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S-extremal strongly modular lattices

par Gabriele NEBE et Kristina SCHINDELAR

Résumé. Un réseau fortement modulaire est dit s-extrémal, s’il maximise le minimum du réseau et son ombre simultanément. La dimension des réseaux s-extrémaux dont le minimum est pair peut être bornée par la théorie des formes modulaires. En particulier de tels réseaux sont extrémaux.

Abstract. S-extremal strongly modular lattices maximize the minimum of the lattice and its shadow simultaneously. They are a direct generalization of the s-extremal unimodular lattices defined in [6]. If the minimum of the lattice is even, then the dimension of an s-extremal lattices can be bounded by the theory of modular forms. This shows that such lattices are also extremal and that there are only finitely many s-extremal strongly modular lattices of even minimum.

1. Introduction.

Strongly modular lattices have been defined in [11] to generalize the notion of unimodular lattices. For square-free $N \in \mathbb{N}$ a lattice $L \subset (\mathbb{R}^n, (,))$ in Euclidean space is called strongly $N$-modular, if $L$ is integral, i.e. contained in its dual lattice

$$L^* = \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \forall \ell \in L\}$$

and isometric to its rescaled partial dual lattices $\sqrt{d}(L^* \cap \frac{1}{d}L)$ for all $d \mid N$. The simplest strongly modular lattice is

$$C_N := \perp_{d \mid N} \sqrt{d}\mathbb{Z}$$

of dimension $\sigma_0(N)$, the number of divisors of $N$. For

$$N \in \mathcal{L} = \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\}$$

which is the set of square-free numbers such that $\sigma_1(N) = \sum_{d \mid N} d$ divides 24, Theorems 1 and 2 in [13] bound the minimum $\min(L) := \min\{(\ell, \ell) \mid$
$0 \neq \ell \in L \}$ of a strongly $N$-modular lattice that is rational equivalent to $C_N^k$ by

$$\min(L) \leq 2 + 2\lfloor \frac{k}{s(N)} \rfloor, \text{ where } s(N) = \frac{24}{\sigma_1(N)}.$$  

For $N \in \{1, 3, 5, 7, 11\}$ there is one exception to this bound: $k = s(N) - 1$ and $L = S^{(N)}$ of minimum 3 (see [13, Table 1]). Lattices achieving this bound are called extremal.

For an odd strongly $N$-modular lattice $L$ let

$$S(L) = L_0^* \setminus L^*$$

denote the shadow of $L$, where $L_0 = \{ \ell \in L \mid (\ell, \ell) \in 2\mathbb{Z} \}$ is the even sublattice of $L$. For even strongly $N$-modular lattices $L$ let $S(L) := L^*$. Then the shadow-minimum of an $N$-modular lattice is defined as

$$\text{smin}(L) := \min\{N(x, x) \mid x \in S(L)\}.$$  

In particular $\text{smin}(L) = 0$ for even lattices $L$. In this paper we show that for all $N \in \mathcal{L}$ and for all strongly $N$-modular lattices $L$ that are rational equivalent to $C_N^k$

$$2\min(L) + \text{smin}(L) \leq k\sigma_1(N) + 2 \quad \text{if } N \text{ is odd and}$$

$$\min(L) + \text{smin}(L) \leq k\sigma_1(N/2)^2 + 1 \quad \text{if } N \text{ is even}$$

with the exceptions $L = S^{(N)}$, $k = s(N) - 1$ ($N \neq 23, 15$ odd) where the bound has to be increased by 2 and $L = O^{(N)}$, $k = s(N)$ and $N$ even, where the bound has to be increased by 1 (see [13, Table 1] for the definition of the lattices $S^{(N)}$, $O^{(N)}$ and also $E^{(N)}$). Lattices achieving this bound are called s-extremal. The theory of modular forms allows us to bound the dimension $\sigma_0(N)k$ of an s-extremal lattice of even minimum $\mu$ by

$$2k < \mu s(N).$$

In particular s-extremal lattices of even minimum are automatically extremal and hence by [12] there are only finitely many strongly $N$-modular s-extremal lattices of even minimum. This is also proven in Section 3, where explicit bounds on the dimension of such s-extremal lattices and some classifications are obtained. It would be interesting to have a similar bound for odd minimum $\mu \geq 3$. Of course for $\mu = 1$, the lattices $C_N^k$ are s-extremal strongly $N$-modular lattices of minimum 1 for arbitrary $k \in \mathbb{N}$ (see [9]), but already for $\mu = 3$ there are only finitely many s-extremal unimodular lattices of minimum 3 (see [10]). The s-extremal strongly $N$-modular lattices of minimum $\mu = 2$ are classified in [9] and some s-extremal lattices of minimum 3 are constructed in [15]. For all calculations we used the computer algebra system MAGMA [2].
2. S-extremal lattices.

For a subset $S \subset \mathbb{R}^n$, which is a finite union of cosets of an integral lattice we put its theta series

$$\Theta_S(z) := \sum_{v \in S} q^{(v,v)}, \quad q = \exp(\pi i z).$$

The theta series of strongly $N$-modular lattices are modular forms for a certain discrete subgroup $\Gamma_N$ of $SL_2(\mathbb{R})$ (see [13]). Fix $N \in \mathcal{L}$ and put

$$g_1^{(N)}(z) := \Theta_{CN}(z) = \prod_{d|N} \Theta_{\mathbb{Z}/d}(dz) = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2dj})(1 + q^{d(2j-1)})^2$$

(see [4, Section 4.4]). Let $\eta$ be the Dedekind eta-function

$$\eta(z) := q^{1/12} \prod_{j=1}^{\infty} (1 - q^{2j})$$

and put $\eta^{(N)}(z) := \prod_{d|N} \eta(dz)$.

If $N$ is odd define

$$g_2^{(N)}(z) := \left( \frac{\eta^{(N)}(z/2) \eta^{(N)}(2z)}{\eta^{(N)}(z)^2} \right)^{s(N)}$$

and if $N$ is even then

$$g_2^{(N)}(z) := \left( \frac{\eta^{(N/2)}(z/2) \eta^{(N/2)}(4z)}{\eta^{(N/2)}(z) \eta^{(N/2)}(2z)} \right)^{s(N)}.$$

The meromorphic function $g_2^{(N)}$ generates the field of modular functions of $\Gamma_N$. It is a power series in $q$ starting with

$$g_2^{(N)}(z) = q - s(N)q^2 + \ldots.$$ 

Using the product expansion of the $\eta$-function we find that

$$q^{-1} g_2^{(N)}(z) = \prod_{d|N} \prod_{j=1}^{\infty} (1 + q^{d(2j-1)})^{-s(N)}.$$

For even $N$ one has to note that

$$q^{-1} g_2^{(N)}(z) = \prod_{d|\frac{N}{2}} \prod_{j=1}^{\infty} \left( \frac{1 + q^{4dj}}{1 + q^{d^2j}} \right)^{s(N)}$$

$$= \prod_{d|\frac{N}{2}} \prod_{j=1}^{\infty} (1 + q^{2d(2j-1)})^{-s(N)}(1 + q^{d(2j-1)})^{-s(N)}.$$
By [13, Theorem 9, Corollary 3] the theta series of a strongly $N$-modular lattice $L$ that is rational equivalent to $C_N^k$ is of the form

\[(2.1) \quad \Theta_L(z) = g_1^{(N)}(z)^k \sum_{i=0}^{b} c_i g_2^{(N)}(z)^i\]

for $c_i \in \mathbb{R}$ and some explicit $b$ depending on $k$ and $N$. The theta series of the rescaled shadow $S := \sqrt{N}S(L)$ of $L$ is

\[(2.2) \quad \Theta_S(z) = s_1^{(N)}(z)^k \sum_{i=0}^{b} c_i s_2^{(N)}(z)^i\]

where $s_1^{(N)}$ and $s_2^{(N)}$ are the corresponding “shadows” of $g_1^{(N)}$ and $g_2^{(N)}$ as defined in [13] (see also [9]).

