoplus_{v \in \mathbb{Q}_{\geq 0}} V^v$ its slope decomposition. We define the Swan conductor $sw(V)$ of $V$ by

$$sw(V) = \sum_{v \in \mathbb{Q}_{\geq 0}} v \cdot \text{rank } V^v.$$

**Lemma 3.4.** $sw(V) \in \mathbb{Z}$.

**Proof.** Since $sw(V) = sw(V \otimes_R R/m_R)$, we may assume that $R$ is a field. Then the lemma is classical. □

Assume further that $R$ contains a primitive $p$-th root of unity. Let $(\rho, V)$ be an object in $\text{Rep}(G, R)$. Let $v \in \mathbb{Q}_{>0}$ and let $V^v$ denote the slope $v$ part of $(\rho, V)$. We have a decomposition

$$V^v = \bigoplus_{\chi \in \text{Hom}(G^v/G^{v+}, R^\times)} V_\chi$$

of $V^v$ by the sub $R[G^v/G^{v+}]$-modules $V_\chi$ on which $G^v/G^{v+}$ acts by $\chi$. The group $G$ acts on the set $\text{Hom}(G^v/G^{v+}, R^\times)$ by conjugation : $(g, \chi)(h) = \chi(g^{-1}hg)$. The action of $g \in G$ on $V$ induces an $R$-linear isomorphism $V_\chi \cong V_{g.\chi}$. Let $X^v$ denote the set of $G$-orbits in the $G$-set $\text{Hom}(G^v/G^{v+}, R^\times)$. Then for any $\Sigma \in X^v$,

$$V^\Sigma = \bigoplus_{\chi \in \Sigma} V_\chi$$

is a sub $R[G]$-module of $V$ and we have

$$V = \bigoplus_{\Sigma \in X^v} V^\Sigma.$$

The object in $\text{Rep}(G, R)$ defined by $V^\Sigma$ is called the refined slope $\Sigma$-part of $(\rho, V)$. $(\rho, V)$ is called pure of refined slope $\Sigma$ if $V = V^\Sigma$. $V \mapsto V^\Sigma$ defines a functor from $\text{Rep}(G, R)$ to itself which preserves short exact sequences. These functors commute with base changes by $R \to R'$.

**Lemma 3.5.** Let $(\rho, V)$ be a non-zero object in $\text{Rep}(G, R)$ which is pure of refined slope $\Sigma \in X^v$, $\chi \in \Sigma$, and $V_\chi \subset \text{Res}^G_{G^v} V$ be the $\chi$-part of $\text{Res}^G_{G^v} V$.

Let $H_\chi \subset G$ be the stabilizing subgroup of $\chi$.

1. $H_\chi$ is a subgroup of $G$ of finite index.
2. $V_\chi$ is stable under the action of $H_\chi$ on $V$.
3. $V$ is, as an object in $\text{Rep}(G, R)$, isomorphic to $\text{Ind}^G_{H_\chi} V_\chi$. 
Proof. Obvious.

\[ \text{Remark 3.6.} \text{ The claim } [G : H] < \infty \text{ also follows from the explicit description of the group } \text{Hom}(G^v/G^+, R) \text{ by Saito [5, p. 3, Thm. 1].} \]

3.4. Character sheaves. Let \( S \) be a scheme of characteristic \( p \), \((R, m_R)\) a complete \( p'\)-coefficient ring, and \( G \) a commutative group scheme over \( S \). An \textit{invertible character } \( R \)-sheaf on \( G \) is a smooth invertible étale \( R \)-sheaf (that is, a pro-system of smooth invertible \( R/m_R^n \)-sheaves in the étale topology) \( \mathcal{L} \) on \( G \) such that \( \mathcal{L} \boxtimes \mathcal{L} \cong \mu^* \mathcal{L} \), where \( \mu : G \times_S G \rightarrow G \) is the group law. We have \( i^* \mathcal{L} \cong \mathcal{L} \), where \( i : G \rightarrow G \) is the inverse morphism. If \( \mathcal{L}_1, \mathcal{L}_2 \) are two invertible character \( R \)-sheaf on \( G \), then so is \( \mathcal{L}_1 \otimes_R \mathcal{L}_2 \).

\[ \text{Lemma 3.7 (Orthogonality relation). Suppose that } S \text{ is quasi-compact and quasi-separated, and that the structure morphism } \pi : G \rightarrow S \text{ is compactifiable. Let } \mathcal{L} \text{ be an invertible character } R \text{-sheaf on } G \text{ such that } \mathcal{L} \otimes_R R/m_R \text{ is non-trivial. Then we have } \mathcal{R}\pi_! \mathcal{L} = 0. \]

Proof. We may assume that \( R \) is a field. Since \( R \text{pr}_1(\mathcal{L} \boxtimes \mathcal{L}) \cong (\pi^* R\pi_! \mathcal{L}) \otimes \mathcal{L} \) and \( R \text{pr}_1(\mu^* \mathcal{L}) \cong \pi^* R\pi_! \mathcal{L} \), we have \( (\pi^* R^i \pi_! \mathcal{L}) \otimes \mathcal{L} \cong \pi^* R^{i} \pi_! \mathcal{L} \) for all \( i \). Hence \( R^i \pi_! \mathcal{L} = 0 \) for all \( i \).

\[ \text{Lemma 3.8. Suppose further that } S \text{ and } G \text{ are noetherian and connected, and that } R \text{ is a finite ring. Let } \mathcal{L} \text{ be a smooth invertible } R \text{-sheaf on } G. \text{ Then } \mathcal{L} \text{ is an invertible character } R \text{-sheaf if and only if there is a finite etale homomorphism } G' \rightarrow G \text{ of commutative } S \text{-group schemes with a constant kernel } H_S \text{ and a homomorphism } \chi : H \rightarrow R^\times \text{ of groups such that } \mathcal{L} \text{ is the sheaf defined by } G' \text{ and } \chi. \]

Proof. This is [11, Lem. 3.2].

4. \( \varepsilon_0 \)-characters

Throughout this section, let \( K \) be a \( p \)-CDVF with residue field \( k \) and \((R, m_R)\) a complete strict \( p' \)-coefficient ring with a positive residue character. In this section, we generalize the theory of local \( \varepsilon_0 \)-constants to that for objects in \( \text{Rep}(W_K, R) \).

We use the following notation: for any \( k \)-algebra \( A \), let \( R_A \) denote \( A(\text{resp. } W(A)) \) when \( K \) is of equal characteristic (resp. mixed characteristic). Then \( O_K \) has a natural structure of \( R_k \)-algebra.

4.1. Additive character sheaves. For two integers \( m, n \in \mathbb{Z} \) with \( m \leq n \), let \( K^{[m,n]} \) denote \( m_R^n/m_R^{n+1} \) regarded as an affine commutative \( k \)-group. More precisely, take a prime element \( \pi_K \) of \( K \). If \( \text{char } K = p \), then \( K^{[m,n]} \) is canonically isomorphic to the affine \( k \)-group which associates every \( k \)-algebra \( A \) the group \( \bigoplus_{i=m}^n A \). If \( \text{char } K = 0 \), let \( e = [K : \text{Frac } W(k)] \) be
the absolute ramification index of $K$. Then $K^{[m,n]}$ is canonically isomorphic to the affine $k$-group which associates every $k$-algebra $A$ the group $\bigoplus_{i=0}^{e-1} W_{1+\frac{m-n-i}{e}}(A)$.

Let $R_0$ be a pro-finite local ring on which $p$ is invertible. Let $ACh(K^{[m,n]}, R_0)$ (resp. $ACh^0(K^{[m,n]}, R_0)$) denote the set of all isomorphism classes of invertible character $R_0$-sheaves (resp. non-trivial invertible character $R_0$-sheaves) on $K^{[m,n]}$. For a $p'$-coefficient ring $(R, m_R)$, let $ACh(K^{[m,n]}, R)$ denote the set $\lim_{\leftarrow R_0} ACh(K^{[m,n]}, R_0)$, where $R_0$ runs over all isomorphism classes of injective local ring homomorphisms $R_0 \hookrightarrow R$ from pro-finite local rings $R_0$ to $R$.

For four integers $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ with $m_1 \leq m_2 \leq n_2$ and $m_1 \leq n_1 \leq n_2$, the canonical morphism $K^{[m_2,n_2]} \to K^{[m_1,n_1]}$ induces a map $ACh(K^{[m_1,n_1]}, R) \to ACh(K^{[m_2,n_2]}, R)$.

**Definition 4.1.** A non-trivial additive character sheaf of $K$ with coefficients in $R$ is an element $\tilde{\psi}$ in

$$\prod_{n \in \mathbb{Z}} \lim_{m \leq -n-1} ACh^0(K^{[m,-n-1]}, R).$$

When $\tilde{\psi} = \lim_{\leftarrow m \leq -n-1} ACh^0(K^{[m,-n-1]}, R)$, the integer $n$ is called the conductor of $\tilde{\psi}$ and is denoted by $\text{ord} \ 	ilde{\psi}$.

Let $a \in K$ with $v_K(a) = v$. The multiplication-by-$a$ map

$$a_{[m,n]} : K^{[m-v,n-v]} \to K^{[m,n]}$$

induces a canonical isomorphism

$$a^*_{[m,n]} : ACh^0(K^{[m,n]}) \cong ACh^0(K^{[m-v,n-v]})$$

and hence an isomorphism

$$\lim_{m \leq -n-1} ACh^0(K^{[m,-n-1]}, R) \cong \lim_{m \leq -n-v-1} ACh^0(K^{[m,-n-v-1]}, R).$$

We denote by $\tilde{\psi}_a$ the image of $\tilde{\psi}$ by this isomorphism.

Let $L$ be a finite separable extension of $K$. The trace map $\text{Tr}_{L/K} : L \to K$ induces the map

$$\text{Tr}_{L/K}^* : ACh(K^{[m,-n-1]}, R) \to ACh(L^{[-e_{L/K} m-v_L(D_{L/K}), -e_{L/K} n-v_L(D_{L/K})-1]}, R).$$

We denote by $\tilde{\psi} \circ \text{Tr}_{L/K}$ the image of $\tilde{\psi}$ by this map. We have $\text{ord}(\tilde{\psi} \circ \text{Tr}_{L/K}) = e_{L/K} \text{ord} \tilde{\psi} + v_L(D_{L/K}).$
Lemma 4.2. Let $k$ be a perfect field of characteristic $p$, and $G = \mathbb{G}_{a,k}$ be the additive group scheme over $k$, $\phi_0 : \mathbb{F}_p \to R_0^\times$ a non-trivial additive character, and $\mathcal{L}_{\phi_0}$ the Artin-Schreier sheaf on $\mathbb{G}_{a,\mathbb{F}_p}$ associated to $\phi_0$. Then for any additive character sheaf $\mathcal{L}$ on $G$, there exists a unique element $a \in k$ such that $\mathcal{L}$ is isomorphic to the pull-back of $\mathcal{L}_{\phi_0}|_G$ by the multiplication-by-$a$ map $G \to G$.

Proof. This follows from Lemma 3.8 and [6, 8.3, Prop. 3].

Corollary 4.3. Let $K$ be a $p$-local field and $R$ a complete strict $p'$-coefficient ring with a positive residue characteristic. Then for any non-trivial continuous additive character $\psi : K \to R^\times$ of conductor $n$. Then there exists a unique non-trivial additive character $R$-sheaf $\tilde{\psi}$ of conductor $n$ such that for any $a \in K$ with $v_K(a) < -n - 1$, we have

$$
\psi(a) = \text{Tr}(\text{Fr}_\sigma; \tilde{\psi}|_{K[v_K(a),-n-1]}),
$$

where $\sigma$ is the $k$-rational point of $K[v_K(a),-n-1]$ corresponding to $a$. Furthermore, $\psi \mapsto \tilde{\psi}$ gives a one-to-one correspondence between the non-trivial continuous $R$-valued additive characters of $K$ of conductor $n$ and the non-trivial additive character $R$-sheaves of conductor $n$.

Proof. The only non-trivial part is the existence of the sheaf $\tilde{\psi}$. When $\text{char } K = p$, take a non-trivial additive character $\phi_0 : \mathbb{F}_p \to R_0^\times$ with values in a pro-finite local subring $R_0$. Then there exists a unique continuous 1-differential $\omega$ on $K$ over $k$ such that $\psi(x) = \phi_0(\text{Tr}_{k/\mathbb{F}_p}(x\omega))$ for all $x \in K$ (Here Res denotes the residue at the closed point of $\text{Spec } \mathcal{O}_K$). Then for all $m < -n - 1$, the map $x \mapsto \text{Res}(x\omega)$ defines a morphism $f : K^{[m,-n-1]} \to \mathbb{G}_{a,k}$ of $k$-groups. The sheaf $\tilde{\psi}|_{K^{[m,-n-1]}}$ is realized as the pull-back of the Artin-Schreier sheaf on $\mathbb{G}_{a,k}$ associated to $\phi_0$.

When $\text{char } K = 0$, fix a non-trivial continuous additive character $\psi_0 : \mathbb{Q}_p \to R^\times$ with $\text{ord } \psi_0 = 0$. For each integer $n \geq 1$, let $\mathbb{Q}_p^{-[n,-1]}$ is canonically isomorphic to the group of Witt covectors $CW_n,\mathbb{F}_p$ of length $n$. Then the morphism $1 - F : CW_n,\mathbb{F}_p \to CW_n,\mathbb{F}_p$ and the character $\phi_0$ defines a non-trivial additive character $R$-sheaf $\tilde{\psi}_0$ of conductor $0$. There exists a unique element $a \in K^\times$ such that $\psi(x) = \psi_0(\text{Tr}_{K/\mathbb{Q}_p}(ax))$ for all $x$. Then the sheaf $\tilde{\psi}$ is realized as $(\tilde{\psi}_0 \circ \text{Tr}_{K/\mathbb{Q}_p})a$.\hfill\qed

Corollary 4.4. Let $K$ be a $p$-CDVF with a residue field $k$.