If $N$ is odd, then

\[s_1^{(N)} = 2^{\sigma_0(N)} q^{\sigma_1(N)/4} (1 + q^2 + \ldots)\]

and

\[s_2^{(N)} = 2^{-s(N)\sigma_0(N)/2} (-q^{-2} + s(N) + \ldots).\]

If $N$ is even, then

\[s_1^{(N)} = 2^{\sigma_0(N)/2} q^{\sigma_1(N)/2} (1 + 2q + \ldots),\]

\[s_2^{(N)} = 2^{-s(N)\sigma_0(N)/2} (-q^{-1} + s(N) + \ldots).\]

**Theorem 2.1.** Let $N \in \mathcal{L}$ be odd and let $L$ be a strongly $N$-modular lattice in the genus of $C_N^k$. Let $\sigma := \text{sm}(L)$ and let $\mu := \text{min}(L)$. Then

\[\sigma + 2\mu \leq k \frac{\sigma_1(N)}{4} + 2\]

unless $k = s(N) - 1$ and $\mu = 3$. In the latter case the lattice $S^{(N)}$ is the only exception (with $\text{min}(S^{(N)}) = 3$ and $\text{sm}(S^{(N)}) = 4 - \sigma_1(N)/4$).

**Proof.** The proof is a straightforward generalization of the one given in [6]. We always assume that $L \neq S^{(N)}$ and put $g_1 := g_1^{(N)}$ and $g_2 := g_2^{(N)}$. Let $m := \mu - 1$ and assume that $\sigma + 2\mu \geq k \frac{\sigma_1(N)}{4} + 2$. Then from the expansion of

\[\Theta_S = \sum_{j=\sigma}^{\infty} b_j q^j = s_1^{(N)}(z)^k \sum_{i=0}^{b} c_i s_2^{(N)}(z)^i\]

in formula (2.2) above we see that $c_i = 0$ for $i > m$ and (2.1) determines the remaining coefficients $c_0 = 1$, $c_1, \ldots, c_m$ uniquely from the fact that

\[\Theta_L = 1 + \sum_{j=\mu}^{\infty} a_j q^j \equiv 1 \pmod{q^{m+1}}.\]
The number of vectors of norm \( k \sigma_4(N)/4 + 2 - 2\mu \) in \( S = \sqrt{N}S(L) \) is

\[
c_m(-1)^{m_2}m^{-\sigma_0(N)s(N)/2+k\sigma_0(N)}
\]

and nonzero, iff \( c_m \neq 0 \). The expansion of \( g_1^{-k} \) in a power series in \( g_2 \) is given by

\[
(2.3) \quad g_1^{-k} = \sum_{i=0}^{m} c_{i} g_2^i - a_{m+1} q^{m+1} g_1^{-k} + \ast q^{m+2} + \ldots = \sum_{i=0}^{\infty} \tilde{c}_i g_2^i
\]

with \( \tilde{c}_i = c_i \) (\( i = 0, \ldots, m \)) and \( \tilde{c}_{m+1} = -a_{m+1} \). Hence Bürmann-Lagrange (see for instance [16]) yields that

\[
c_m = \frac{1}{m!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left( \frac{\partial}{\partial q} (g_1^{-k}) (q g_2^{-1})^m \right)_{q=0} = \frac{-k}{m} \left( \text{coeff. of } q^{m-1} \text{ in } (g_1'/g_1)/f_1 \right)
\]

with \( f_1 = (q^{-1} g_2)^m g_1^k \). Using the product expansion of \( g_1 \) and \( g_2 \) above we get

\[
f_1 = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2dj}) (1 + q^{d(2j-1)})^{2k-s(N)m}.
\]

Since

\[
g_1'/g_1 = \sum_{d|N} \frac{\partial}{\partial q} \frac{\theta_3(dz)}{\theta_3(dz)}
\]

is alternating as a sum of alternating power series, the series \( P := g_1'/g_1/f_1 \) is alternating, if \( 2k - s(N)m \geq 0 \). In this case all coefficients of \( P \) are nonzero, since all even powers of \( q \) occur in \( (1 - q^2)^{-1} \) and \( g_1'/g_1 \) has a non-zero coefficient at \( q^1 \). Otherwise write

\[
P = g_1' \prod_{d|N} \prod_{j=1}^{\infty} \frac{(1 + q^{d(2j-1)})^{s(N)m-2k-2}}{(1 - q^{2dj})^{k+1}}.
\]

If \( 2k - s(N)m < -2 \) then \( P \) is a positive power series in which all \( q \)-powers occur. Hence \( c_m < 0 \) in this case. If the minimum \( \mu \) is odd then this implies that \( b_\sigma < 0 \) and hence the nonexistence of an \( s \)-extremal lattice of odd minimum for \( s(N)m - 2 > 2k \). Assume now that \( 2k - s(N)m = -2 \), i.e. \( k = s(N)m/2 - 1 \). By the bound in [13] one has

\[
m + 1 \leq 2\left\lfloor \frac{k}{s(N)} \right\rfloor + 2 = 2\left\lfloor \frac{m}{2} \right\rfloor - \frac{1}{s(N)} + 2.
\]

This is only possible if \( m \) is odd. Since \( g_1' \) has a non-zero constant term, \( P \) contains all even powers of \( q \). In particular the coefficient of \( q^{m-1} \) is positive. The last case is \( 2k - s(N)m = -1 \). Then clearly \( m \) and \( s(N) \) are
odd and $P = GH^{(m-1)/2}$ where

$$G = g' \prod_{d \mid N} \prod_{j=1}^{\infty} (1 + q^{d(2j-1)})^{-1}(1 - q^{2dj})^{-(s(N)+1)/2}$$

and

$$H = \prod_{d \mid N} \prod_{j=1}^{\infty} (1 - q^{2dj})^{-s(N)}.$$

If $m$ is odd then the coefficient of $P$ at $q^{m-1}$ is

$$\int_{-1+iy_0}^{1+iy_0} e^{-(m-1)\pi iz} G(e^{\pi iz})H(e^{\pi iz})^{(m-1)/2} dz$$

which may be estimated by the saddle point method as illustrated in [8, Lemma 1]. In particular this coefficient grows like a constant times

$$\frac{e^{(m-1)/2}}{m^{1/2}}$$

where $c = F(y_0)$, $F'(y) = e^{2\pi y}H(e^{-2\pi y})$ and $y_0$ is the first positive zero of $F'$. Since $c > 0$ and also $F''(y_0) > 0$ and the coefficient of $P$ at $q^{m-1}$ is positive for the first few values of $m$ (we checked 10000 values), this proves that $b_{m} > 0$ also in this case.