(1) Suppose that $\text{char } K = 0$. Let $K_0 = \text{Frac } W(k)$ the maximal absolutely unramified subfield of $K$ and let $K_00 = \text{Frac } W(\mathbb{F}_p)$. Fix a non-trivial additive character sheaf $\tilde{\psi}_0$ on $K_00$. Then for any non-trivial additive character sheaf $\tilde{\psi}$ on $K$, there exists a unique element $a \in K^\times$ with

$$
v_K(a) = \text{ord } \tilde{\psi} - v_L(D_{K/K_0}) - e_{K/K_0} \cdot \text{ord } \tilde{\psi}_0
$$
such that for all $m \in \mathbb{Z}$ with $m \leq \text{ord} \tilde{\psi}_0 - 1$, the sheaf

$$
\tilde{\psi}|_{K^{[m e_K/K_0 + e_K/K_0 \cdot \text{ord} \tilde{\psi}_0 - \text{ord} \tilde{\psi} - 1]}}
$$

is the pull-back of $\tilde{\psi}_0$ by the morphism

$$
K^{[m e_K/K_0 + e_K/K_0 \cdot \text{ord} \tilde{\psi}_0 - \text{ord} \tilde{\psi} - 1]} \xrightarrow{\alpha} K^{[e_L(D_{K/K_0}) + m e_K/K_0 - \text{ord} \tilde{\psi}_0 - 1]} \xrightarrow{\text{Tr}_{K_0/K}} K_0^{[m, \text{ord} \tilde{\psi}_0 - 1]}. 
$$

(2) If $\text{char} K = p > 0$, take a prime element $\pi_K$ in $K$ and set $K_0 = \mathbb{F}_p((\pi_K))$. Fix a non-trivial additive character sheaf $\tilde{\psi}_0$ on $K_0$. Then for any non-trivial additive character sheaf $\tilde{\psi}$ on $K$, there exists a unique element $a \in K^\times$ with $\nu_K(a) = \text{ord} \psi - \text{ord} \psi_0$ such that for all $m \in \mathbb{Z}$ with $m \leq -\text{ord} \tilde{\psi}_0 - 1$, the sheaf

$$
\tilde{\psi}|_{K^{[m + \text{ord} \tilde{\psi}_0 - \text{ord} \psi - \text{ord} \tilde{\psi} - 1]}}
$$

is the pull-back of $\tilde{\psi}_0$ by the morphism

$$
K^{[m + \text{ord} \tilde{\psi}_0 - \text{ord} \psi - \text{ord} \tilde{\psi} - 1]} \xrightarrow{\alpha} K^{[m, \text{ord} \tilde{\psi}_0 - 1]} \xrightarrow{\text{ev}} K_0^{[m, \text{ord} \tilde{\psi}_0 - 1]}.
$$

\[\square\]

4.2. A map from the Brauer group. There is a canonical map $\partial : \text{Br}(K) \rightarrow H^1(k, \mathbb{Q}/\mathbb{Z})$. Let us recall its definition: we have

$$
\text{Br}(K) = \bigcup_L \text{Br}(L/K),
$$

where $L$ runs over all unramified finite Galois extension of $K$ in a fixed separable closure of $K$ and $\text{Br}(L/K) := \ker(\text{Br}(K) \rightarrow \text{Br}(L))$. We define $\partial$ to be the composition

$$
\text{Br}(K) \cong \varprojlim L, \text{Inf} H^2(\text{Gal}(L/K), L^\times) \rightarrow \varprojlim L, \text{Inf} H^2(\text{Gal}(L/K), \mathbb{Z}) \cong \varprojlim L, \text{Inf} H^1(\text{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \cong H^1(k, \mathbb{Q}/\mathbb{Z}).
$$

By local class field theory, the following lemma holds.

**Lemma 4.5.** Suppose that $k$ is finite. Then the invariant map $\text{inv} : \text{Br}(K) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ of local class field theory is equal to the composition

$$
\text{Br}(K) \xrightarrow{\partial} H^1(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{ev}} \mathbb{Q}/\mathbb{Z}
$$

where $\text{ev} : H^1(k, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}$ denotes the evaluation map at $\text{Fr}_k$. \[\square\]
Let $\chi : W_K \to R^\times$ be a character of $W_K$ of finite order $n$, and let $a \in K^\times$. Take a generator $\zeta \in R^\times$ of $\text{Im} \chi$. Let $L$ be the finite cyclic extension of $K$ corresponding to $\text{Ker} \chi$, Let $\sigma \in \text{Gal}(L/K)$ be the generator of $\text{Gal}(L/K)$ such that $\chi(\sigma) = \zeta$. Then the cyclic algebra $(a, L/K, \sigma)$ defines an element $[(a, L/K, \sigma)]$ in $\text{Br}(K)$. We identify $H^1(k, \mathbb{Z}/n\mathbb{Z})$ with $\text{Hom}(G_K, \mu_n(\mathbb{Q}_l))$ by the isomorphism $\mathbb{Z}/n\mathbb{Z} \to \mu_n(R)$, $1 \mapsto \zeta$, and regard $\partial_n([((a, L/K, \sigma)])$ as a character of $G_K$ of finite order. This character does not depend on the choice of $\zeta$, and is denoted by $\chi_a : G_K \to R^\times$. It is well-known that $(\chi, a) \mapsto \chi[a]$ is biadditive with respect to $\chi$ and $a$. If $R'$ is another complete strict $p'$-coefficient ring and if $h : R \to R'$ is a local homomorphism, then we have $\chi_a \otimes_R R' \cong (h \circ \chi)_a$.

**Corollary 4.6.** Suppose that $k$ is finite. Let $\chi$ be a character of $G_K$ of finite order, and let $a \in K^\times$. Then we have

$$\chi[a](\text{Fr}) = \chi(\text{rec}(a)).$$

The following lemma is easily proved:

**Lemma 4.7.** Let $\chi : W_K \to R^\times$ be an unramified character of $W_K$ of finite order. Then, for $a \in K^\times$, we have $\chi[a] = \chi^\otimes_{v_K}(a)$.

Let $\chi : W_K \to R^\times$ be an arbitrary character of $W_K$. Then there exists an unramified character $\chi_1$ and a character $\chi_2$ such that $\chi_2 \mod m^n_R$ is of finite order for all $n \in \mathbb{Z}_{>0}$ and that $\chi = \chi_1 \otimes_R \chi_2$. For $a \in K^\times$ define $\chi[a] : W_k \to R^\times$ by

$$\chi[a] := \chi_1^\otimes_{v_K}(a) \otimes_R (\lim_{n} (\chi_2 \mod m^n_R)[a]).$$

This does not depend on the choice of $\chi_1$ and $\chi_2$. By definition, we have $\chi[aa'] \cong \chi[a'] \otimes_R \chi[a']$ and $(\chi \otimes_R \chi')[a] \cong \chi[a] \otimes_R \chi'[a]$.

**Lemma 4.8.** If $a \in 1 + m_K$, then the character $\chi[a]$ is finite of a $p$-power order.

**Proof.** We may assume that $\chi \mod m^n_R$ is of finite order for all $n \in \mathbb{Z}_{>0}$. Let $L_n$ be the finite cyclic extension of $K$ corresponding to $\text{Ker} (\chi \mod m^n_R)$. There exists a $p$-power $N$ such that $a^N \in 1 + m^{sw}(\chi)$. Since $1 + m^{sw}(\chi)$ is contained in $N_{L_n/K}(L_n^\times)$, the character $\chi[a^N]$ is trivial. This completes the proof.

4.3. Serre-Hazewinkel’s geometric class field theory. For any finite separable extension $L$ of $K$, let $U_L$, $U_{L,n}$, and $U_L^{(n)}$ denote the affine commutative $k$-group schemes which represent the functors which associate to each
4.4. Local $\bar{\varepsilon}_0$-character for rank one objects. Let $\bar{\psi}$ be an additive character sheaf of $K$. Let $(R_0, m_{R_0})$ be a pro-finite local subring of $R$ such that $R_0 \hookrightarrow R$ is a local homomorphism and that $\bar{\psi}$ is defined over $R_0$.

In this subsection we attach, for every rank one object $(\chi, V)$ in $\text{Rep}(W_K, R_0)$, a rank one object $\bar{\varepsilon}_0, R(V, \bar{\psi})$ in $\text{Rep}(W_K, R)$, which we call the local $\bar{\varepsilon}_0$-character of $V$.

For each integer $m \in \mathbb{Z}$, let $K^{[m, \infty]}$ denote the affine $k$-scheme $\lim_{\leftarrow} K^{[m,n]}$. This represents the functor associating for any $k$-algebra $A$ the set $R_A \otimes_{R_k} m_{K}^{m}$. Take a prime element $\pi_K$ of $K$ and let $\pi_K^{-m} : K^{[m, \infty]} \to K^{[0, \infty]}$ be the morphism defined by the multiplication by $\pi_K^{-m}$. The inverse image of $U_K \subset K^{[0, \infty]}$ by $\pi_K^{-m}$ is an open subscheme of $K^{[m, \infty]}$ which we denote by $K^{v=m}$. This does not depend on the choice of $\pi_K$.

For $m, n \in \mathbb{Z}$, the multiplication map defines a morphism $K^{v=n} \times K^{v=m} \to K^{v=m+n}$ of $k$-schemes. This defines a structure of commutative $k$-group scheme on the disjoint union $\coprod_{m} K^{v=m}$. There is a canonical exact sequence

$$1 \to U_K \to \prod_{m} K^{v=m} \to \mathbb{Z} \to 0,$$

where $\mathbb{Z}$ is a constant $k$-group scheme.

Now we shall define, for every rank one object $\chi$ in $\text{Rep}(W_K, R_0)$, a character sheaf $\mathcal{L}_\chi$ on $\coprod_{m} K^{v=m}$.

(1) First assume that $\chi$ is unramified, let $\mathcal{L}_\chi'$ be the invertible $R_0$-sheaf on $\text{Spec}(k)$ corresponding to $\chi$. Define an invertible $R_0$-sheaf $\mathcal{L}_\chi$ on $\coprod_{m} K^{v=m}$.
by
\[ \mathcal{L}_\chi|_{K^{v=m}} = \pi^m_* (\mathcal{L}'_\chi)^{\otimes m}, \]
where \( \pi^m: K^{v=m} \to \text{Spec}(k) \) is the structure morphism. It is easily checked that \( \mathcal{L}_\chi \) is a character sheaf.

(2) Next assume that \( \chi \) is a character of the Galois group of a finite separable totally ramified abelian extension \( L \) of \( K \). Consider the norm map \( \Pi_m L^{v=m} \to \Pi_m K^{v=m} \). It is surjective and the group of the connected components of its kernel is canonically isomorphic to \( \text{Gal}(L/K) \). Hence we have a canonical group extension of \( \Pi_m K^{v=m} \) by \( \text{Gal}(L/K) \). Define \( \mathcal{L}_\chi \) to be the character sheaf on \( \Pi_m K^{v=m} \) defined by this extension and \( \chi \).

(3) Assume that \( \chi \) is of finite order. Then \( \chi \) is a tensor product \( \chi = \chi_1 \otimes R_0 \chi_2 \), where \( \chi_1 \) is unramified and \( \chi_2 \) is of the form in (2). Define \( L^{\chi} \) to be \( L^{\chi_1} \otimes R_0 L^{\chi_2} \).

Let \( L/K \) be a finite abelian extension such that \( \chi \) factors through \( \text{Gal}(L/K) \). Let \( L_0 \) be the maximal unramified subextension of \( L/K \). From the norm map \( L^{v=1} \to L_0^{v=1} \) and the canonical morphism \( L_0^{v=1} \cong k_L \to K^{v=1} \), we obtain a canonical etale \( \text{Gal}(L/K) \)-torsor on \( K^{v=1} \). The following lemma is easily proved.

**Lemma 4.9.** \( \mathcal{L}_\chi|_{K^{v=1}} \) is isomorphic to the smooth \( R_0 \)-sheaf defined by \( T \) and \( \chi \).

**Corollary 4.10.** The sheaf \( \mathcal{L}_\chi \) does not depend on the choice of \( \chi_1 \) and \( \chi_2 \).

(4) General case. For each \( n \in \mathbb{Z}_{>0} \), \( \chi_n := \chi \mod m_{R_0}^n \) is a character of finite order. Define \( \mathcal{L}_\chi \) to be \( (\mathcal{L}_{\chi_n})_n \).

**Corollary 4.11.** Let \( \chi_1, \chi_2 \) be two rank one objects in \( \text{Rep}(W_K, R_0) \). Then we have an isomorphism \( \mathcal{L}_{\chi_1} \otimes R_0 \mathcal{L}_{\chi_2} \cong \mathcal{L}_{\chi_1} \otimes R_0 \mathcal{L}_{\chi_2} \).

**Lemma 4.12.** Let \( s = \text{sw}(\chi) \) be the Swan conductor of \( \chi \). Then the restriction of \( \mathcal{L}_\chi \) to \( U_K \) is the pull-back of a character sheaf \( \mathcal{T}_\chi \) on \( U_{K,s+1} \). Furthermore, if \( s \geq 1 \), the restriction of \( \mathcal{T}_\chi \otimes R_0/m_{R_0} \) to \( U_{K,s}^s \) is non-trivial.

**Proof.** We may assume that \( \chi \) is of the form of (2). Let \( L \) be the finite extension of \( K \) corresponding to \( \text{Ker} \chi \), \( \pi_L \) a prime element in \( L \). For the first (resp. the second) assertion, it suffices to prove that there does not exist (resp. there exists) \( \sigma \in \text{Gal}(L/K) \) with \( \sigma \neq 1 \) such that \( \sigma(\pi_L)/\pi_L \) lies in the neutral component of the kernel of the map \( U_L \to U_K \to U_{K,s+1}, \) which is easy to prove.

**Lemma 4.13.** For \( a \in K^\times \), let \( \overline{a}: \text{Spec}(k) \to U_K \) be the \( k \)-rational point of \( U_K \) defined by \( a \). Then \( \overline{a}_* \mathcal{L}_\chi \) is isomorphic to \( \chi[a] \).
Proof. We may assume that \( \chi \) is of the form of case (1) or (2). In case (1), the assertion follows from Lemma 4.7. In case (2), let \( L \) be the finite extension of \( K \) corresponding to \( \text{Ker} \chi \). Take a generator \( \sigma \in \text{Gal}(L/K) \) and let us consider the cyclic algebra \((a, L/K, \sigma)\). This is isomorphic to a matrix algebra over a central division algebra \( D = D(a, L/K, \sigma) \) over \( K \). The valuation of \( K \) is canonically extended to a valuation of \( D \). Let \( \mathcal{O}_D \) denote the valuation ring of \( D \), \( k_D \) the residue field of \( D \). There is a maximal commutative subfield of \( D \) which is isomorphic (as a \( k \)-algebra) to a subextension of \( L/K \). Since \( L/K \) is totally ramified, \( k_D \) is a commutative field. Let \( \pi_D \) be a prime element of \( D \). The conjugation by \( \pi_D \); \( x \mapsto \pi_D^{-1}x\pi_D \) defines an automorphism \( \tau \) of \( k_D \) over \( k \). It is checked that the fixed field of \( \tau \) is equal to \( k \). Hence \( k_D/k \) is a cyclic extension whose Galois group is generated by \( \tau \). Let \( K_D \) be the unramified extension of \( K \) corresponding to \( k_D/k \). Then \( D \otimes_K K_D \) is split. Hence there exists an element \( b \in (\mathcal{O}_{LKD})^\times \) such that \( a = N_{LKD/K_D}(b) \).

Consider the following commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & (K_D)^\times & \longrightarrow & GL_n(K_D) & \longrightarrow & PGL_n(K_D) & \longrightarrow & 1 \\
& \downarrow v_{K_D} & & \downarrow \frac{1}{n}v_{K_D} \circ \text{det} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0.
\end{array}
\]

By \([9, X, \S\ 5]\), \((a, L/K, \sigma)\) gives a canonical element in \( H^1(\text{Gal}(K_D/K), PGL_n(K_D)) \) whose image by the canonical map

\[
H^1(\text{Gal}(K_D/K), PGL_n(K_D)) \rightarrow H^1(\text{Gal}(K_D/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{Inf}} H^1(k, \mathbb{Q}/\mathbb{Z})
\]

is equal to \( \partial([a, L/K, \sigma]) \).

By definition, \((a, L/K, \sigma) = \bigoplus_{i=0}^{n-1} L \cdot \alpha_i \) with \( \alpha^n = a, \alpha \sigma = \sigma(x) \alpha \) for \( x \in K \). Let \( \iota : LKD \hookrightarrow \text{End}_{K_D}(LK_D) \) be the canonical homomorphism. Let \( \varphi : (a, L/K, \sigma) \otimes_K K_D \cong \text{End}_{K_D}(LK_D) \) be the \( K_D \)-algebra isomorphism defined by \( \varphi(x) = \iota(x) \) for \( x \in LKD \) and by \( \varphi(\alpha) = \iota(b) \cdot \sigma \). It is easily checked that the composition

\[
\tau \circ \varphi \circ \tau^{-1} \circ \varphi^{-1} : \text{End}_{K_D}(LK_D) \xrightarrow{\varphi^{-1}} (a, L/K, \sigma) \otimes_K K_D \\
\xrightarrow{\tau^{-1}} (a, L/K, \sigma) \otimes_K K_D \\
\xrightarrow{\varphi} \text{End}_{K_D}(LK_D) \cong \text{End}_K(L) \otimes_K K_D \\
\xrightarrow{\tau} \text{End}_K(L) \otimes_K K_D \cong \text{End}_{K_D}(LK_D)
\]

is a \( K_D \)-algebra automorphism which is identity on \( \iota(LKD) \) and which sends \( \sigma \) to \( \frac{\tau(b)}{\sigma(b)} \cdot \sigma \). By Skolem-Noether theorem, there exists an element \( c \in LKD^\times \) such that \( \frac{\tau(b)}{\sigma(b)} = \frac{\sigma(c)}{c} \) and that \( \tau \circ \varphi \circ \tau^{-1} \circ \varphi^{-1} \) is the conjugation by \( \iota(c) \). Then \( \chi_{(a)} \) is the inflation of the character of \( \text{Gal}(k_D/k) \) which sends \( \tau \) to \( \zeta^\chi\cdot c \). Hence the assertion follows. \( \square \)
Let $s = \text{sw}(\chi)$ be the Swan conductor of $\chi$ and set $m = -s - \text{ord } \tilde{\psi} - 1$. Then the character $\tilde{\psi}$ defines an invertible character $R_0$-sheaf $\tilde{\psi}[m, -\text{ord } \tilde{\psi} - 1]$ on $K^{[m, -\text{ord } \tilde{\psi} - 1]}$. The sheaf $\mathcal{L}_\chi$ is a pull-back of a character sheaf $\tilde{\mathcal{L}}_\chi$ on $\prod_{m'} K^{v=m'/U_K}$. Let $i : K^{v=m}/U_K^{(s+1)} \hookrightarrow K^{[m, -\text{ord } \tilde{\psi} - 1]}$ be the canonical inclusion and let $f : K^{v=m}/U_K^{(s+1)} \to \text{Spec } (k)$ be the structure morphism. Define the $\tilde{\varepsilon}_0$-character $\tilde{\varepsilon}_{0,R}(\chi, \tilde{\psi})$ to be the rank one object in $\text{Rep}(W_k, R)$ corresponding to the invertible $R_0$-sheaf

$$\tilde{\varepsilon}_{0,R}(\chi, \tilde{\psi}) = \det R_0(Rf_i((\mathcal{L}_\chi|_{K^{v=m}/U_K^{(s+1)}})^{\otimes -1} \otimes R_0 i^* \tilde{\psi}[m, -\text{ord } \tilde{\psi} - 1])[s + 1](\text{ord } \tilde{\psi})).$$

Here [ ] denotes a shift in the derived category and ( ) is a Tate twist.

**Proposition 4.14.** Let $\mathcal{F} := (\tilde{\mathcal{L}}_\chi|_{K^{v=m}/U_K^{(s+1)}})^{\otimes -1} \otimes R_0 i^* \tilde{\psi}[m, -\text{ord } \tilde{\psi} - 1])$.

1. Suppose that $s = 0$. Then $R^if_*\mathcal{F} = 0$ for $i \neq 1$ and $R^if_*\mathcal{F}$ is an invertible $R$-sheaf on $\text{Spec } (k)$.
2. Suppose that $s = 2b - 1$ is odd and $\geq 1$. Let $f' : K^{v=m}/U_K^{(s+1)} \to K^{v=m}/U_K^{(b)}$ be the canonical morphism. Then $R^if'_*\mathcal{F} = 0$ for $i \neq 2b$ and there exists a $k$-rational point $P$ in $K^{v=m}/U_K^{(b)}$ such that $R^{2b}f'_*\mathcal{F}$ is supported on $P$ whose fiber is free of rank one.
3. Suppose that $s = 2b$ is even and $\geq 2$. Let $f' : K^{v=m}/U_K^{(s+1)} \to K^{v=m}/U_K^{(b+1)}$ be the canonical morphism. Then $R^if'_*\mathcal{F} = 0$ for $i \neq 2b - 2$ and there exists a $k$-rational point $P$ in $K^{v=m}/U_K^{(b)}$ such that $R^{2b-2}f'_*\mathcal{F}$ is supported on the fiber $A \cong \mathbb{A}^1_k$ at $P$ by the canonical morphism

$$K^{v=m}/U_K^{(b+1)} \to K^{v=m}/U_K^{(b)}$$

and that $R^{2b-2}f'_*\mathcal{F}|_A$ is a smooth invertible $R$-sheaf on $A$, whose swan conductor at infinity is equal to 2.

**Proof.** The assertions (1) and (2) are easy and their proofs are left to the reader. We will prove (3). We may assume that $k$ is algebraically closed.

For any closed point $Q$ in $K^{v=m}/U_K^{(s+1)}$, the pull-back of $\mathcal{F}$ by the multiplication-by-$Q$ map $U_K^{(b+1)}/U_K^{(s+1)} \hookrightarrow K^{v=m}/U_K^{(s+1)}$ is an invertible character $R_0$-sheaf, which we denote by $\mathcal{L}_Q$. 
There exists a unique $k$-rational point $P$ in $K^{v=m}/U^{(b)}_K$ such that $L_Q$ is trivial if and only if $Q$ lies in the fiber $A \cong \mathbb{A}^1_k$ at $P$ by the canonical morphism $K^{v=m}/U^{(b+1)}_K \to K^{v=m}/U^{(b)}_K$. By the orthogonality relation of character sheaves, $R^if_i\mathcal{F} = 0$ for $i \neq 2b-2$, $R^{2b-2}f_2\mathcal{F}$ is supported on $A$ and $\mathcal{G} = R^{2b-2}f_2\mathcal{F}|_A$ is a smooth invertible $R$-sheaf on $A$. Take a closed point $P_0$ in $A \subset K^{v=m}/U^{(b+1)}_K$ and identify $A$ with $U^{(b)}_K/U^{(b+1)}_K \cong G_{a,k}$ by $P_0$. The sheaf $\mathcal{G}$ is has the following property: there exists a non-trivial invertible character sheaf $\mathcal{L}_1$ on $G_{a,k}$ such that $\mathcal{G} \boxtimes \mathcal{G} \cong \alpha^* \mathcal{G} \boxtimes \mu^* \mathcal{L}_1$, where $\alpha, \mu : G_{a,k} \times G_{a,k} \to G_{a,k}$ denote the addition map and the multiplication map respectively.

If $p \neq 2$, then let $f : G_{a,k} \to G_{a,k}$ denote the map defined by $x \mapsto x^2/2$. Then $\mathcal{G} \boxtimes f^* \mathcal{L}_1$ is an invertible character sheaf on $G_{a,k}$. Hence the swan conductor of $\mathcal{G}$ at infinity is equal to 2.

It remains to consider the case $p = 2$. Let $W_{2,k}$ be the $k$-group of Witt vectors of length two. Let $\mathcal{G}'$ be the invertible sheaf on $W_{2,k}$ defined by $\mathcal{G}' = a_0^* \mathcal{G} \boxtimes a_1^* \mathcal{L}$, where $a_i : W_{2,k} \to G_{a,k}$ are $k$-morphisms defined by $\langle x_0, x_1 \rangle \mapsto x_i$.

Then the sheaf $\mathcal{G}'$ is an invertible character sheaf on $W_{2,k}$. There exists an element $a \in W_{2}(k)^\times$ such that the pull-back $a^* \mathcal{G}'$ of $\mathcal{G}'$ by the multiplication-by-$a$ map is trivial on the finite etale covering $1 - F : W_{2,k} \to W_{2,k}$ of $W_{2,k}$. Since $\mathcal{G}$ is isomorphic to the pull-back of $\mathcal{G}'$ by the Teichmüller map $G_{a,k} \to W_{2,k}$, the assertion of the lemma follows from direct computation. \hfill \Box

**Lemma 4.15.** For $a \in K^\times$, we have

$$\bar{\varepsilon}_{0,R}(\chi, \bar{\psi}_a) = \chi[a] \otimes_R \bar{\varepsilon}_{0,R}(\chi, \bar{\psi}_a) \otimes_R R(v_K(a)).$$

**Proof.** It follows from Lemma 4.13. \hfill \Box

**4.5. $\tilde{\lambda}_R$-characters.** Let $L$ be a finite separable extension, and $\bar{\psi}$ an additive character sheaf on $K$.

Let $V = \text{Ind}_{W_L}^{W_K} 1 \in \text{Rep}(W_K, R)$, and $V_C = \text{Ind}_{W_L}^{W_K} 1 \in \text{Rep}(W_K, \mathbb{C})$. Since $V_C^0 + \text{det} V_C - ([L : K] + 1)1_C$ is an real virtual representation of $W_K$ of virtual rank 0, we can define a canonical element

$$\text{sw}_2(V_C^0) \in 2Br(K)$$

as in [2, 1.4.1]. Let $\text{sw}_2(R(V_C^0))$ be the rank one object in $\text{Rep}(W_k, R)$ induced by $\partial_2(\text{sw}_2(V_C^0)) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ and the map $\mathbb{Z}/2\mathbb{Z} \to R^\times$, $n \mapsto (-1)^n$.

Next we define $\bar{\varepsilon}_R(\text{det} V, \bar{\psi})$. When $\text{det} V$ is unramified, we denote by the same symbol $\text{det} V$ the rank one object in $\text{Rep}(W_k, R)$ corresponding to $\text{det} V$ and set $\bar{\varepsilon}_R(\text{det} V, \bar{\psi}) := (\text{det} V)^{\otimes \text{ord} \bar{\psi}} \otimes_R R(-\text{ord} \bar{\psi})$. When $\text{det} V$ is not unramified, we set $\bar{\varepsilon}_R(\text{det} V, \bar{\psi}) := \bar{\varepsilon}_{0,R}(\text{det} V, \bar{\psi})$. 

\vspace{1cm}

**Local $\varepsilon_0$-characters in torsion rings**

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Definition 4.16. Define the rank one object \( \tilde{\lambda}_R(L/K, \tilde{\psi}) \) in \( \text{Rep}(W_k, R) \) by

\[
\tilde{\lambda}_R(L/K, \tilde{\psi}) := \text{sw}_2 R(V_C^0) \otimes_R \tilde{\varepsilon}_R(\det V, \tilde{\psi}) \otimes -1 \otimes_R \text{det} \left( \text{Ind}_{W_kL}^{W_k} 1 \right) \\
\otimes_R R \left( \frac{1}{2} (v_{L/K}(d_{L/K}) - a(\det V_C)) - \text{ord} \tilde{\psi} \right),
\]

where \( a(\det V_C) \) is the Artin conductor of \( \det V_C \).

The following lemma is easily checked:

Lemma 4.17. Suppose that \( k \) is finite. Let \( \psi : K \to R^\times \) be the additive character corresponding to \( \tilde{\psi} \). Then we have

\[
\tilde{\lambda}_R(L/K, \tilde{\psi})(\text{Fr}_k) = (-1)^{v_K(d_{L/K}) + f_{L/K} + 1} \lambda_R(L/K, \psi).
\]

\( \square \)

4.6. Local \( \varepsilon_0 \)-characters of representations of \( G_K \) whose images are finite. Let \( R \) be an algebraically closed field of positive characteristic \( \neq p \). In this subsection we shall define, for an object \( (\rho, V) \) in \( \text{Rep}(W_K, R) \) such that \( \text{Im} \rho \) is finite, a rank one object \( \varepsilon_{0,R}(V, \tilde{\psi}) \) in \( \text{Rep}(W_k, R) \), which is called the local \( \varepsilon_0 \)-character of \( V \).

Let \( L \) be the finite Galois extension of \( K \) corresponding to the kernel of \( \rho \) and let \( G = \text{Gal}(L/K) \).

By Brauer’s theorem for modular representations (cf. [8]), there exist subgroups \( H_1, \cdots, H_m \) of \( G \), characters \( \chi_1, \cdots, \chi_m \) of \( H_1, \cdots, H_m \) and integers \( n_1, \cdots, n_m \in \mathbb{Z} \) such that \( \rho = \sum_i n_i \text{Ind}_{H_i}^G \chi_i \) as a virtual representation of \( G \) over \( R \). Let \( K_i \) be the subextension of \( L/K \) corresponding to \( H_i \).

Define \( \varepsilon_{0,R}(V, \tilde{\psi}) \) by

\[
\varepsilon_{0,R}(V, \tilde{\psi}) = \bigotimes_i (\varepsilon_{0,R}(\chi_i, \tilde{\psi} \circ \text{Tr}_{K_i/K}) \otimes \tilde{\lambda}_R(K_i/K, \tilde{\psi}))^{\otimes n_i}.
\]

Lemma 4.18. The sheaf \( \varepsilon_{0,R}(V, \tilde{\psi}) \) does not depend on the choice of \( H_i \), \( \chi_i \) and \( n_i \).

A proof of this lemma is given in the next two subsections.

The following two lemmas are easily proved.