To treat the even $N \in \mathcal{L}$, we need two easy (probably well known) observations:

**Lemma 2.1.** Let

$$f(q) := \prod_{j=1}^{\infty} (1 + q^{2j-1})(1 + q^{2(2j-1)}).$$

Then the $q$-series expansion of $1/f$ is alternating with non zero coefficients at $q^a$ for $a \neq 2$.

**Proof.**

$$1/f = \prod_{j=1}^{\infty} (1 + q^{2j-1} + q^{2(2j-1)} + q^{3(2j-1)})^{-1} = \prod_{j=1}^{\infty} \sum_{\ell=0}^{\infty} q^{4\ell(2j-1)} - q^{(4\ell+1)(2j-1)}$$

is alternating as a product of alternating series. The coefficient of $q^a$ is non-zero, if and only if $a$ is a sum of numbers of the form $4\ell(2j - 1)$ and $(4\ell + 1)(2j - 1)$ with distinct $\ell$. One obtains 0 and 1 with $\ell = 0$ and $j = 1$ and $3 = 1(2 \cdot 2 - 1)$ and $6 = 1 + 5$. Since one may add arbitrary multiples of 4, this shows that the coefficients are all non-zero except for the case that $a = 2$. 

□
Lemma 2.2. Let \( g_{1} := g_{1}^{(N)} \) for even \( N \) such that \( N/2 \) is odd and denote by \( g'_{1} \) the derivative of \( g_{1} \) with respect to \( q \). Then \( \frac{g'_{1}}{g_{1}} \) is an alternating series with non-zero coefficients for all \( q^{a} \) with \( a \not\equiv 1 \pmod{4} \). The coefficients for \( q^{a} \) with \( a \equiv 1 \pmod{4} \) are zero.

Proof. Using the product expansion

\[
g_{1} = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2jd})(1 + q^{(2j-1)d})^{2}
\]

we calculate

\[
g'_{1}/g_{1} = \sum_{d|N} \sum_{j=1}^{\infty} \frac{2(2j-1)dq^{d(2j-1)-1}}{1 - q^{d(2j-1)}} - \frac{2djq^{2dj-1}}{1 - q^{2dj}} - \frac{4djq^{Adj-1}}{1 - q^{Adj}}
\]

\[
+ \frac{2(4j-2)dq^{d(4j-2)-1}}{1 - q^{d(4j-2)}}
\]

\[
= \sum_{d|N} \sum_{j=1}^{\infty} \frac{(4j-2)dq^{d(2j-1)d-1}}{1 + q^{(2j-1)d}} - \frac{8djq^{4dj-1}}{1 - q^{4dj}}
\]

\[
+ \frac{(4j-2)d(q^{(4j-2)d-1} - 3q^{(8j-4)d-1})}{1 - q^{(8j-4)d}}
\]

\[
= \sum_{d|N} \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} -8j\ell q^{4j\ell-1} - 3(4j-2)dq^{(8j-4)d\ell-1}
\]

\[
+ (4j-2)dq^{(2j-1)d(4\ell-2)-1} = (-1)^{\ell}(4j-2)dq^{(2j-1)d\ell-1}
\]

Hence the coefficient of \( q^{a} \) is positive if \( a \) is even and negative if \( a \equiv -1 \pmod{4} \). The only cancellation that occurs is for \( a \equiv 1 \pmod{4} \). In this case the coefficient of \( q^{a} \) is zero. \( \square \)

Theorem 2.2. Let \( N \in \mathcal{L} \) be even and let \( L \) be a strongly \( N \)-modular lattice in the genus of \( C_{N}^{k} \). Let \( \sigma := \text{smin}(L) \) and \( \mu := \text{min}(L) \). Then

\[
\sigma + \mu \leq k \frac{\sigma_{1}(N/2)}{2} + 1
\]

unless \( k = s(N) \) and \( \mu = 3 \) where this bound has to be increased by 1. In these cases \( L \) is the unique lattice \( L = O^{(N)} \) (from [13, Table 1]) of minimum 3 described in [9, Theorem 3].

Proof. As in the proof of Theorem 2.1 let \( g_{1} := g_{1}^{(N)} \) and \( g_{2} := g_{2}^{(N)} \), \( m := \mu - 1 \) and assume that \( \sigma + \mu \geq k \frac{\sigma_{1}(N/2)}{2} + 1 \). Again all coefficients \( c_{i} \) in (2.2) and (2.1) are uniquely determined by the conditions that \( \text{smin}(L) \geq k \frac{\sigma_{1}(N/2)}{4} - m \) and \( \Theta_{L} \equiv 1 \pmod{q^{m+1}} \). The number of vectors of norm
\[ k \frac{\sigma_1(N/2)}{2} - m \text{ in } S = \sqrt{N}S(L) \text{ is } c_m(-1)^m 2^{\sigma_0(N)k/2 - ms(N)}. \]

As in the proof of Theorem 2.1 the formula of Bürmann-Lagrange yields that

\[ c_m = \frac{-k}{m} \text{ (coeff. of } q^{m-1} \text{ in } (g_1'/g_1)/f_1) \]

with \( f_1 \) as in the proof of Theorem 2.1. We have

\[ f_1 = \prod_{d \mid N} f(dq)^{2k-s(N)m} \prod_{j=1}^{\infty} (1-q^{2dj})^k (1-q^{4dj})^k \]

where \( f \) is as in Lemma 2.1. If \( 2k - s(N)m > 0 \) then \( 1/f_1 \) is alternating by Lemma 2.1 and \( g_1'/g_1 \) is alternating (with a non-zero coefficient at \( q^3 \)) by Lemma 2.2 and we can argue as in the proof of Theorem 2.1. Since \( k > 0 \) all even coefficients occur in the product

\[ \prod_{j=1}^{\infty} (1-q^{2j})^{-k} \]

hence all coefficients in \( (g_1'/g_1)/f_1 \) are non-zero. If \( 2k - s(N)m = 0 \) similarly the only zero coefficient in \( (g_1'/g_1)/f_1 \) is at \( q^1 \) yielding the exception stated in the Theorem. Now assume that \( 2k - s(N)m < 0 \) and write

\[ P = (g_1'/g_1)/f_1 = g_1' \prod_{d \mid N} \frac{f(dq)^{s(N)m-2k-2}}{\prod_{j=1}^{\infty}((1-q^{2dj})(1-q^{4dj}))^{k+1}}. \]

If \( 2k - s(N)m < -2 \) then \( P \) is a positive power series in which all \( q \)-powers occur and hence \( c_m < 0 \). If the minimum \( \mu \) is odd then this implies that \( b_\sigma < 0 \) and hence the nonexistence of an \( s \)-extremal lattice of odd minimum for \( s(N)m - 2 > 2k \). Assume now that \( 2k - s(N)m = -2 \), i.e. \( k = s(N)m/2 - 1 \). Then again \( m \) is odd and since \( g_1' \) has a non-zero constant term \( P \) contains all even powers of \( q \). In particular the coefficient of \( q^{m-1} \) is positive. The last case is \( 2k - s(N)m = -1 \) and dealt with as in the proof of Theorem 2.1.