Lemma 4.19. Let \( \chi \) be an unramified rank one object in \( \text{Rep}(W_K, R) \) of finite order. Then

\[
\varepsilon_{0,R}(V \otimes \chi, \tilde{\psi}) \cong \varepsilon_{0,R}(V, \tilde{\psi}) \otimes_R \chi^{\otimes \text{sw}(V) + \text{rank} V \cdot (\text{ord} \tilde{\psi} + 1)}.
\]

\( \square \)

Lemma 4.20. Let

\[
0 \to V' \to V \to V'' \to 0
\]
be a short exact sequence of objects in \( \text{Rep}(W_K, R) \) with finite images. Then we have
\[
\tilde{e}_{0,R}(V, \tilde{\psi}) \cong \tilde{e}_{0,R}(V', \tilde{\psi}) \otimes \tilde{e}_{0,R}(V'', \tilde{\psi}).
\]

4.7. \( A \)-structures. For any ring \( A \) of characteristic \( p \), let \( R_A \) denote \( W(A) \) (resp. \( A \)) if \( K \) is of mixed characteristic (resp. of equal characteristic). When \( A \) is a subring of \( k \), we regard \( R_A \) as a subalgebra of \( \mathcal{O}_K \) in canonical way.

If \( \text{char} K = 0 \), a prime element \( \pi_K \) of \( K \) is called \( A \)-admissible if the minimal polynomial of \( \pi_K \) over \( \text{Frac} W(k) \) has coefficients in \( R_A \) and has the constant term in \( pR_A' \).

If \( \text{char} K = p \), any prime element \( \pi_K \) of \( K \) is called \( A \)-admissible.

**Lemma 4.21.** For any prime element \( \pi'_K \) of \( K \) and for any positive integer \( N \in \mathbb{Z}_{>0} \), there exists a finitely generated \( \mathbb{F}_p \)-subalgebra \( A \subset k \) and an \( A \)-admissible prime element \( \pi_K \) of \( K \) congruent to \( \pi'_K \) modulo \( m_K^N \).

Before proving this lemma, we prove the following lemma:

**Lemma 4.22.** Let \( K \) be a \( p \)-CDVF, and \( f(T) \in \mathcal{O}_K[T] \) a polynomial. Suppose that \( x_0 \in \mathcal{O}_K \) satisfies \( f'(x_0) \neq 0 \) and \( f(x_0) \in m_K^{2v_K(f'(x_0))+1} \). Then there exists a unique element \( x \in \mathcal{O}_K \) such that \( x \equiv x_0 \mod m_K^{v_K(f'(x))+1} \) and that \( f(x) = 0 \). Moreover we have \( v_K(f'(x)) = v_K(f'(x_0)) \).

**Proof.** We prove the lemma using induction. Put \( v = v_K(f'(x_0)) \). It suffices to prove the following statement:

If \( n > 2v \) and if an element \( y_0 \in \mathcal{O}_K \) satisfies \( y_0 \equiv x_0 \mod m_K^{v+1} \) and \( f(y_0) \in m_K^n \), then there exists an element \( y \in \mathcal{O}_K \) satisfying \( y \equiv y_0 \mod m_K^{v-1} \) and \( f(y) \in m_K^{n+1} \). Furthermore, the class of such \( y \) modulo \( m_K^{v-1} \) is unique.

Since \( y_0 \equiv x_0 \mod m_K^{v+1} \), we have
\[
f'(y_0) \equiv f'(x_0) \neq 0 \mod m_K^{v+1}.
\]
Hence \( f(y_0) / f'(y_0) \) is an element in \( m_K^{-v} \). Then the polynomial \( f(y_0 + f'(y_0)T) \) is congruent to \( f(x) + f(x)T \mod m_K^{v+1} \). Hence the assertion follows. □

**Proof of Lemma 4.21.** We may assume that \( \text{char} K = 0 \). Take an arbitrary prime element \( \pi'_K \) of \( K \). Let \( f(T) = T^e + \sum_{i=0}^{e-1} a_i T^i \) be the minimal polynomial of \( \pi_K \) over \( K_0 = \text{Frac} W(k) \). Set \( M = \max \{ N, v_K(D_K/K_0) + 1 \} \). For each \( i \), write \( a_i \in W(k) \) as the form of a Witt vector \( a_i = (0, a_{i,1}, a_{i,2}, \ldots) \). Define \( a'_i \in W(k) \) by \( a'_i = (0, a_{i,1}, \ldots, a_{i,M+v_K(D_K/K_0)}, 0, \ldots) \) and let \( g(T) = T^e + \sum_{i=0}^{e-1} a'_i T^i \). By Lemma 4.22, there exists a root \( \pi_K \) of \( g(T) \) such that \( \pi_K \equiv \pi'_K \mod m_K^M \).
Let $A$ be the $\mathbb{F}_p$-subalgebra of $k$ generated by
\[ \{a_{i,j} : 0 \leq i \leq e - 1, 1 \leq j \leq M + v_K(D_K/K_0)\} \cup \{a_{0,1}^{-1}\}. \]
Then $\pi_K$ is $A$-admissible.

For an $A$-admissible prime element $\pi_K$ of $K$, let $\widetilde{R}_{A, \pi_K}$ denote the subalgebra $R_A[[\pi_K]]$ of $\mathcal{O}_K$.

There exists a canonical morphism $\widetilde{R}_{A, \pi_K} \to A$. The following lemma is easily checked:

**Lemma 4.23.**

1. Let any $n \in \mathbb{Z}_{>0}$, $x \in \widetilde{R}_{A, \pi_K} \cap m^n_K$, and $y \in k$ the class of $\frac{x}{\pi^n_K}$. Then there exists a positive integer $m$ satisfying $y^{p^m} \in A$.
2. An element in $\widetilde{R}_{A, \pi_K}$ is invertible if and only if its canonical image in $A$ is invertible.
3. If $A$ is perfect, then $\widetilde{R}_{A, \pi_K} \cap m^n_K = \pi^n_K \cdot \widetilde{R}_{A, \pi_K}$.

In a manner similar to that in the proof of Lemma 4.22, we have:

**Lemma 4.24.** Let $f(T) \in \widetilde{R}_{A, \pi_K}[T]$ be a polynomial. Let $v$ be a positive integer. Suppose that $x_0 \in \widetilde{R}_{A, \pi_K}$ satisfies $f'(x_0) \in \pi^v_K(\widetilde{R}_{A, \pi_K})^\times$ and $f(x_0) \in \pi^{nv+1}_K \widetilde{R}_{A, \pi_K}$. Then there exists a unique element $x$ in $\widetilde{R}_{A, \pi_K}$ such that $x \equiv x_0 \mod \pi^{v+1}_K \widetilde{R}_{A, \pi_K}$ and that $f(x) = 0$. Furthermore we have $f'(x) \in \pi^v_K(\widetilde{R}_{A, \pi_K})^\times$.

Let $m, n \in \mathbb{Z}$ be two integers with $m \leq n$. When $\text{char} K = p$, let $\widetilde{R}_{A, \pi_K}^{[m,n]}$ denote the affine $A$-group scheme which associates to any $A$-algebra $A'$ the group
\[ \{\sum_{i=m}^{n} a_i \pi_i^i ; a_i \in A'\}. \]
There exists a canonical isomorphism of $k$-groups $\widetilde{R}_{A, \pi_K}^{[m,n]} \otimes_A k \cong K^{[m,n]}$.

When $\text{char} K = 0$, let $K_0 = \text{Frac} W(k)$ and $e = [K : K_0]$. Let $\widetilde{R}_{A, \pi_K}^{[m,n]}$ denote the affine $A$-group scheme which associates to any $A$-algebra $A'$ the group
\[ \bigoplus_{i=0}^{e-1} W_{1+\lfloor \frac{n-m-i}{e} \rfloor}(A'). \]
Then the multiplication by $\pi_K^{-m}$ induces a canonical isomorphism of $k$-groups $\widetilde{R}_{A, \pi_K}^{[m,n]} \otimes_A k \cong K^{[m,n]}$. 
Definition 4.25. Let $L$ be a finite separable totally ramified extension of $K$ of degree $d$. Let $A$ be a subring of $k$, $\pi_K$ an $A$-admissible prime element. A prime element $\pi_L$ of $L$ is called $(A, \pi_K)$-admissible over $K$ if the minimal monic polynomial $f(T) \in \mathcal{O}_K[T]$ of $\pi_L$ satisfies the following two conditions:

- $f(T) \in T^d + \pi_K \bar{R}_{A, \pi_K}[T]$ and $f(0) \in \pi_K(\bar{R}_{A, \pi_K})^\times$.
- Set $f(\pi_L + T) = T^d + \sum_{i=1}^{d-1} a_i T^i$. For each $i$, let $b_i = N_{L/K}(a_i)$ and $v_i = v_K(b_i)$. Then for any $i$ such that $(i, v_K(a_i))$ is a vertex of the Newton polygon of $f(\pi_L + T)$, $b_i \in \pi^v_K(\bar{R}_{A, \pi_K})^\times$.

Definition 4.26. Let $L/K$ be a finite Galois extension. Let $L_0$ be the maximal unramified subextension of $L/K$. A pre $A$-structure of $L/K$ consists of the following data $(\pi_K, B, \pi_L)$:

- $\pi_K$ is an $A$-admissible prime element in $K$.
- $B$ is a finite etale $A$-subalgebra of $k_L$ such that $B \otimes_A k \cong k_L$.
- $\pi_L$ is a prime element of $L$ which is $(B, \pi_K)$-admissible over $L_0$ such that all $\text{Gal}(L/K)$-conjugates of $\pi_L$ belong to $\pi_L(\bar{R}_{B, \pi_L})^\times$.

Definition 4.27. An $\mathbb{F}_p$-algebra $A$ is good perfect if $A$ is isomorphic to the perfection of a smooth $\mathbb{F}_p$-algebra.

Lemma 4.28. For any finite Galois extension $L/K$ as above, there exists a good perfect $\mathbb{F}_p$-subalgebra $A$ of $k$ and a pre $A$-structure $(\pi_K, B, \pi_L)$ of $L/K$.

Proof. By Lemma 4.21, there exists a subring $A_1$ of $k$ which is finitely generated over $\mathbb{F}_p$ and an $A_1$-admissible prime element $\pi_K$ of $K$. Since $k_L$ is a finite separable extension of $k$, there exists a monic polynomial $g(T) \in k[T]$ such that $k_L \cong k[T]/(g(T))$ as a $k$-algebra. Let $A_2$ be the subring of $k$ obtained by adjoining all coefficients of $g(T)$ and by inverting the discriminant of $g(T)$. Then there exists a finite etale $A_2$-subalgebra $B_2$ of $k_L$ such that $B_2 \otimes_{A_2} k \cong k_L$.

By Lemma 4.22, there exist a finitely generated $B_2$-subalgebra $B_3$ of $k_L$ and a prime element $\pi_L$ of $L$ such that the minimal polynomial $f(T) \in L[T]$ of $\pi_L$ over $L_0$ has coefficients in $\bar{R}_{B_3, \pi_K}$. There exists a finitely generated $B_3$-subalgebra $B_4$ of $k_L$ such that $\pi_L$ is $(B_4, \pi_K)$-admissible. By Lemma 4.23 (2) and Lemma 4.24, there exists a finitely generated $B_4$-subalgebra $B_5$ of $k$ such that $(\pi_L, B_5^{\text{perf}}, \pi_K)$ satisfies the third condition of Definition 4.26.

Take a finite set of generators $y_1, \ldots, y_n \in B_5$ of the $B_2$-algebra $B_5$. Let $A_5$ be the $A_2$-subalgebra of $k$ obtained by adjoining all coefficients of the minimal polynomials of all $y_i$ over $k$. There exists a non-empty affine open subscheme $\text{Spec}(A_6)$ of $\text{Spec}(A_5)$ which is smooth over $\mathbb{F}_p$. Let $B_6$ be the subring of $k_L$ generated by $A$ and $B_2$. Since $B_6$ is etale over $\mathbb{F}_p$, $B$ is regular.
In particular $B_6$ is normal. Hence $B_6$ contains all $y_i$ and $\pi_L$ is $(B_6^{\text{perf}}, \pi_K)$-admissible. We put $A = A_6^{\text{perf}}$ and $B = B_6^{\text{perf}}$. Then $(\pi_K, B, \pi_L)$ is a pre $A$-structure of $L/K$. \hfill \Box

**Lemma 4.29.** Let $(\pi_K, B, \pi_L)$ be a pre $A$-structure of $L/K$. Then for any ring homomorphism from $A$ to a perfect field $k'$ of characteristic $p$, $K_{k'} := (\bar{R}_A, \pi_K \widehat{\otimes}_R R_{k'})[\frac{1}{\pi_K}]$ is a $p$-CDVF with residue field $k'$. Decompose $B \otimes_A k'$ into a direct product $\prod_i k_i'$ of a finite number of finite separable extensions of $k'$. Then $L_{k'} = (\bar{R}_B, \pi_K \widehat{\otimes}_R R_{k'})[\frac{1}{\pi_K}]$ is a direct product $\prod_i L_{k_i'}$, where $L_{k_i'}$ is a finite separable extension of $K_{k_i'}$ with residue field $k_i$. The scheme $\text{Spec}(L_{k_i'})$ is an etale $\text{Gal}(L/K)$-torsor on $\text{Spec}(K_{k_i'})$. Furthermore, for each $i$ and for each $n \geq 0$, the lower numbering ramification subgroup $\text{Gal}(L_{k_i'}(K_{k_i'}))_n$ is canonically identified with $\text{Gal}(L/K)_n$.

**Proof.** All assertions are clear except the last one.

Let $L_0$ be the maximal unramified subextension of $L/K$ and $f(T) \in L_0[T]$ the minimal polynomial of $\pi_L$ over $L_0$. Then last assertion holds because the Herbrand function of $L/L_0$ is completely determined by the Newton polygon of $f(T + \pi_L) \in L[T]$. \hfill \Box

Let $L/K$ be a finite totally ramified abelian extension of $p$-CDVF. Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer such that the Herbrand function $\psi_{L/K}(v)$ is linear for $v > n$. Let $N_{L/K} : U_{L, \psi_{L/K}(n)+1} \rightarrow U_{K,n+1}$ be the morphism of affine $k$-group schemes induced by norms, $B_n$ the neutral component of the kernel of $N_{L/K}$, and $V_{L/K,n}$ the quotient group $U_{L, \psi_{L/K}(n)+1}/B_n$. Then the kernel of the canonical morphism $\beta_{L/K,n} : V_{L,K,n} \rightarrow U_{K,n+1}$ is canonically isomorphic to the constant group scheme $\text{Gal}(L/K)$.

Let $(\pi_K, A, \pi_L)$ be a pre $A$-structure of $L/K$. Let $U_{K,A}$ (resp. $U_{L,A}$) be the $A$-group scheme $(\bar{R}_A, \pi_K)^\times$ (resp. $(\bar{R}_A, \pi_L)^\times$). We define $U_{K,n+1,A}$ and $U_{L,\psi_{L/K}(n)+1,A}$ for $n \in \mathbb{Z}_{\geq 0}$ in a similar way. There exists norm maps $U_{L,A} \rightarrow U_{K,A}$ and $U_{L,\psi_{L/K}(n)+1,A} \rightarrow U_{K,n+1,A}$ which are homomorphisms of affine $A$-group schemes.