□

From the proof of Theorem 2.1 and 2.2 we obtain the following bound on the minimum of an \( s \)-extremal lattice which is sometimes a slight improvement of the bound (1.1).

**Corollary 2.1.** Let \( L \) be an \( s \)-extremal strongly \( N \)-modular lattice in the genus of \( C^k_N \) with odd minimum \( \mu := \min(L) \). Then

\[ \mu < \frac{2k + 2}{s(N)} + 1. \]
3. S-extremal lattices of even minimum.

In this section we use the methods of [8] to show that there are only finitely many s-extremal lattices of even minimum. The first result generalizes the bound on the dimension of an s-extremal lattice of even minimum that is obtained in [6] for unimodular lattices. In particular such s-extremal lattices are automatically extremal. Now [12, Theorem 5.2] shows that there are only finitely many extremal strongly $N$-modular lattices which also implies that there are only finitely many such s-extremal lattices with even minimum. To get a good upper bound on the maximal dimension of an s-extremal strongly $N$-modular lattice, we show that the second (resp. third) coefficient in the shadow theta series becomes eventually negative.

**Theorem 3.1.** Let $N \in \mathcal{L}$ and let $L$ be an s-extremal strongly $N$-modular lattice in the genus of $C^k_N$. Assume that $\mu := \min(L)$ is even. Then

$$s(N)(\mu - 2) \leq 2k < \mu s(N).$$

**Proof.** The lower bound follows from (1.1). As in the proof of Theorem 2.1 we obtain the number $a_\mu$ of minimal vectors of $L$ as

$$a_\mu = \frac{k}{\mu - 1} \left( \text{coeff. of } q^{\mu - 1} \text{ in } (g'_1/g_1)/f_2 \right)$$

with

$$f_2 = (q^{-1}g_2)^\mu g_1^k.$$

If $N$ is odd, then

$$f_2 = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2dj})^k (1 + q^{d(2j-1)})^{2k - s(N)\mu}$$

and for even $N$ we obtain

$$f_2 = \prod_{d|\frac{N}{2}} f(dq)^{2k - s(N)\mu} \prod_{j=1}^{\infty} (1 - q^{2dj})^k (1 + q^{4dj})^k$$

where $f$ is as in Lemma 2.1. If $2k - s(N)\mu \geq 0$ then in both cases $(g'_1/g_1)/f_2$ is an alternating series and since $\mu - 1$ is odd the coefficient of $q^{\mu - 1}$ in this series is negative. Therefore $a_\mu$ is negative which is a contradiction. \qed

We now proceed as in [8] and express the first coefficients of the shadow theta series of an s-extremal $N$-modular lattice.

**Lemma 3.1.** Let $N \in \mathcal{L}$, $s_1 := s_1^{(N)}$ and $s_2 := s_2^{(N)}$. Then $s_1^k \sum_{i=0}^{m} c_is_2^i$ starts with $(-1)^{m} 2^{s_1(N)(k-ms(N)/2)} q^{k\sigma_1(N)/4 - 2m}$ times

$$c_m - (2^{s_1(N)\sigma_0(N)/2} c_{m-1} + (s(N)m - k)c_m)q^2.$$
if $N$ is odd, and with $(-1)^m 2\sigma_0(N)k/2- ms(N)\sigma_0(N)/4 k\sigma_1(N/2)^2-m$ times
\[ c_m - (2s(N)\sigma_0(N)/4 c_{m-1} + (s(N)m - 2k)c_m)q \]
\[ + (2s(N)\sigma_0(N)/2 c_{m-2} + 2s(N)\sigma_0(N)/4 (s(N)(m-1) - 2k)c_{m-1} \]
\[ + (s(N)^2 m(m-1)/2 - 2kms(N) + 2k(k-1) + 2s(N)\sigma_0(N)/4 m(s(N)+1)c_m)q^2 \]
if $N$ is even.

**Proof.** If $N$ is odd then
\[ s_1 = 2\sigma_0(N)q^{\sigma_1(N)/4}(1 + q^2) + \ldots \]
\[ s_2 = 2^{-s(N)\sigma_0(N)/2}(-q^{-2} + s(N)) + \ldots \]
and for even $N$
\[ s_1 = 2\sigma_0(N)/2 q^{\sigma_1(N/2)/2}(1 + 2q + 0q^2) + \ldots \]
\[ s_2 = 2^{-s(N)\sigma_0(N)/4}(-q^{-1} + s(N)) - \frac{s(N)+1}{4}q + \ldots \]
Explicit calculations prove the lemma. \qed

We now want to use [8, Lemma 1] to show that the coefficients $c_m$ and $c_{m-1}$ determined in the proof of Theorem 2.1 for the theta series of an $s$-extremal lattice satisfy $(-1)^m c_j > 0$ and $c_m/c_{m-1}$ is bounded.

If $L$ is an $s$-extremal lattice of even minimum $\mu = m + 1$ in the genus of $C^k_N$, then Theorem 3.1 yields that
\[ k = \frac{s(N)}{2}(m-1) + b \text{ for some } 0 \leq b < s(N). \]
Let
\[ \psi := \psi^{(N)} := \prod_{j=1}^{\infty} \prod_{d|N} (1 - q^{2jd}) \text{ and } \varphi := \varphi^{(N)} := \prod_{j=1}^{\infty} \prod_{d|N} (1 + q^{(2j-1)d}) \]
Then
\[ c_{m-\ell} = \frac{-k}{m-\ell} \text{ coeff. of } q^{m-\ell-1} \text{ in } g_1^{(N)} \psi_{-k-1}^{s(N)(m-\ell)-2(k+1)} \]
\[ = \frac{-k}{m-\ell} \text{ coeff. of } q^{m-\ell-1} \text{ in } G^{(b)}_\ell H^{m-\ell-1} \]
where
\[ G^{(b)}_\ell = g_1^{(N)} \psi_{b-1-\ell s(N)/2}^{s(N)/2-2b-2+(1-\ell)s(N)} = G^{(0)}_\ell (\psi^{-1} \phi^{-2})^b \]
and
\[ H = \psi^{-s(N)/2} = 1 + \frac{s(N)}{2} q^2 + \ldots . \]
In particular the first two coefficients of $H$ are positive and the remaining coefficients are nonnegative. Since also odd powers of $q$ arise in $G^{(b)}_\ell$ the coefficient $\beta_{m-\ell-1}$ of $q^{m-\ell-1}$ in $G^{(b)}_\ell H^{m-\ell-1}$ is by Cauchy’s formula
\[ \beta_{m-\ell-1} = \frac{1}{2} \int_{-1+iy}^{1+iy} e^{-\pi i (m-\ell-1)z} G^{(b)}_\ell (e^{\pi iz}) H^{m-\ell-1}(e^{\pi iz}) dz \]
for arbitrary \( y > 0 \).

Put \( F(y) := e^{\pi y} H(e^{-\pi y}) \) and let \( y_0 \) be the first positive zero of \( F' \). Then we check that \( d_1 := F(y_0) > 0 \) and \( d_2 := F''(y_0)/F(y_0) > 0 \). Now \( H \) has two saddle points in \([-1 + iy_0, 1 + iy_0]\) namely at \( \pm 1 + iy_0 \) and \( iy_0 \). By the saddle point method (see [1, (5.7.2)]) we obtain

\[
\beta_{m-\ell-1} \sim d_1^{\ell} (G^{(b)}_{\ell}) - (1)^{\ell} (1) (G^{(b)}_{\ell}) - (e^{-\pi y_0}) \times (2\pi (m - \ell - 1)d_2)^{-1/2}
\]

as \( m \) tends to infinity. In particular

\[
c_m \sim d_1 G^{(b)}_0 (e^{-\pi y_0}) + (1)^{m-1} G^{(b)}_0 (e^{-\pi y_0}) \frac{G^{(b)}_1 (e^{-\pi y_0}) + (1)^m G^{(b)}_1 (e^{-\pi y_0})}{G^{(b)}_1 (e^{-\pi y_0}) - G^{(b)}_1 (e^{-\pi y_0})} c_{m-1}.
\]

**Lemma 3.2.** For \( N \in \mathcal{L} \) and \( b \in \{0, \ldots, s(N) - 1\} \) let \( k := \frac{s(N)}{2} (m-1) + b = js(N) + b, \ G^{(b)}_{\ell}, \ d_1, d_2, y_0 \) be as above where \( m = 2j + 1 \) is odd. Then \( c_{2j+1}/c_{2j} \) tends to

\[
Q(N, b) := d_1 \frac{G^{(b)}_0 (e^{-\pi y_0}) + G^{(b)}_0 (e^{-\pi y_0})}{G^{(b)}_1 (e^{-\pi y_0}) - G^{(b)}_1 (e^{-\pi y_0})} \in \mathbb{R}_{<0}
\]

if \( j \) goes to infinity.

By Lemma 3.1 the second coefficient \( b_{\sigma+2} \) in the shadow theta series of a putative \( s \)-extremal strongly \( N \)-modular lattice of even minimum \( \mu = m + 1 \) in the genus of \( G_N^k \) \( (k = \frac{s(N)}{2} (m-1) + b \) as above) is a positive multiple of

\[
2^{s(N)} \frac{\sigma_0 (N)}{2} c_{m-1} + (s(N)m - k)c_m \sim (2^{s(N)} \frac{\sigma_0 (N)}{2} + Q(N, b) \frac{s(N)(m + 1) - 2b}{2}) c_{m-1}
\]

when \( m \) tends to infinity. In particular this coefficient is expected to be negative if

\[
\mu = m + 1 > B(N, b) := \frac{2}{s(N)} (b + \frac{2^{s(N)} \sigma_0 (N)/2}{-Q(N, b)}).
\]

Since all these are asymptotic values, the actual value \( \mu_- (N, b) \) of the first even minimum \( \mu \) where \( b_{\sigma+2} \) becomes negative may be different. In all cases, the second coefficient of the relevant shadow theta series seems to remain negative for even minimum \( \mu \geq \mu_- (N, b) \).