**Definition 4.30.** Let notation be as above. A pre $A$-structure $(\pi_K, A, \pi_L)$ of is called *good* if there exists an affine $A$-group scheme $V_{L/K,n,A}$ and homomorphisms

\[ \gamma_{L/K,n,A} : U_{L,\psi_{L/K}(n)+1,A} \rightarrow V_{L/K,n,A}, \quad \beta_{L/K,n,A} : V_{L/K,n,A} \rightarrow U_{K,n+1,A} \]

of $A$-group schemes which satisfy the following four conditions:

- The composition $\beta_{L/K,n,A} \circ \gamma_{L/K,n,A}$ is equal to the norm map.
- The morphism $\beta_{L/K,n,A}$ is finite etale.
• The homomorphisms
\[ U_{L,\psi_{L/K}(n)+1,A} \otimes_A k \xrightarrow{\gamma_{L/K,n,A}} V_{L/K,n,A} \otimes_A k \xrightarrow{\beta_{L/K,n,A}} U_{K,n+1,A} \otimes_A k \]
are canonically identified with the homomorphisms
\[ U_{L,\psi_{L/K}(n)+1} \to V_{L/K,n} \xrightarrow{\beta_{L/K,n}} U_{K,n+1}. \]
• For any ring homomorphism \( A \to B \), the homomorphism
\[ \Gamma(V_{L/K,n,A}, O_{V_{L/K,n,A}}) \otimes_A B \to \Gamma(U_{L,\psi_{L/K}(n),A}, O_{U_{L,\psi_{L/K}(n),A}}) \otimes_A B \]
induced by \( \gamma_{L/K,n,A} \) is injective.

**Lemma 4.31.** There exists a good perfect subring \( A \) of \( k \) and a good pre \( A \)-structure \( (\pi_K, A, \pi_L) \) of \( L/K \).

**Proof.** By Lemma 4.28, there exists a good perfect subring \( A_1 \) of \( k \) and a pre \( A_1 \)-structure \( (\pi_K, A_1, \pi_L) \) of \( L/K \). Let us denote the coordinate rings of \( U_{K,n+1,A_1}, U_{L,\psi_{L/K}(n)+1,A_1} \) and \( V_{L/K,n} \) by \( C_{K,A_1}, C_{L,A_1} \) and \( C_V \), respectively. There exist canonical injective ring homomorphisms
\[ C_{K,A_1} \otimes_{A_1} k \to C_V \to C_{L,A_1} \otimes_{A_1} k. \]
We regard \( C_V \) as a subring of \( C_{L,A_1} \otimes_{A_1} k \) by the latter homomorphism. The rings \( C_{K,A_1} \) and \( C_{L,A_1} \) are each isomorphic to polynomial rings over \( A_1 \) with finitely many variables. So we write \( C_{K,A_1} = A_1[x_1, x_2, \cdots, x_m] \), and \( C_{L,A_1} = A_1[y_1, y_2, \cdots, y_m] \). The ring \( C_V \) is finite free as a \( C_{K,A_1} \otimes_{A_1} k \)-module. Take a \( C_{K,A_1} \otimes_{A_1} k \)-basis \( b_1, \ldots, b_{n'} \) of \( C_V \). There exist \( n' \) monomials \( s_1, \ldots, s_{n'} \) of \( y_1, \ldots, y_{m'} \) such that the matrix \((c_{ij})\) of coefficients of \( s_i \) in \( b_j \in A_1[y_1, y_2, \cdots, y_{m'}] \) is invertible. Let \( I \) be the kernel of the homomorphism \( \varphi : C_{K,A_1} \otimes_{A_1} k[z_1, \cdots, z_{n'}] \to C_V \) which sends \( z_i \) to \( b_i \). Take a generator \( f_1, \ldots, f_{n'} \) of \( I \). Then the image of the determinant of the matrix \((\partial f_i / \partial z_j)\) by \( \varphi \) belongs to \( B^\times \).

There exists a finitely generated \( A_1 \)-algebra \( A \) which satisfies the following seven properties:

• For all \( i, b_i \in C_{L,A_1} \otimes_{A_1} A \).
• The determinant of the matrix \((c_{ij})\) belongs to \( A^\times \).
• For all \( i_1, i_2, b_{i_1} b_{i_2} \in \sum_{j} C_{K,A_1} \otimes_{A_1} A \cdot b_j \).
• For all \( i, f_i \in C_{K,A_1} \otimes_{A_1} A[z_1, \cdots, z_{n'}] \).
• The image of the determinant of the matrix \((\partial f_i / \partial z_j)\) by \( \varphi \) belongs to \((\bigoplus_i (C_{K,A_1} \otimes_{A_1} A \cdot b_i))^\times \).
• Let \( \Delta : C_V \to C_V \otimes_k C_V \cong \bigoplus_{i_1,i_2}(C_{K,A_1} \otimes_{A_1} C_{K,A_1}) \otimes_{A_1} k \cdot b_{i_1} \otimes b_{i_2} \).
be the morphism induced by the group law of $V_{L,K,n}$. Then $\Delta(b_j) \in (C_{K,A_1} \otimes A_1, \pi_M) \otimes A_1 A \cdot b_i \otimes b_j$.

- Let $S : C_V \to C_V$ be the morphism induced by the inverse morphism of $V_{L,K,n}$. Then $S(b_i) \in \bigoplus_j C_{K,A_1} \otimes A_1 A \cdot b_j$.

Put

$$C_{V,A} = \bigoplus_i C_{K,A_1} \otimes A_1 A \cdot b_i.$$ 

This is a finite etale $C_{K,A_1} \otimes A_1 A$-algebra. There exists a canonical structure of $A$-group scheme on $V_{L/K,n} = \text{Spec} (C_{V,A})$. It is easily checked that this $A$ and $V_{L/K,n,A}$ satisfy the desired properties. $\square$

**Definition 4.32.** Let $L/K$ be a finite separable extension, An $A$-structure of $L/K$ is a pair $(B, (\pi_M)_M)$ which satisfies the following conditions:

- $B$ is a finite etale $A$-subalgebra of $k_L$ such that $B \otimes_A k \cong k_L$.
- $(\pi_M)_M$ is a system of a prime element $\pi_M$ of $M$, where $\pi_M$ runs over all subextensions of $L/K$.
- For any two subextensions $M_1$, $M_2$ of $L/K$ with $M_2 \supset M_1$, let $B_{k_{M_i}} (i = 1, 2)$ be the finite etale $A$-subalgebra of $B$ corresponding to the residue field $k_{M_i}$ of $M_i$. Then $(\pi_{M_1}, B_{k_{M_2}}, \pi_{M_2})$ is a pre $B_{k_{M_1}}$-structure of $M_2/M_1$.
- For any two subextensions $M_1$, $M_2$ of $L/K$ with $M_2 \supset M_1$ such that $M_2/M_1$ is a totally ramified abelian extension, the pre $B_{k_{M_1}}$-structure $(\pi_{M_1}, B_{k_{M_1}}, \pi_{M_2})$ of $M_2/M_1$ is a good pre $B_{k_{M_1}}$-structure of $M_2/M_1$.

By Lemma 4.28 and Lemma 4.31, we have:

**Lemma 4.33.** For any finite separable extension $L/K$ as above, there exists a good perfect $\mathbb{F}_p$-subalgebra $A$ of $k$ and an $A$-structure $(B, (\pi_M)_M)$ of $L/K$. $\square$

**Definition 4.34.** Let $L/K$ be a finite separable extension of $p$-CDVF's, $R_0$ a pro-finite $p'$-coefficient ring, and $\tilde{\psi}$ a non-trivial additive character $R_0$-sheaf on $K$. Let $N > \text{ord} \tilde{\psi}$ be an integer. A pre $A$-structure $(\pi_K, B, \pi_L)$ of $L/K$ is called $(N, \tilde{\psi})$-admissible if the following two conditions are satisfied:

- The sheaf $\tilde{\psi}|_{K[\sim N, -\text{ord} \tilde{\psi} - 1]}$ is the pull back of an invertible character sheaf on the $A$-group scheme $\tilde{R}^{\sim N, -\text{ord} \tilde{\psi} - 1}_{A, \pi_K}$.
- There exists a unit $b \in A^\times$ such that when $K[\sim \text{ord} \tilde{\psi} - 1, -\text{ord} \tilde{\psi} - 1]$ is identified with $G_{a,k}$ by using $\pi_K$, the sheaf $\tilde{\psi}|_{K[\sim \text{ord} \tilde{\psi} - 1, -\text{ord} \tilde{\psi} - 1]}$ is the pull-back of a non-trivial Artin-Schreier sheaf on $G_{a, \mathbb{F}_p}$ by the multiplication-by-$b$ map $G_{a,k} \to G_{a,k} \to G_{a, \mathbb{F}_p}$. 
Definition 4.35. Let $L/K$, $R_0$ and $\tilde{\psi}$ be as above. Let $N$ be a sufficiently large integer. An $A$-structure $(B, (\pi_M)_M)$ of $L/K$ is called $(N, \tilde{\psi})$-admissible if for any two intermediate extensions $M_1, M_2$ of $L/K$ with $M_2 \supset M_1$, the pre $B_{k,M_2}$-structure $(\pi_{M_1}, B_{M_2}, \pi_{M_2})$ of $M_2/M_1$ is $(N, \tilde{\psi} \circ \text{Tr})$-admissible.

Proposition 4.36. Let $L$ be a finite separable extension of $K$, $R_0$ a finite $p'$-coefficient ring, $\tilde{\psi}$ a non-trivial additive character $R_0$-sheaf on $K$, and $N \in \mathbb{Z}$ an integer larger than $\text{ord} \tilde{\psi} + \text{sw}(\chi) + 1$. Let $A$ be a smooth $\mathbb{F}_p$-algebra, and let $(B, (\pi_M)_M)$ be a $(N, \tilde{\psi})$-admissible $A$-structure of $L/K$. Let $\psi_A$ be an invertible character sheaf on the $A$-group scheme $R^\text{-admissible}_{A,\pi_K}$ which comes from an object in $\text{Rep}(W_{K,R_0})$ which comes from an object in $\text{Rep}(\text{Gal}(L/K), R_0)$. Let $N$ be an integer larger than $\text{ord} \tilde{\psi} + \text{sw}(\chi) + 1$. Let $A$ be a smooth $\mathbb{F}_p$-algebra, and let $(B, (\pi_M)_M)$ be a $(N, \tilde{\psi})$-admissible $A$-structure of $L/K$. Let $\psi_A$ be an invertible character sheaf on the $A$-group scheme $R^\text{-admissible}_{A,\pi_K}$ whose pull-back on $K[-N, -\text{ord} \tilde{\psi} - 1]$ is isomorphic to $\tilde{\psi}|_{K[-N, -\text{ord} \tilde{\psi} - 1]}$.

Then there exists a smooth invertible $R_0$-sheaf $\bar{\varepsilon}_{0,R_0,A}(\chi, \tilde{\psi})$ on $\text{Spec } (A)$ which satisfies the following property:

For any ring homomorphism from $A$ to a perfect field $k'$ of characteristic $p$, let $K_{k'}$ be as in Lemma 4.29. Let $\chi_{k'}$ be the rank one object in $\text{Rep}(W_{K_{k'}, R_0})$ which comes from a character of $\text{Gal}(L/K)$ and $\chi$. Let $\psi_{k'}$ be the pull-back of $\psi_A$ by the canonical morphism

$$R^\text{-admissible}_{A,\pi_K} K[-N, -\text{ord} \tilde{\psi} - 1] \cong R^\text{-admissible}_{A,\pi_K} \otimes_A k' \to R^\text{-admissible}_{A,\pi_K} K[-N, -\text{ord} \tilde{\psi} - 1].$$

Then the pull-back of $\bar{\varepsilon}_{0,R_0,A}(\chi, \tilde{\psi})$ on $\text{Spec } (k')$ is the smooth invertible $R_0$-sheaf on $\text{Spec } (k')$ corresponding to $\bar{\varepsilon}_{0,R}(\chi_{k'}, \psi_{k'})$. 
Proof. Decompose \( \chi \) into the tensor product \( \chi = \chi_1 \otimes \chi_2 \) of two rank one objects \( \chi_1 \), \( \chi_2 \) in \( \text{Rep}(W_K, R_0) \), both of which come from objects in \( \text{Rep}(\text{Gal}(L/K), R_0) \), such that \( \chi_1 \) is an unramified and that the extension \( K'/K \) corresponding to \( \ker \chi_2 \) is a totally ramified cyclic abelian extension. The etale \( A \)-algebra \( B \) and \( \chi_1 \) induces a smooth invertible sheaf \( \chi_{1,A} \) on \( \text{Spec} \( A \) \).

Let \( s = \text{sw}(\chi_2) \). For \( m \in \mathbb{Z} \), let \( K_{s,A}^{v=m} \) (resp. \( (K'_{s,A})^{v=m} \)) denote the \( A \)-scheme \( U_{K,s+1,A} \) (resp. \( U_{K',\psi_{K'/K}(s)+1,A} \)). The scheme \( \prod_{m \in \mathbb{Z}} K_{s,A}^{v=m} = U_{K,s+1,A} \times \mathbb{Z} \) (resp. \( \prod_{m \in \mathbb{Z}} (K'_{s,A})^{v=m} = U_{K',\psi_{K'/K}(s)+1,A} \times \mathbb{Z} \)) has a canonical structure of an \( A \)-group scheme. The multiplication by \( \pi_K^m \) (resp. \( \pi_{K'}^m \)) induces a canonical isomorphism \( K_{s,A}^{v=m} \otimes_A k \cong K^{v=m}/U_{K,\psi_{K'/K}(s)+1} \) (resp. \( (K'_{s,A})^{v=m} \otimes_A k \cong (K')^{v=m}/U_{K',\psi_{K'/K}(s)+1} \)).