For odd \( N \in \mathcal{L} \) the values of \( B(N, b) \) and \( \mu_- (N, b) \) are given in the following tables:
<table>
<thead>
<tr>
<th>$N = 1$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 3$</th>
<th>$b = 4$</th>
<th>$b = 5$</th>
<th>$b = 6$</th>
<th>$b = 7$</th>
<th>$b = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(1,b)$</td>
<td>-380</td>
<td>-113</td>
<td>-43.8</td>
<td>-18.4</td>
<td>-8</td>
<td>-3.53</td>
<td>-1.57</td>
<td>-0.71</td>
<td>-0.33</td>
</tr>
<tr>
<td>$B(1,b)$</td>
<td>0.9</td>
<td>3.1</td>
<td>7.96</td>
<td>18.8</td>
<td>43</td>
<td>97.1</td>
<td>217.4</td>
<td>480.4</td>
<td>1036.6</td>
</tr>
<tr>
<td>$\mu_-(1,b)$</td>
<td>6</td>
<td>6</td>
<td>12</td>
<td>20</td>
<td>44</td>
<td>96</td>
<td>216</td>
<td>478</td>
<td>1032</td>
</tr>
<tr>
<td>$k_-(1,b)$</td>
<td>48</td>
<td>49</td>
<td>122</td>
<td>219</td>
<td>508</td>
<td>1133</td>
<td>2574</td>
<td>5719</td>
<td>12368</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = 1$</th>
<th>$b = 9$</th>
<th>$b = 10$</th>
<th>$b = 11$</th>
<th>$b = 12$</th>
<th>$b = 13$</th>
<th>$b = 14$</th>
<th>$b = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(1,b)$</td>
<td>-0.16</td>
<td>-0.08</td>
<td>-0.05</td>
<td>-0.04</td>
<td>-0.03</td>
<td>-0.027</td>
<td>-0.026</td>
</tr>
<tr>
<td>$B(1,b)$</td>
<td>2131.3</td>
<td>4012.4</td>
<td>6597.4</td>
<td>9240.4</td>
<td>11239.4</td>
<td>12433.6</td>
<td>13049.1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = 1$</th>
<th>$b = 16$</th>
<th>$b = 17$</th>
<th>$b = 18$</th>
<th>$b = 19$</th>
<th>$b = 20$</th>
<th>$b = 21$</th>
<th>$b = 22$</th>
<th>$b = 23$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(1,b)$</td>
<td>-0.026</td>
<td>-0.025</td>
<td>-0.025</td>
<td>-0.025</td>
<td>-0.025</td>
<td>-0.025</td>
<td>-0.025</td>
<td>-0.025</td>
</tr>
<tr>
<td>$B(1,b)$</td>
<td>13342</td>
<td>13477</td>
<td>13538</td>
<td>13565</td>
<td>13577</td>
<td>13582</td>
<td>13585</td>
<td>13586</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = 3$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 3$</th>
<th>$b = 4$</th>
<th>$b = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(3,b)$</td>
<td>-15.6</td>
<td>-2</td>
<td>-0.45</td>
<td>-0.2</td>
<td>-0.16</td>
<td>-0.15</td>
</tr>
<tr>
<td>$B(3,b)$</td>
<td>1.36</td>
<td>11</td>
<td>47.6</td>
<td>107.13</td>
<td>137.07</td>
<td>144.34</td>
</tr>
<tr>
<td>$\mu_-(3,b)$</td>
<td>6</td>
<td>12</td>
<td>44</td>
<td>100</td>
<td>126</td>
<td>130</td>
</tr>
<tr>
<td>$k_-(3,b)$</td>
<td>12</td>
<td>31</td>
<td>128</td>
<td>297</td>
<td>376</td>
<td>389</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = 5$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
<th>$b = 3$</th>
<th>$N = 7$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
<th>$b = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(5,b)$</td>
<td>-5</td>
<td>-0.73</td>
<td>-0.31</td>
<td>-0.25</td>
<td>$Q(7,b)$</td>
<td>-2.88</td>
<td>-0.51</td>
<td>-0.32</td>
</tr>
<tr>
<td>$B(5,b)$</td>
<td>1.6</td>
<td>11</td>
<td>27</td>
<td>33.5</td>
<td>$B(7,b)$</td>
<td>1.85</td>
<td>11</td>
<td>17.8</td>
</tr>
<tr>
<td>$\mu_-(5,b)$</td>
<td>6</td>
<td>12</td>
<td>22</td>
<td>24</td>
<td>$\mu_-(7,b)$</td>
<td>6</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>$k_-(5,b)$</td>
<td>8</td>
<td>21</td>
<td>42</td>
<td>47</td>
<td>$k_-(7,b)$</td>
<td>6</td>
<td>13</td>
<td>17</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$N = 11$</th>
<th>$b = 0$</th>
<th>$b = 1$</th>
<th>$N = 15$</th>
<th>$b = 0$</th>
<th>$N = 23$</th>
<th>$b = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(11,b)$</td>
<td>-1.72</td>
<td>-0.45</td>
<td>$Q(15,b)$</td>
<td>-2.03</td>
<td>$Q(23,b)$</td>
<td>-1.08</td>
</tr>
<tr>
<td>$B(11,b)$</td>
<td>2.33</td>
<td>9.8</td>
<td>$B(15,b)$</td>
<td>3.93</td>
<td>$B(23,b)$</td>
<td>3.69</td>
</tr>
<tr>
<td>$\mu_-(11,b)$</td>
<td>6</td>
<td>6</td>
<td>$\mu_-(15,b)$</td>
<td>6</td>
<td>$\mu_-(23,b)$</td>
<td>6</td>
</tr>
<tr>
<td>$k_-(11,b)$</td>
<td>4</td>
<td>5</td>
<td>$k_-(15,b)$</td>
<td>2</td>
<td>$k_-(23,b)$</td>
<td>2</td>
</tr>
</tbody>
</table>
For even \( N \in \mathcal{L} \) the situation is slightly different. Again \( k = b + \frac{s(N)}{2}(m - 1) \) for some \( 0 \leq b < s(N) \). From Lemma 3.1 the second coefficient \( b_{\sigma+1} \) in the s-extremal shadow theta series is a nonzero multiple of \( 2^{s(N)}\sigma_0(N)/4c_{m-1} + (s(N) - 2b)c_m \) and in particular its sign is asymptotically independent of \( m \). Therefore we need to consider the third coefficient \( b_{\sigma+2} \), which is by Lemma 3.1 for odd \( m \) a positive multiple of 
\[
-a^2c_{m-2} + a(2k - s(m - 1))c_{m-1}
+ (2kms - s^2m(m - 1)) - 2k(k - 1) - am\frac{s + 1}{4}c_m
\]
where for short \( a := 2^{s\sigma_0(N)/4} \) and \( s := s(N) \). For \( k = \frac{s(N)}{2}(m - 1) + b \) this becomes 
\[
-a^2c_{m-2} + 2abc_{m-1} + (m(2b(b - 1) - s) - a\frac{s + 1}{4} + s\frac{s + 2}{2}) + 2s + s^2)c_m.
\]
Since the quotients \( c_{m-1}/c_{m-2} \) and \( c_m/c_{m-2} \) are bounded, there is an explicit asymptotic bound \( B(N; b) \) for \( \mu = m + 1 \) after which this coefficient should become negative. Again, the true values \( \mu_-(N, b) \) differ and the results are displayed in the following table.

<table>
<thead>
<tr>
<th>( N = 2 )</th>
<th>( b = 0 )</th>
<th>( b = 1 )</th>
<th>( b = 2 )</th>
<th>( b = 3 )</th>
<th>( b = 4 )</th>
<th>( b = 5 )</th>
<th>( b = 6 )</th>
<th>( b = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B(2, b)</td>
<td>-4.9</td>
<td>10</td>
<td>52.5</td>
<td>170.1</td>
<td>382.6</td>
<td>575.9</td>
<td>677.7</td>
<td>725.7</td>
</tr>
<tr>
<td>( \mu_-(2, b) )</td>
<td>16</td>
<td>22</td>
<td>54</td>
<td>166</td>
<td>374</td>
<td>564</td>
<td>666</td>
<td>716</td>
</tr>
<tr>
<td>( k_-(2, b) )</td>
<td>56</td>
<td>81</td>
<td>210</td>
<td>659</td>
<td>1492</td>
<td>2253</td>
<td>2662</td>
<td>2863</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( N = 6 )</th>
<th>( b = 0 )</th>
<th>( b = 1 )</th>
<th>( N = 14 )</th>
<th>( b = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B(6, b)</td>
<td>1</td>
<td>33.58</td>
<td>B(14, b)</td>
<td>2</td>
</tr>
<tr>
<td>( \mu_-(6, b) )</td>
<td>10</td>
<td>28</td>
<td>( \mu_-(14, b) )</td>
<td>10</td>
</tr>
<tr>
<td>( k_-(6, b) )</td>
<td>8</td>
<td>27</td>
<td>( k_-(14, b) )</td>
<td>4</td>
</tr>
</tbody>
</table>

### 3.1. Explicit classifications.
In this section we classify the s-extremal strongly \( N \)-modular lattices \( L_N(\mu, k) \) rational equivalent to \( C_N^k \) for certain \( N \) and even minimum \( \mu \). For \( N \in \{11, 14, 15, 23\} \) a complete classification is obtained. For convenience we denote the uniquely determined modular form that should be the theta series of \( L_N(\mu, k) \) by \( \theta_N(\mu, k) \) and its shadow by \( \sigma_N(\mu, k) \).

Important examples are the unique extremal even strongly \( N \)-modular lattices \( E^{(N)} \) of minimum 4 and with \( k = s(N) \) from [13, Table 1]. For odd
$N$, these lattices are s-extremal since $2\mu + \sigma = 8 = s(N)\sigma_1(N)/4 + 2$ and hence $E'(N) = L_N(4, s(N))$.

Theorem 3.1 suggests to write $k = \frac{s(N)(\mu - 2)}{2} + b$ for some $0 \leq b \leq s(N) - 1$ and we will organize the classification according to the possible $b$. Note that for every $b$ the maximal minimum $\mu$ is bounded by $\mu - (N, b)$ above.

If $N = 14, 15$ or $23$, then $s(N) = 1$ and hence Theorem 3.1 implies that $k = \frac{\mu - 2}{2}$. For $N = 15, 23$ the only possibility is $k = 1$ and $\mu = 4$ and $L_N(4, 1) = E'(N)$. The second coefficient of $\sigma_{14}(4, 1)$ and $\sigma_{14}(8, 3)$ is negative, hence the only s-extremal strongly 14-modular lattice with even minimum is $L_{14}(6, 2)$ of minimum 6. The series $\sigma_{14}(6, 2)$ starts with $8q^3 + 8q^5 + 16q^6 + \ldots$. Therefore the even neighbour of $L_{14}(6, 2)$ in the sense of [13, Theorem 8] is the unique even extremal strongly 14-modular lattice of dimension 8 (see [14, p. 160]). Constructing all odd 2-neighbours of this lattice, it turns out that there is a unique such lattice $L_{14}(6, 2)$. Note that $L_{14}(6, 2)$ is an odd extremal strongly modular lattice in a jump dimension and hence the first counterexample to conjecture (3) in the Remark after [13, Theorem 2].

For $N = 11$ and $b = 0$ the only possibility is $\mu = 4$ and $k = 2 = s(N)$ whence $L_{11}(4, 2) = E^{(11)}$. If $b = 1$ then either $\mu = 2$ and $L_{11}(2, 1) = \begin{pmatrix} 21 \\ 16 \end{pmatrix}$ or $\mu = 4$. An explicit enumeration of the genus of $C_{11}^3$ with the Kneser neighbouring method [7] shows that there is a unique lattice $L_{11}(4, 3)$.

Now let $N = 7$. For $b = 0$ again the only possibility is $k = s(N)$ and $L_7(4, 3) = E^{(7)}$. For $b = 1$ and $b = 2$ one obtains unique lattices $L_7(2, 1)$ (with Grammatrix $\begin{pmatrix} 21 \\ 14 \end{pmatrix}$ ) $L_7(4, 4)$ and $L_7(4, 5)$. There is no contradiction for the existence of lattices $L_7(6, 7)$, $L_7(6, 8)$, $L_7(8, 10)$, $L_7(8, 11)$, though a complete classification of the relevant genera seems to be difficult. For the lattice $L_7(6, 8)$ we tried the following: Both even neighbours of such a lattice are extremal even 7-modular lattices. Starting from the extremal 7-modular lattice constructed from the structure over $\mathbb{Z}[^2] \mathbb{Z}$ of the Barnes-Wall lattice as described in [14], we calculated the part of the Kneser 2-neighbouring graph consisting only of even lattices of minimum 6 and therewith found 126 such even lattices 120 of which are 7-modular. None of the edges between such lattices gave rise to an s-extremal lattice. The lattice $L_7(10, 14)$ does not exist because $\theta_7(10, 14)$ has a negative coefficient at $q^{13}$.

Now let $N := 6$. For $k = \mu - 2$ the second coefficient in the shadow theta series is negative, hence there are no lattices $L_6(\mu, \mu - 2)$ of even minimum $\mu$. For $k = \mu - 1 < 27$ the modular forms $\theta_6(\mu, \mu - 1)$ and $\sigma_6(\mu, \mu - 1)$ seem to have nonnegative integral coefficients. The lattice $L_6(2, 1)$ is unique and already given in [9]. For $\mu = 4$ the even neighbour of any lattice $L_6(4, 3)$ (as defined in [13, Theorem 8]) is one of the five even extremal strongly...
6-modular lattices given in [14]. Constructing all odd 2-neighbours of these lattices we find a unique lattice $L_6(4,3)$ as displayed below.

For $N = 5$ the lattice $L_5(4,4) = E^{(5)}$ is is the only s-extremal lattice of even minimum $µ$ for $k = 2(µ - 2)$, because $µ_-(5,0) = 6$. For $k = 2(µ - 2) + 1$ the shadow series $σ_5(2,1)$, $σ_5(4,5)$ and $σ_5(6,9)$ have non-integral respectively odd coefficients so the only lattices that might exist here are $L_5(8,13)$ and $L_5(10,17)$. The s-extremal lattice $L_5(2,2) = \frac{(21)}{13} \bot \frac{(21)}{13}$ is unique. The theta series $θ_5(2,3)$ starts with $1 + 20q^3 + \ldots$, hence $L_5(2,3) = S^{(5)}$ has minimum 3. The genus of $C_5^6$ contains 1161 isometry classes, 3 of which represent s-extremal lattices of minimum 4 and whose Grammatrices $L_5(4,6)_{a,b,c}$ are displayed below. For $k = 7$ a complete classification of the genus of $C_5^6$ seems to be out of range. A search for lattices in this genus that have minimum 4 constructs the example $L_5(4,7)_a$ displayed below of which we do not know whether it is unique. For the remaining even minima $µ < µ_-(5, b)$ we do not find a contradiction against the existence of such s-extremal lattices.

For $N = 3$ and $b = 0$ again $E^{(3)} = L_3(4,6)$ is the unique s-extremal lattice. For $k = 3(µ - 2) + 1$, the theta series $θ_3(8,19)$ and $θ_3(10,25)$ as well as their shadows seem to have integral non-negative coefficients, whereas $σ_3(4,7)$ and $σ_3(6,13)$ have non-integral coefficients. The remaining theta-series and their shadows again seem to have integral non-negative coefficients. The lattices of minimum 2 are already classified in [9]. In all cases $L_3(2,b) (2 \leq b \leq 5)$ is unique but $L_3(2,5) = S^{(3)}$ has minimum 3.

Now let $N := 2$. For $b = 0$ and $b = 1$ the second coefficient in $σ_2(µ, 4(µ - 2) + b)$ is always negative, proving the non-existence of such s-extremal lattices. The lattices of minimum 2 are already classified in [9]. There is a unique lattice $L_2(2,2) \cong D_4$, no lattice $L_2(2,3)$ since the first coefficient of $σ_2(2,3)$ is 3, unique lattices $L_2(2,b)$ for $b = 4, 5$ and 7 and two such lattices $L_2(2,6)$.

For $N = 1$ we also refer to the paper [6] for the known classifications. Again for $b = 0$, the Leech lattice $L_1(4,24) = E^{(1)}$ is the unique s-extremal lattice. For $µ = 2$, these lattices are already classified in [5]. The possibilities for $b = k$ are $8, 12, 14 \leq b \leq 22$. For $µ = 4$, the possibilities are either $b = 0$ and $k = 24$ or $8 \leq b \leq 23$ whence $32 \leq k \leq 47$ since the other shadow series have non-integral coefficients. The lattices $L_1(4,32)$ are classified in [3]. For $µ = 6$ no such lattices are known. The first possible dimension is 56, since the other shadow series have non-integral coefficients.