The norm map \( N_{K'/K} : K^\ell \rightarrow K \) induces a homomorphism \( N_{K'/K,A} : \prod_{m \in \mathbb{Z}} (K'_{s,A})^{v=m} \rightarrow \prod_{m \in \mathbb{Z}} K_{s,A}^{v=m} \) of \( A \)-group schemes. By the definition of good pre \( A \)-structure, there exists an affine \( A \)-group scheme \( \prod_{m} V_{K'/K,s,A}^{v=m} \) and homomorphisms

\[
\gamma_{K'/K,s,A} : \prod_{m \in \mathbb{Z}} (K'_{s,A})^{v=m} \rightarrow \prod_{m \in \mathbb{Z}} V_{K'/K,s,A}^{v=m}
\]

and

\[
\beta_{K'/K,s,A} : \prod_{m \in \mathbb{Z}} V_{K'/K,s,A}^{v=m} \rightarrow \prod_{m \in \mathbb{Z}} K_{s,A}^{v=m}
\]

of \( A \)-group schemes such that \( \beta_{K'/K,s,A} \circ \gamma_{K'/K,s,A} = N_{K'/K,A} \), that \( \beta_{K'/K,s,A} \) is finite etale and the kernel of \( \beta_{K'/K,s,A} \) is isomorphic to the constant group scheme \( \text{Gal}(K'/K) \), and that for any \( A \)-algebra \( B \), the homomorphism

\[
\Gamma(V_{K'/K,s,A}^{v=m}, \mathcal{O}_{V_{K'/K,s,A}^{v=m}}) \otimes_A B \rightarrow \Gamma((K'_{s,A})^{v=m}, \mathcal{O}_{(K'_{s,A})^{v=m}}) \otimes_A B
\]

is injective. Let \( \mathcal{L}_{\chi_2,A} \) be the invertible character sheaf on \( \prod_{m \in \mathbb{Z}} K_{s,A}^{v=m} \) defined by \( \beta_{K'/K,s,A} \) and \( \chi_2 \).

Let \( m_0 = -s - \text{ord} \psi - 1 \) and let \( \mathcal{L}_{\chi,A} \) be the invertible sheaf \( \mathcal{L}_{\chi_2,A}|K_{s,A}^{v=m_0} \otimes \pi^* \chi_{1,A}^{\otimes m_0} \) on \( K_{s,A}^{v=m_0} \), where \( \pi : K_{s,A}^{v=m_0} \rightarrow \text{Spec} \( A \) \) is the structure morphism.

Let \( \bar{\varepsilon}_{0,R_0,A}(\chi, \bar{\psi}) \) be the invertible sheaf

\[
\bar{\varepsilon}_{0,R_0,A}(\chi, \bar{\psi}) = \text{det} R_0(\bar{\psi}^{-1}[s + 1](\text{ord} \bar{\psi})).
\]

It is easy to prove that this \( \bar{\varepsilon}_{0,R_0,A}(\chi, \bar{\psi}) \) has desired properties. \( \Box \)

Corollary 4.39. Let \( L \), \( R_0 \), and \( \bar{\psi} \) be as the above proposition. Let \( N \) be a sufficiently large integer. Let \( A \) be a good perfect \( \mathbb{F}_p \)-algebra, and let
(B, (π_M)_M) be a (N, ψ)-admissible A-structure of L/K. Let ̂ψ_A be an invertible character sheaf on the A-group scheme ̂R_{A, π_K}[−N, −ord ψ − 1] whose pull-back on K[−N, −ord ψ − 1] is isomorphic to ̂ψ|_{K[−N, −ord ψ − 1]}.

Then there exists a smooth invertible R₀-sheaf ̂λ_{R₀, A}(L/K, ̂ψ) on Spec (A) which satisfies the following property:

For any ring homomorphism from A to a perfect field k' of characteristic p, let K_{k'} be as in Lemma 4.29. Let ̂ψ_{k'} be the pull-back of ̂ψ_A by the canonical morphism

K_{k'}[−N, −ord ψ − 1] ⊅ R_{A, π_K} ⊗ k' → R_{A, π_K}.

Then the pull-back of ̂λ_{R₀, A}(L/K, ̂ψ) on Spec (k') is the smooth invertible R₀-sheaf on Spec (k') corresponding to ̂λ_R(L_{k'}/K_{k'}, ̂ψ_{k'}). □

4.8. Proof of Lemma 4.18. Suppose that

ρ = ∑ i n_i \text{Ind}_{H_i}^{G_i} \chi_i = ∑ j n'_j \text{Ind}_{H'_j}^{G'_j} \chi'_j.

Take a sufficiently large integer N ∈ Z. Take a good perfect \mathbb{F}_p-subalgebra A of k and a (ψ, N)-admissible A-structure (B, (π_M)_M) of L/K. Take an invertible character sheaf ̂ψ_A on the A-group scheme ̂R_{A, π_K}[−N, −ord ψ − 1] whose pull-back on K'[−N, −ord ψ − 1] is isomorphic to ̂ψ|_{K'[−N, −ord ψ − 1]}.

We define, in a canonical way, smooth invertible R-sheaves ̂ε_{0, R, A}(χ_i, ̂ψ ∘ Tr_{K_i/K}), ̂ε_{0, R, A}(χ'_j, ̂ψ ∘ Tr_{K'_j/K}), ̂λ_{R, A}(K'_i/K, ̂ψ), and ̂λ_{R, A}(K'_j/K, ̂ψ) on Spec (A) whose pull-backs on Spec (k) correspond to ̂ε_{0, R}(χ_i, ̂ψ ∘ Tr_{K_i/K}), ̂ε_{0, R}(χ'_j, ̂ψ ∘ Tr_{K'_j/K}), ̂λ_R(K'_i/K, ̂ψ), and ̂λ_R(K'_j/K, ̂ψ) respectively.

Let ̂ε_{0, R, A}(V, ̂ψ) and ̂ε'_{0, R, A}(V, ̂ψ) be two invertible sheaves on Spec (A) defined by

̂ε_{0, R, A}(V, ̂ψ) := \bigotimes_i (ε_{0, R, A}(χ_i, ̂ψ ∘ Tr_{K_i/K}) ⊗ ̂λ_R(K_i/K, ̂ψ))^{n_i}.

and

̂ε'_{0, R, A}(V, ̂ψ) := \bigotimes_j (ε_{0, R, A}(χ'_j, ̂ψ ∘ Tr_{K'_j/K}) ⊗ ̂λ_R(K'_j/K, ̂ψ))^{n'_j}.

Let x ∈ Spec (A) be a closed point. We denote by κ(x) (resp. i_x) the residue field at x (resp. the canonical inclusion) i_x : x ← Spec (A). Let K_x denote the field K_{κ(x)} introduced in Lemma 4.29 for the ring homomorphism A → κ(x). The canonical homomorphism W_{K_x} → Gal(L/K) and V defines an object in Rep(W_{K_x}, R) which we denote by V_x. Then we
have an isomorphism \( i^*_x \varepsilon_{0,R,A}(V, \tilde{\psi}) \cong i^*_x \varepsilon'_{0,R,A}(V, \tilde{\psi}) \), since the main result of [10] implies that both sides, as characters of \( W_{n(x)} \), send the geometric Frobenius at \( x \) to \((-1)^{\text{rank} \ V \otimes_{\text{sw}(V)} \varepsilon_{0,R}(V_x, \psi_x)} \), where \( \psi_x \) denotes the additive character of \( K_x \) corresponding to the specialization of \( \tilde{\psi}_A \) at \( x \). Let \( A_0 \subset A \) be a smooth \( \mathbb{F}_p \)-subalgebra of \( A \) whose perfection equals \( A \). By the standard argument using Chebotarev’s theorem which states that the geometric Frobenii at the closed points are dense in \( \pi_1(\text{Spec}(A_0)) \) ([7, Theorem 7]), we conclude that \( \varepsilon_{0,R,A}(V, \tilde{\psi}) \cong \varepsilon'_{0,R,A}(V, \tilde{\psi}) \). This completes the proof. \( \square \)

4.9. Local \( \varepsilon_0 \)-characters of representations of \( G_K \) (field coefficients). Let \( R \) be an algebraically closed field of positive characteristic \( \neq p \). In this subsection we shall define, for an object \((\rho, V)\) in \( \text{Rep}(W_K, R) \), a rank one object \( \varepsilon_{0,R}(V, \tilde{\psi}) \) in \( \text{Rep}(W_K, R) \), which is called the local \( \varepsilon_0 \)-character of \( V \).

First assume that \( V \) is irreducible. Then \( V \) is of the form \( V = V' \otimes_R \chi \), where \( V' \) is an object in \( \text{Rep}(W_K, R) \) such that the image of \( W_K \) in \( GL_R(V') \) is finite, and \( \chi \) is an unramified rank one object in \( \text{Rep}(W_K, R) \). Define \( \varepsilon_{0,R}(V, \tilde{\psi}) \) to be \( \varepsilon_{0,R}(V', \tilde{\psi}) \otimes_R \chi^{\otimes_{\text{sw}(V)} \text{rank} \ V \otimes_{\text{sw}(V)} \text{ord} \ \tilde{\psi} + 1} \). By Lemma 4.19, \( \varepsilon_{0,R}(V, \tilde{\psi}) \) is independent of the choice of \( V' \) and \( \chi \).

For a general \( V \), let \( V_1, \ldots, V_n \) denote the Jordan-Hölder constituents of \( V \) counted with multiplicity. Define \( \varepsilon_{0,R}(V, \tilde{\psi}) \) to be \( \bigotimes_i \varepsilon_{0,R}(V_i, \tilde{\psi}) \).

4.10. Local \( \varepsilon_0 \)-characters for torsion coefficients (totally wild case). Let \((R, m_R)\) be a complete strict \( p' \)-coefficient ring with a positive residue characteristic and \( \tilde{\psi} \) a non-trivial additive character \( R \)-sheaf on \( K \).

Assume that \( p \neq 2 \). For \( x \in K^\times \) with \( v_K(x) + \text{ord} \ \tilde{\psi} = 2b + 1 \) is odd, define a quadratic Gauss sum sheaf \( \tau_{K,\tilde{\psi}}(x) \) by

\[
\tau_{K,\tilde{\psi}}(x) = \varepsilon_{0,R}(\chi_{-\frac{x}{2}}, \tilde{\psi}) \otimes_R \text{R}(\text{ord} \ \tilde{\psi}),
\]

where \( \chi_{-\frac{x}{2}} : W_K \to R^\times \) is the composition of the quadratic character \( W_K \to \{ \pm 1 \} \) corresponding to the quadratic extension \( K(\sqrt{-\frac{x}{2}}) \) of \( K \) and the canonical map \( \{ \pm 1 \} \to R^\times \). The sheaf \( \tau_{K,\tilde{\psi}}(x) \) does not depend on the choice of \( y \) and depends only on the class of \( x \in \{ x \in K^\times | v_K(x) + \text{ord} \ \tilde{\psi} \equiv 1 \mod 2 \} \) in \((K^\times / 1 + m_K) \otimes \mathbb{Z}/2\mathbb{Z}\). Thus we can define \( \tau_{K,\tilde{\psi}}(x) \) for \( x \in \{ x \in (K^\times / 1 + m_K) \otimes \mathbb{Z}[\frac{1}{p}] | v_K(x) + \text{ord} \ \tilde{\psi} \in 1 + 2\mathbb{Z}[\frac{1}{p}] \} \).

Remark 4.40. Suppose that \( k \) is finite. Let \( \psi : K \to R^\times \) be the additive character of \( K \) corresponding to \( \tilde{\psi} \). Then we have

\[
\tau_{K,\tilde{\psi}}(x)(\text{Fr}_k) = -\tau_{K,\tilde{\psi}}(x),
\]

where \( \tau_{K,\tilde{\psi}}(x) \) is the quadratic Gauss sum defined in [10, § 7.3].
Definition 4.41. Let \( v \in \mathbb{Q}_{>0} \), \( G = W_K \), and let \( \chi \in \text{Hom}(G^v/G^{v+}, R^\times) \). Let \( K_\chi \) be the extension of \( K \) corresponding to the stabilizing subgroup of \( \chi \), and \( k_\chi \) the residue field of \( K_\chi \). Let \( r \in \mathbb{Z} \) be an integer such that \( rv \in \mathbb{Z} \). Define the Gauss sum part sheaf \( g_R(\chi, \overline{\psi})^{\otimes_r} \) of \( \overline{\varepsilon}_{0,R}\text{-constant} \) to be the object in \( \text{Rep}(W_k, R) \) defined by

\[
g_R(\chi, \overline{\psi})^{\otimes_r} = (\overline{\lambda_R(K_\chi/K, \overline{\psi}))^{\otimes_r}} \otimes_R R(-[k_\chi : k] \text{ord } (\overline{\psi} \circ \text{Tr}_{K_\chi/K}))\]

\[
\otimes_R \left\{ \begin{array}{ll}
R(-[k_\chi : k]r \cdot \frac{1+w}{2}) & \text{if } p = 2 \text{ or } p \neq 2 \text{ and } \text{ord}_2(v) \leq 0, \\
R(-[k_\chi : k]r \cdot \frac{w}{2}) \otimes_R \overline{\tau}_{K_\chi, \overline{\psi} \circ \text{Tr}_{K_\chi/K}(\sigma_{\overline{\psi}}(\chi)))^{\otimes_r} & \text{if } p \neq 2 \text{ and } \text{ord}_2(v) > 0,
\end{array} \right.
\]

where \( w = e_{K_\chi/K}v \).

Let \((\rho, V)\) be an object in \( \text{Rep}(W_K, R) \) which is pure of slope \( v \) and of refined slope \( \Sigma \). As in [10, § 7], for \( w \in \mathbb{Q} \), let \( N_K^w \) denote the set

\[
N_K^w := \{ x \in \overline{K} | v_K(x) \geq w \}/\{ x \in \overline{K} | v_K(x) > w \},
\]

endowed with the canonical \( W_K \)-action. By [5, p. 3, Thm. 1] there is a canonical isomorphism from \( \text{Hom}(G^v/G^{v+}, R) \) to the set of all isomorphism classes of character sheaves on \( N_K^v \) considered as a \( \overline{K} \)-group scheme. This isomorphism and the additive character sheaf \( \overline{\psi} \) induce a canonical isomorphism \( \sigma_{\overline{\psi}} : \text{Hom}(G^v/G^{v+}, \mathbb{Z}/p\mathbb{Z}) \to N_K^{-v-\text{ord } \overline{\psi} - 1} \) in a similar way as in [10, § 7].

Take a \( \chi \in \Sigma \), and let \( V' \) be the \( \chi \)-part of \( V \). Then \( V' \) is an object in \( \text{Rep}(W_{K_\chi}, R) \).

Let \( K_\chi \) be the extension of \( K \) corresponding to the stabilizing subgroup of \( \chi \), and \( k_\chi \) the residue field of \( K_\chi \). We consider the element \( \sigma_{\overline{\psi}}(\chi) \in N_K^{-v-\text{ord } \overline{\psi} - 1} \) as an element in \( (K_\chi^\times / 1 + m_{K_\chi}) \otimes_{\mathbb{Z}} \mathbb{Z} \frac{1}{p} \).

Definition 4.42. Let \( R \) be a \( p' \)-coefficient ring, and \( V \) an object in \( \text{Rep}(G, R) \) which is pure of refined slope \( \Sigma \). We define the refined \( \overline{\psi} \)-Swan conductor \( \text{rsw}_{\overline{\psi}}(V) \in K^\times / 1 + m_K \) by

\[
\text{rsw}_{\overline{\psi}}(V) = N_{K_\chi/K}(\sigma_{\overline{\psi}}(\chi))^{-\text{rank}_V} \cdot \frac{1}{\text{rank}_{K_\chi/K}}.
\]

For an arbitrary object \( W \) in \( \text{Rep}(G, R) \), define \( \text{rsw}_{\overline{\psi}}(W) \in K^\times / 1 + m_K \) by

\[
\text{rsw}_{\overline{\psi}}(W) = \prod_{\Sigma'} \text{rsw}_{\overline{\psi}}(W^{\Sigma'}),
\]

where

\[
W = W^0 \oplus \bigoplus_{\Sigma'} W^{\Sigma'}
\]
is the refined slope decomposition of $W$.

**Lemma 4.43.** The $p$-power map $\Hom_{\text{cont}}(W_k, R^\times) \to \Hom_{\text{cont}}(W_k, R^\times)$ is surjective.

**Proof.** It suffices to prove that the $p$-power map $H^1(k, R^\times) \to H^1(k, R^\times)$ is surjective. Since $H^2(k, \mathbb{Z}/p\mathbb{Z}) = \{0\}$ by Artin-Schreier theory, the map $H^1(k, (R/m_R)^\times) \to H^1(k, (R/m_R)^\times)$ induced by the $p$-power map $(R/m_R)^\times \to (R/m_R)^\times$ is surjective. Since the $p$-power map $1 + m_R \to 1 + m_R$ is a homeomorphism, it suffices to prove that the natural map $H^1(k, R^\times) \to H^1(k, (R/m_R)^\times)$ is surjective. Let $\ell$ be the residue characteristic of $R$. Then the composition

$$H^1(k, (R/m_R)^\times) \cong H^1(k, \mathbb{F}_\ell^\times) \to H^1(k, W(\mathbb{F}_\ell)^\times) \to H^1(k, R^\times)$$

gives the right inverse of the last map. This complete the proof. \qed

Let $\overline{\Hom}(W_k, R^\times)$ be the quotient of $\Hom_{\text{cont}}(W_k, R^\times)$ by the subgroup of the characters of $p$-power orders. For an object $V$ in $\text{Rep}(G, R)$ which is pure of refined slope $\Sigma$, define the $\overline{\varepsilon}_0$-character of $V$ to be an element in $\overline{\Hom}(W_k, R^\times)$ defined by

$$\overline{\varepsilon}_{0,R}(V, \overline{\psi}) := ((\text{det} V'[\sigma_q(\chi)] \circ \text{Ver}_{W_k}^R)^{\otimes -1} \otimes g_R(\chi, \overline{\psi})^{\otimes \text{rank} V'}.$$  

Here $\text{Ver}_{W_k}^R$ is the transfer map.

**Lemma 4.44.** Suppose that $R$ is a field. Then $\overline{\varepsilon}_{0,R}(V, \overline{\psi})$ is equal to the class of $\varepsilon_{0,R}(V, \overline{\psi})$ in $\overline{\Hom}(W_k, R^\times)$.

**Proof.** We may assume that $(\rho, V)$ is irreducible. Twisting $V$ by an unramified character, we may assume that the image of $\rho$ is finite. Take a representative $\overline{\varepsilon}_{0,R}(V, \overline{\psi}) \in \Hom_{\text{cont}}(W_k, R^\times)$ of $\overline{\varepsilon}_{0,R}(V, \overline{\psi})$. Take a finite Galois extension $L$ of $K$ such that $\rho$ factors through the finite quotient $G = \text{Gal}(L/K)$ of $W_K$. By Brauer’s theorem of modular representations, we may assume that $V$ is of the form $V = \text{Ind}_{H/K}^G \chi$ for a subgroup $H \subset G$ and a character $\chi$ of $H$. Let $K'$ be the subextension of $L/K$ corresponding to $H$. There exist a sufficiently large integer $N \in \mathbb{Z}$ and a good perfect $\mathbb{F}_p$-subalgebra $A$ of $k$ and a $(N, \psi)$-admissible $A$-structure $(B, (\pi_M)_M)$ of $L/K$ such that we can define smooth invertible $R$-sheaves $\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})$ and $\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})$ on $\text{Spec}(A)$ whose pull-backs on $\text{Spec}(k)$ correspond to $\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})$ and $\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})$, respectively. Then for any closed point $x \in \text{Spec}(A)$, there exists an integer $n_x$ such that $i_x^*\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})^{\otimes p^{nx}} \cong i_x^*\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})^{\otimes p^{nx}}$, where $i_x : x \hookrightarrow \text{Spec}(A)$ is the canonical inclusion. By the construction of $\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})$ and $\overline{\varepsilon}_{0,R,A}(V, \overline{\psi})$, there exists a positive integer $n \in \mathbb{Z}_{>0}$ such that $n_x \leq n$ for every closed
\[
\varepsilon_0, R(V, \tilde{\psi}) = \tilde{\varepsilon}_0, R(V, \tilde{\psi}) \text{ in } \text{Hom}(W_k, R^\times)
\]
and
\[
\varepsilon_0, R(V, \tilde{\psi}) \mod m_R = \tilde{\varepsilon}_0(V \otimes_R R/m_R, \tilde{\psi}).
\]

**Proposition 4.46** (cf. [10, Prop. 8.3]). Let \( R \) be a strict \( p' \)-coefficient ring, \( R_0 \) a pro-finite subring of \( R \), and \( V \neq \{0\} \) a totally wild object in \( \text{Rep}(G, R_0) \). Then for every tamely ramified object \( W \) in \( \text{Rep}(G, R_0) \), we have
\[
\varepsilon_0, R(V \otimes_R W, \tilde{\psi}) = (\det W)_{[\text{rsw}_\psi(V)]} \otimes \varepsilon_0, R(V, \tilde{\psi})^{\text{rank } W}.
\]

**Proof.** We may assume that \( R_0 \) is finite and that \( V \) is pure of refined slope \( \Sigma \). Take \( \chi \in \Sigma \) and let \( K_\chi \) be the finite separable extension of \( K \) corresponding to \( \chi \).

Take a sufficiently large Galois extension \( L \) of \( K \) containing \( K_\chi \) such that \( V \) and \( W \) come from objects in \( \text{Rep}(\text{Gal}(L/K), R_0) \).

Take a sufficiently large integer \( N \in \mathbb{Z} \). Take a good perfect \( \mathbb{F}_p \)-subalgebra \( A \) of \( k \) and a \((N, \tilde{\psi})\)-admissible \( A \)-structure \((B, (\pi_M)_M)\) of \( L/K \). Then we can define, in a canonical way, smooth invertible \( R \)-sheaves of \( \varepsilon_0, R, A(V, \tilde{\psi}) \) and \( \varepsilon_0, R, A(V \otimes W, \tilde{\psi}) \) on \( \text{Spec}(A) \) whose pull-backs on \( \text{Spec}(k) \) are identified with \( \varepsilon_0, R(V, \tilde{\psi}) \) and \( \varepsilon_0, R(V \otimes W, \tilde{\psi}) \) respectively.

Let \( v \in \mathbb{Q}_{>0} \) be the slope of \( V \). Let \( K' \) and \( L' \) be the subextensions of \( L/K_\chi \) corresponding to \( \text{Gal}(L/K_\chi)^{F_{K_\chi}/K^{v}} \) and \( \text{Gal}(L/K_\chi)^{F_{K_\chi}/K^{v}}, \) respectively. Let \( v_{K'} = \psi_{K'/K}(v) \) and \( v_{L'} = \psi_{L'/K}(v) \). Consider the homomorphism
\[
\alpha : L'/K', v_{K'} : m_{L'}^{v_{L'}}/m_{L'}^{v_{L'}+1} \to m_{K'}^{v_{K'}}/m_{K'}^{v_{K'}+1}
\]
defined by \( N_{L'/K'}(1 + x) = 1 + \alpha : L'/K', v_{K'} \). It is easily checked that the homomorphism \( \alpha : L'/K', v_{K'} \) is induced from a morphism
\[
\tilde{\alpha} : \tilde{R}_{B_{K'} \tilde{\pi}_{K'}}^{[v_{L'}, v_{L'}]} \to \tilde{R}_{B_{K'} \tilde{\pi}_{K'}}^{[v_{K'}, v_{K'}]}
\]
of \( B_{K'} \)-group schemes. By localizing \( A \) if necessary, we may assume that \( \tilde{\alpha} \) is finite etale and \( \text{Ker } \tilde{\alpha} \) is constant. Put \( A_\chi = A_{K\chi} \). Using the character sheaf on \( \tilde{R}_{B_{K'} \tilde{\pi}_{K'}}^{[v_{K'}, v_{K'}]} \) and the sheaf \( \tilde{\psi}_{A_\chi} \), we can define the \( A_\chi \)-valued point of the group scheme \( \prod_{m \in \mathbb{Z}_{1/\tilde{p}}} \mathbb{G}_m, A_\chi \) which induces \( \sigma_\chi \). Using this, we can define the refined swan conductor \( \text{rs}_{\psi, A}(V) \) of \( V \) as a \( A \)-valued point of the group scheme \( \prod_{m \in \mathbb{Z}_{1/\tilde{p}}} \mathbb{G}_m, A_\chi \).
On the other hand, we have an invertible character sheaf $(\det W)_A$ on $\prod_{m \in \mathbb{Z}} \mathbb{G}_m.A$ corresponding to $\det W$. Thus we obtain an invertible sheaf $(\det W)_A,\varepsilon_{\psi},A(V)$ on $\text{Spec}(A)$ which induces $(\det W)_{\varepsilon_{\psi},A(V)}$. Then we have
\[
i_z^* \varepsilon_{0,R,A}(V \otimes R, \tilde{W}, \tilde{\psi}) = i_z^*(\det W)_A,\varepsilon_{\psi},A(V) \otimes \varepsilon_{0,R,A}(V, \tilde{\psi})_{\otimes \text{rank } W}
\]
for all closed point $i_z : z \to \text{Spec}(A)$. Hence the assertion follows. □

4.11. Local $\varepsilon_0$-characters for torsion coefficients (tame case). Let $R$ be a complete strict $p'$-coefficient ring with a positive residue characteristic.

Define the $k$-algebra $\text{Gr}^\bullet K$, $\text{Gr}^{>0} K$ and $\text{Gr}^ullet K$ in a similar way to that in [10, § 10.1]. Let $\ell$ be the residue characteristic of $R$, and set $R_0 := W(\mathbb{F}_\ell(\mu_\ell))$. Let $\phi_0$ be a non-trivial additive character $R_0$-sheaf on $K^{[-1,-1]}$. Let $L_{\phi_0}$ be the invertible $R_0$-sheaf on $\text{Spec}(\text{Gr}^{>0} K)$ corresponding to $\phi_0$ by the canonical isomorphism $K^{[-1,-1]} \cong \text{Spec}(\text{Gr}^{>0} K)$.

Define schemes $X_0$, $X$, and $X_m$, groups $G$, $I$, and $I_m$ and a smooth $R_0$-sheaf $L_{\phi_0}'$ on $X_0$ in a similar way to that in [10, § 10.1]. Put $W_m := H^1_k(X_m, L_{\phi_0}')$. $W_m$ is a free $R_0[I_m]$-module of rank one with a semi-linear action of $\text{Gal}(X_m/X_0)$.

**Definition 4.47.** Let $(\rho, V)$ is a tamely ramified object in $\text{Rep}(W_{K}, R)$. Let $\tilde{\psi}_0$ be a non-trivial additive character $R$-sheaf on $K^{[0,0]}$.

For each $n \in \mathbb{Z}_{>0}$, take a sufficiently divisible $m$ such that $W^n_K$ acts on $V \otimes R / R/m^n_R$ via the quotient $I_m$.

Define the $R/m^n_R[W_K]$-module $\varepsilon_{0,R/m^n_R}(V \otimes R / R/m^n_R, \tilde{\psi}_0, \phi_0)$ by
\[
\varepsilon_{0,R/m^n_R}(V \otimes R / R/m^n_R, \tilde{\psi}_0, \phi_0) := R/m^n_R(\text{rank } V) \otimes R/m^n_R(V \otimes R / R/m^n_R \otimes R_0 W_m)_{I_m} \otimes R/m^n_R(\rho, V)_{\text{Gr}_R} \otimes R_0 L_{\phi_0}, \tilde{\psi}' \otimes^{-1}
\]
where $\tilde{\psi}'$ is an additive character $R_0$-sheaf on $\text{Gr}^\bullet K$ of conductor $-1$ which is the pull-back of $\tilde{\psi}$ by the canonical morphism
\[
\text{Gr}^\bullet K^{[-n,0]} \cong K^{[0,0]} \times_k \cdots K^{[1,1]} \times_k K^{[n,n]} \xrightarrow{\text{pr}_1} K^{[0,0]} \xrightarrow{-1} K^{[0,0]}.
\]

Define $\varepsilon_{0,R}(V, \tilde{\psi}_0, \phi_0)$ by the projective limit
\[
\varepsilon_{0,R}(V, \tilde{\psi}_0, \phi_0) := \lim_{\substack{\longrightarrow \cr m}} \varepsilon_{0,R/m^n_R}(V \otimes R / R/m^n_R, \tilde{\psi}_0, \phi_0).
\]

**Lemma 4.48.** The character $\varepsilon_{0,R}(V, \tilde{\psi}_0, \phi_0)$ does not depend on the choice of $\phi_0$. 
Proof. We may assume that $V$, $\tilde{\psi}_0$ and $\phi_0$ are defined over a finite $p'$-coefficient ring $R_0$.