Since for odd $N$ the value $µ_-(N, 0) = 6$ and the s-extremal lattices of minimum 4 with $k = s(N)$ are even and hence isometric to $E^{(N)}$ we obtain the following theorem.
Theorem 3.2. Let $L$ be an extremal and $s$-extremal lattice rational equivalent to $C_N^k$ for some $N \in \mathcal{L}$ such that $k$ is a multiple of $s(N)$. Then $\mu := \min(L)$ is even and $k = s(N)(\mu - 2)/2$ and either $\mu = 4$, $N$ is odd and $L = E^{(N)}$ or $\mu = 6$, $N = 14$ and $L = L_{14}(6, 2)$.

For $N \in \{11, 14, 15, 23\}$ the complete classification of $s$-extremal strongly $N$-modular lattices in the genus of $C_N^k$ is as follows:

<table>
<thead>
<tr>
<th>N</th>
<th>23</th>
<th>15</th>
<th>14</th>
<th>11</th>
<th>11</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>k</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>lattice</td>
<td>$E^{(23)}$</td>
<td>$E^{(15)}$</td>
<td>$E^{(14)}$</td>
<td>$L_{11}(2, 1)$</td>
<td>$E^{(11)}$</td>
<td>$L_{11}(4, 3)$</td>
</tr>
</tbody>
</table>

For the remaining $N \in \mathcal{L}$, the results are summarized in the following tables. The last line, labelled with #, displays the number of lattices, where we display $-$ if there is no such lattice, $?$ if we do not know such a lattice, $+$ if there is a lattice, but the lattices are not classified. We always write $k = \ell s(N) + b$ with $0 \leq b \leq s(N) - 1$ such that $\mu = \min(L) = 2\ell + 2$ by Theorem 3.1 and $\dim(L) = k\sigma_0(N)$.

$N = 7$, $s(N) = 3$, $k = \ell s(N) + b$

<table>
<thead>
<tr>
<th>b</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$\geq 2$</td>
<td>0 1 2 3 4 0 1 2 3 4 4</td>
</tr>
<tr>
<td>min</td>
<td>$\geq 4$</td>
<td>2 4 6 8 4 10 3 4 6 8 10</td>
</tr>
<tr>
<td>#</td>
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<td>1</td>
</tr>
</tbody>
</table>

$N = 6$, $s(N) = 2$, $k = \ell s(N) + b$

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$\geq 1$</td>
<td>0 1 2 $\leq \ell \leq 12$ 13</td>
</tr>
<tr>
<td>min</td>
<td>$\geq 4$</td>
<td>2 4 6 $\leq \mu \leq 26$ 28</td>
</tr>
<tr>
<td>#</td>
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<td>1</td>
</tr>
</tbody>
</table>

$N = 5$, $s(N) = 4$, $k = \ell s(N) + b$

<table>
<thead>
<tr>
<th>b</th>
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</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$\geq 2$</td>
<td>0 1 2 3 4 5</td>
</tr>
<tr>
<td>min</td>
<td>$\geq 6$</td>
<td>2 4 6 8 10 12</td>
</tr>
<tr>
<td>#</td>
<td>1</td>
<td>-</td>
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</table>

<table>
<thead>
<tr>
<th>b</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>0 1 2 $\leq \ell \leq 9$ 10 0 1 2 $\leq \ell \leq 10$ 11</td>
<td></td>
</tr>
<tr>
<td>min</td>
<td>2 4 6 $\leq \mu \leq 20$ 22 3 4 6 $\leq \mu \leq 22$ 24</td>
<td></td>
</tr>
<tr>
<td>#</td>
<td>1 3</td>
<td>-</td>
</tr>
</tbody>
</table>
\[ N = 3, \ s(N) = 6, \ k = \ell s(N) + b \]

<table>
<thead>
<tr>
<th>b</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>\ell</td>
<td>≥ 2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>\min</td>
<td>≥ 6</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
<tr>
<td>#</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

\[ N = 2, \ s(N) = 8, \ k = \ell s(N) + b \]

<table>
<thead>
<tr>
<th>b</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>\ell</td>
<td>≥ 1</td>
<td>≥ 1</td>
<td>0</td>
<td>1 ≤ \ell ≤ 25</td>
</tr>
<tr>
<td>\min</td>
<td>≥ 4</td>
<td>≥ 4</td>
<td>2</td>
<td>4 ≤ \mu ≤ 52</td>
</tr>
<tr>
<td>#</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>?</td>
</tr>
</tbody>
</table>

Grammatrices of the new s-extremal lattices:

\[ L_{14}(6, 2) = \begin{pmatrix} 6 & 3 & 0 & 2 & 3 & 3 & 1-2 \\ 3 & 6 & 3 & 2 & 3 & 3 & 3-2 \\ 0 & 3 & 6 & 0 & 3 & 2 & 2-3 \\ 2 & 2 & 0 & 6 & 2 & 1 & 1-3 \\ -3 & -3 & -3 & 2 & 6 & 3 & 3-3 \\ 3 & 3 & 2 & 1 & 3 & 7 & 4-2 \\ -1 & -3 & 2 & 1 & 3 & 4 & 7-1 \\ -2 & 2 & 3 & 3 & 3 & 2 & 1-7 \end{pmatrix}, \quad L_{11}(4, 3) = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 & 1-1 \\ 0 & 4 & 0 & 2 & 2 & 1-2 \\ 0 & 0 & 4 & 2 & 2 & 1-2 \\ 0 & 0 & 4 & 2 & 2 & 1-3 \\ 0 & 0 & 4 & 2 & 2 & 1-4 \\ 2 & 2 & 2 & 5 & 0 & 0-3 \\ 2 & 2 & 2 & 5 & 0 & 0-4 \\ 2 & 2 & 2 & 5 & 0 & 0-5 \\ 2 & 2 & 2 & 5 & 0 & 0-6 \end{pmatrix} \]

\[ L_{7}(4, 4) = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 & 2 & 2-1 \\ 0 & 0 & 4 & 2 & 2 & 1-2 \\ 0 & 0 & 4 & 2 & 2 & 1-3 \\ 0 & 0 & 4 & 2 & 2 & 1-4 \\ 0 & 0 & 4 & 2 & 2 & 1-5 \\ 0 & 0 & 4 & 2 & 2 & 1-6 \\ 0 & 0 & 4 & 2 & 2 & 1-7 \\ 0 & 0 & 4 & 2 & 2 & 1-8 \end{pmatrix}, \quad L_{7}(4, 5) = \begin{pmatrix} 4 & 0 & 0 & 2 & 1 & 1 & 1-1 \\ 0 & 4 & 0 & 2 & 1-2 \\ 0 & 4 & 0 & 2 & 1-3 \\ 0 & 4 & 0 & 2 & 1-4 \\ 0 & 4 & 0 & 2 & 1-5 \\ 0 & 4 & 0 & 2 & 1-6 \\ 0 & 4 & 0 & 2 & 1-7 \\ 0 & 4 & 0 & 2 & 1-8 \end{pmatrix} \]
References

S-extremal strongly modular lattices


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