Let $\phi'_0 : k \to R_0^\times$ be another additive character. Define the representation $W'_m$ by

$$W'_m := H_\epsilon^1(X, (\pi_m* R_0) \otimes R_0 \tilde{\rho}'_{\phi'_0}) \in \text{Rep}(\text{Gal}(X_m/ X_0), R_0).$$

Then we have a canonical isomorphism

$$H_\epsilon^1(X, \tilde{V} \otimes R_0 \tilde{\rho}'_{\phi'_0}) \cong (V \otimes R_0 W'_m)_{I_m}.$$ 

There exists a unique element $a \in k^\times$ such that $\phi'(x) = \phi(ax)$ for all $x \in k$. Take an element $\alpha \in \overline{k}$ satisfying $\alpha^m = a$. Then the map $X_m \to X_m$ induced by the multiplication-by-$\alpha$ map $m_K/m_K^2 \to m_K/m_K^2$ induces an isomorphism $\varphi : W_m \cong W'_m$ of $R_0[I_m]$-modules. Let $\sigma \in W_k$. Let $[\frac{\sigma^{-1}(\alpha)}{\alpha}] \in I_m$ be the element corresponding to $\frac{\sigma^{-1}(\alpha)}{\alpha} \in \mu_m(\overline{k})$ by the canonical isomorphism $I_m \cong \mu_m(\overline{k})$. It is easily checked that the action of $\sigma$ on $W_m$ is identified with the action of $\sigma \cdot [\frac{\sigma^{-1}(\alpha)}{\alpha}]$. Hence the proposition follows from Proposition 4.46. \hfill \square

4.12. Local $\varepsilon_0$-characters for torsion coefficients (general case).

Let $R$ be a complete strict $p'$-coefficient ring with positive residue characteristic and $\tilde{\psi}$ a non-trivial additive character $R$-sheaf on $K$. For every object $V$ in $\text{Rep}(W_K,R)$, let $V = V^0 \oplus \bigoplus \Sigma V^\Sigma$ be the refined slope decomposition of $V$ and set

$$\varepsilon_0,R(V, \tilde{\psi}) := \varepsilon_0,R(V^0, \tilde{\psi}) \otimes \bigotimes \varepsilon_0,R(V^\Sigma, \tilde{\psi}).$$

Then $\varepsilon_0,R(V, \tilde{\psi})$ has the following properties:

1. The isomorphism class of $\varepsilon_0,R(V, \tilde{\psi})$ depends only on the isomorphism class of $(\rho, V)$.
2. Let $R'$ be another strict $p'$-coefficient ring, and $h : R \to R'$ a local ring homomorphism. Then we have

$$\varepsilon_0,R(V, \tilde{\psi}) \otimes_R R' \cong \varepsilon_0,R'(V \otimes_R R', \tilde{\psi} \otimes_R R').$$
3. Suppose that there exists an exact sequence

$$0 \to V' \to V \to V'' \to 0$$

in $\text{Rep}(W_K, R)$. Then we have

$$\varepsilon_0,R(V, \tilde{\psi}) \cong \varepsilon_0,R(V', \tilde{\psi}) \otimes_R \varepsilon_0,R(V'', \tilde{\psi}).$$
4. Suppose that the residue field $k$ of $K$ is finite. Let $\psi : K \to R^\times$ be the additive character canonically corresponding to $\tilde{\psi}$. Then

$$\varepsilon_0,R(V, \tilde{\psi})(\text{Fr}_k) = (-1)^{\text{rank}V + \text{sw}(V)} \cdot \varepsilon_0,R(V, \psi).$$
(5) Let $a \in K^\times$. Then we have
\[
\varepsilon_{0,R}(V, \tilde{\psi}_a) = \det(V)_{[a]} \otimes_R R(-v_K(a) \cdot \text{rank } V) \otimes_R \varepsilon_{0,R}(V, \tilde{\psi}).
\]

(6) Let $W$ be an object in $\text{Rep}(W_K, R)$ on which $W_K$ acts via $W_K/W_K^0 \cong W_k$. Then we have
\[
\varepsilon_{0,R}(V \otimes W, \tilde{\psi}) = (\det W)^{(\text{sw}(V) + \text{rank } V \cdot (\text{ord } \tilde{\psi} + 1))} \otimes_R \varepsilon_{0,R}(V, \tilde{\psi})^\text{rank } W.
\]

(7) Suppose that the coinvariant $(V)_{W_K^0}$ is zero. Let $V^*$ be the $R$-linear dual of $V$. Then we have
\[
\varepsilon_{0,R}(V, \tilde{\psi}) \otimes \varepsilon_{0,R}(V^*, \tilde{\psi}) = (\det V)_{[-1]} \otimes_R R(-\text{sw}(V) - \text{rank } V \cdot (2\text{ord } \tilde{\psi} + 1)).
\]

**Theorem 4.49.** Let $L/K$ be a finite separable extension of $p$-CDVFś, $R$ a complete strict $p'$-coefficient ring with a positive residue characteristic, $\tilde{\psi}$ a non-trivial additive character $R$-sheaf. Then for every object $(\rho, V)$ in $\text{Rep}(W_L, R)$, we have
\[
\varepsilon_{0,R}(\text{Ind}_{W_L}^{W_K} V, \tilde{\psi}) = \tilde{\lambda}_R(L/K, \tilde{\psi})^{\text{rank } V} \otimes_R \varepsilon_{0,R}(V, \tilde{\psi} \circ \text{Tr}_{L/K}).
\]

**Proof.** Take a sufficiently large finite Galois extension $L'$ of $K$ containing $L$ such that $\rho$ factors through $W_K/W_{L'}$. Take a sufficiently large integer $N \in \mathbb{Z}$. Take a good perfect $\mathbb{F}_p$-subalgebra $A$ of $k$ and a $(N, \tilde{\psi})$-admissible $A$-structure $(B, (\pi_M)_M)$ of $L'/K$. Then we can define, in a canonical way, smooth invertible $R$-sheaves $\varepsilon_{0,R,A}(\text{Ind}_{W_K}^{W_L} V, \tilde{\psi})$, $\varepsilon_{0,R,A}(V, \tilde{\psi} \circ \text{Tr}_{L/K})$, and $\tilde{\lambda}_{R,A}(L/K, \tilde{\psi})$ on $\text{Spec } (A)$ whose pull-backs on $\text{Spec } (k)$ are identified with $\varepsilon_{0,R}(\text{Ind}_{W_K}^{W_L} V, \tilde{\psi})$, $\varepsilon_{0,R}(V, \tilde{\psi} \circ \text{Tr}_{L/K})$, and $\tilde{\lambda}_R(L/K, \tilde{\psi})$ respectively.

Then for any closed point $x \in \text{Spec } (A)$, we have
\[
i_x^* \varepsilon_{0,R,A}(\text{Ind}_{W_K}^{W_L} V, \tilde{\psi}) \cong i_x^* \varepsilon_{0,R,A}(V, \tilde{\psi} \circ \text{Tr}_{L/K}) \otimes i_x^* \tilde{\lambda}_R(L/K, \tilde{\psi})^{\text{rank } V},
\]
where $i_x : x \hookrightarrow \text{Spec } (A)$ be the canonical inclusion. Hence we have $\varepsilon_{0,R,A}(\text{Ind}_{W_K}^{W_L} V, \tilde{\psi}) \cong \varepsilon_{0,R,A}(V, \tilde{\psi} \circ \text{Tr}_{L/K}) \otimes \tilde{\lambda}_R(L/K, \tilde{\psi})^{\text{rank } V}$. This completes the proof.  

Let $k$ be a perfect field of characteristic $p$, $X_0$ a proper smooth connected curve over $k$, $U_0 \subset X_0$ a non-empty open subscheme of $X_0$, $j_0 : U_0 \hookrightarrow X_0$ the inclusion, $X = X_0 \otimes_k \bar{k}$, $U = U_0 \otimes_k \bar{k}$, $j = j_0 \times \text{id} : U \hookrightarrow X$, $R$ a strict $p'$-coefficient ring, $R_0 \subset R$ a finite subring, and $F$ a smooth $R_0$-flat $R_0$-sheaf on $U_0$. Define the global $\varepsilon$-character $\varepsilon_{R_0}(U_0, F)$ by
\[
\varepsilon_{R_0}(U_0, F) := \det(R \Gamma_c(U, F))^{\otimes -1} = \det(R \Gamma_c(X, j_0^! F))^{\otimes -1}.
\]
Let $\omega \in \Gamma(U_0, \Omega^1_{U_0/k})$ be a non-zero differential on $U$. Fix a non-trivial additive character $R_0$-sheaf $\tilde{\psi}$ on $\mathbb{G}_{a,k}$. For a closed point $x \in X$, let $\kappa(x)$ be the residue field at $x$, $K_x$ the completion of the function field of $X$ at $x$, and $F_x$ the isomorphism class in $\text{Rep}(W_{K_x}, R)$ corresponding to the
pull-back of \( \mathcal{F} \) by the canonical morphism \( \text{Spec}(K_x) \to U \). Define the additive character \( R \)-sheaf \( \tilde{\psi}_{\omega,x} \) on \( K_x \) to be the pull-back of \( \tilde{\psi}|_{G_{a,\kappa(x)}} \) by the morphism \( K_x^{\left[ m, -\text{ord}_x(\omega)-1 \right]} \to G_{a,\kappa(x)} \) defined by \( a \mapsto \text{Res}(a \omega) \).

**Theorem 4.50.** With the above notations, we have

\[
\tilde{\varepsilon}_{R_0}(U_0, \mathcal{F}) = R(-\frac{1}{2} \chi(X) \cdot \text{rank}(\mathcal{F})) \otimes_R \bigotimes_{x \in X_0 - U_0} \tilde{\varepsilon}_{0,R}(\mathcal{F}_x, \tilde{\psi}_{\omega,x}),
\]

where \( \chi(X) \) is the Euler number of \( X \).

**Proof.** We may assume that \( X_0 = \mathbb{P}^1_k \).

Let \( K_0 \) denote the function field of \( \mathbb{P}^1 \). For a closed point \( x \) on \( \mathbb{P}^1_k \), let \( K_x \) be the completion of \( K \) at \( x \). Take a sufficiently large integer \( N \). Take a finite etale Galois covering \( V_0 \) of \( U_0 \) such that the sheaf \( \mathcal{F} \) and the sheaves \( \tilde{\psi}_{\omega,x}|_{K_x^{[-N, -\text{ord}_x(\omega)-1]}} \) is constant on \( V_0 \).

Let \( \overline{V}_0 \) be the smooth completion of \( V_0 \). The morphism \( f \) is canonically extension to the morphism \( \overline{f} : \overline{V}_0 \to \mathbb{P}^1_k \). Let \( L_0 \) denote the function field of \( V_0 \). For a closed point \( y \) on \( \overline{V}_0 \), let \( L_y \) denote the completion of \( L_0 \) at \( y \).

There exists a datum

\[
(A, (i_{A,x})_x, U_A, \omega_A, V_A, (B_y, (\pi_{M_y})_{M_y})_y)
\]

which satisfies the following conditions:

- \( A \) is a good perfect \( \mathbb{F}_p \)-subalgebra of \( k \),
- In \((i_{A,x})_x\), \( x \) runs over all points in \( \mathbb{P}^1_k - U_0 \). For each such \( x \), \( i_{A,x} \) is a closed \( A \)-immersion \( \text{Spec}(A_x) \hookrightarrow \mathbb{P}^1_A \) from a finite etale \( A \)-subalgebra of \( \kappa(x) \) to \( \mathbb{P}^1_A \) which is equal to \( i_x : x \hookrightarrow \mathbb{P}^1_k \) after tensored with \( k \) over \( A \).
- \( U_A := \mathbb{P}^1_A - \bigcup_x i_x(\text{Spec}(A_x)) \).
- \( \omega_A \in \Gamma(U_A, \Omega^{1}_{U_A/A}) \) is a 1-differential on \( U_A \) such that \( \omega_A|_{U_0} = \omega \).
- \( V_A \to U_A \) is a finite etale morphism such that \( V_A \otimes_A k \cong V_0 \) as \( U_0 \)-schemes.
- In \((B_y, (\pi_{M_y})_{M_y})_y\), \( y \) runs over all pairs of closed point \( y \in Y_0 \) with \( x = \overline{f}(y) \not\in U_0 \). For each such \( y \), \((B_y, (\pi_{M_y})_{M_y})\) is an \((N, \tilde{\psi}_{\omega,x})\)-admissible \( A_x \)-structure of \( L_y/K_x \).

Let \( \mathcal{F}_A \) be the smooth etale \( R_0 \)-sheaf on \( U_A \) corresponding to \( \mathcal{F} \). Let \( \mathcal{F}_{A,x} \) be the object in

\[
\text{Rep}(\pi_1(\text{Spec}(A_x((\pi_{K_x})))), R_0)
\]

corresponding to \( \mathcal{F}_x \).

By using \( \mathcal{F}_{A,x} \), we define a smooth invertible \( R_0 \)-sheaf \( \tilde{\varepsilon}_{0,R_0,A_x}(\mathcal{F}_x, \tilde{\psi}_{\omega,x}) \) on \( \text{Spec}(A_x) \) whose pull-back on \( \text{Spec}(k) \) is the sheaf corresponding to \( \tilde{\varepsilon}_{0,R}(\mathcal{F}_x, \tilde{\psi}_{\omega,x}) \). We also define the smooth invertible \( R_0 \)-sheaf \( \tilde{\varepsilon}_{R_0,A}(U, \mathcal{F}) \)
on Spec (A) to be det_{R_0} Rf_i(U_A, \mathcal{F}_A), where f : U_A \to \text{Spec}(A) is the structure morphism.

Let z ∈ Spec (A) be a closed point, and i_z : Spec (κ(z)) ↪ Spec (A) the canonical inclusion. Set U_z := U_A \otimes_A κ(z). Then U_z is an open subscheme of \mathbb{P}_k^1.

Let i_{U,z} : U_z ↪ U_A be the canonical inclusion and set \mathcal{F}_z = i_{U,z}^* \mathcal{F} and ω_z = i_{U,z}^* ω_A. Then we have

\[ \text{Tr}(-Fr_z; i_{U,z}^* \varepsilon_{R_0,A}(U, \mathcal{F})) = \varepsilon_{R_0}(U_z, \mathcal{F}_z). \]

For x ∈ \mathbb{P}_k^1 - U_0, put z_x = z ×_\text{Spec}(A) \text{Spec}(A_x). For all point y ∈ z_x, let i_{z,y} : y ↪ z_x ↪ \text{Spec}(A_x) be the canonical inclusion. Then we have

\[ \text{Tr}(Fr_y; i_{z,y}^* \varepsilon_{0,R,A_x}(\mathcal{F}_x, \bar{\psi}_{\omega,x})) = (-1)^{\text{rank} \mathcal{F}_x + \text{sw}_y(\mathcal{F}_x)} \varepsilon_{0,R}(\mathcal{F}_{z,y}, \psi_{\omega,x,y}). \]

By [11, Thm. 4.1], we have

\[ \varepsilon_{R_0}(U_z, \mathcal{F}_z) = (\varepsilon_{\kappa(z)})^{\text{rank} \mathcal{F}_z} \prod_{x \in \mathbb{P}_k^1 - U_z} \varepsilon_{0,R}(\mathcal{F}_{z,x}, \psi_{\omega,x}). \]

From this we have

\[ i_{U,z}^* \varepsilon_{R_0,A}(U, \mathcal{F}) \cong R_0(-\frac{1}{2} \chi(X) \text{rank} \mathcal{F}) \otimes_{R_0} \bigotimes_x i_{z,x}^* \pi_x, \varepsilon_{0,R,A_x}(\mathcal{F}_x, \bar{\psi}_{\omega,x,x}), \]

where \pi_x : \text{Spec}(A_x) → \text{Spec}(A) is the structure morphism.

Hence the theorem follows from the standard argument using Chebotarev’s theorem (cf. proof of Lemma 4.18).

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**References**


Local $\varepsilon_0$-characters in torsion rings

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