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Résumé. Soit $E$ une courbe elliptique définie sur $\mathbb{F}_q$, le corps fini à $q$ éléments. Nous montrons que pour une constante $\eta > 0$ dépendant seulement de $q$, il existe une infinité d’entiers positifs $n$ tels que l’exposant de $E(\mathbb{F}_q^n)$, le groupe des points $\mathbb{F}_q^n$-rationnels sur $E$, est au plus $q^n \exp(-n^\eta/\log\log n)$. Il s’agit d’un analogue d’un résultat de R. Schoof sur l’exposant du groupe $E(\mathbb{F}_p)$ des points $\mathbb{F}_p$-rationnels, lorsqu’une courbe elliptique fixée $E$ est définie sur $\mathbb{Q}$ et le nombre premier $p$ tend vers l’infini.

Abstract. Let $E$ be an elliptic curve defined over $\mathbb{F}_q$, the finite field of $q$ elements. We show that for some constant $\eta > 0$ depending only on $q$, there are infinitely many positive integers $n$ such that the exponent of $E(\mathbb{F}_q^n)$, the group of $\mathbb{F}_q^n$-rational points on $E$, is at most $q^n \exp(-n^\eta/\log\log n)$. This is an analogue of a result of R. Schoof on the exponent of the group $E(\mathbb{F}_p)$ of $\mathbb{F}_p$-rational points, when a fixed elliptic curve $E$ is defined over $\mathbb{Q}$ and the prime $p$ tends to infinity.

1. Introduction

Let $E$ be an elliptic curve defined over $\mathbb{F}_q$, the finite field of $q$ elements, where $q$ is a prime power, defined by a Weierstrass equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$ 

We consider extensions $\mathbb{F}_q^n$ of $\mathbb{F}_q$ and, accordingly, we consider the sets $E(\mathbb{F}_q^n)$ of $\mathbb{F}_q^n$-rational points on $E$ (including the point at infinity $\mathcal{O}$).

We recall that $E(\mathbb{F}_q^n)$ forms an abelian group (with $\mathcal{O}$ as the identity element). The cardinality $\#E(\mathbb{F}_q^n)$ of this group satisfies the Hasse–Weil inequality

$$|\#E(\mathbb{F}_q^n) - q^n - 1| \leq 2q^{n/2}$$ 

(see [2, 13, 14] for this, and other general properties of elliptic curves).

It is well-known that the group of $\mathbb{F}_q^n$-rational points $E(\mathbb{F}_q^n)$ is of the form

$$\mathcal{E}(\mathbb{F}_q^n) \cong \mathbb{Z}/L\mathbb{Z} \times \mathbb{Z}/M\mathbb{Z},$$
where the integers $L$ and $M$ are uniquely determined with $M \mid L$. In particular, $\#E(\mathbb{F}_{q^n}) = LM$. The number $\ell(q^n) = L$ is called the exponent of $E(\mathbb{F}_{q^n})$, and is the largest possible order of points $P \in E(\mathbb{F}_{q^n})$.

Trivially, from the definition (1.2), and from the equation (1.1), we see that the inequality

$$\ell(q^n) \geq \left(\#E(\mathbb{F}_{q^n})\right)^{1/2} \geq (q^n + 1 - 2q^{n/2})^{1/2} = q^{n/2} - 1$$

holds for all $q$ and $n$.

For a fixed elliptic curve $E$ which is defined over $\mathbb{Q}$ that admits no complex multiplication, it has been shown by Schoof [11] that the inequality

$$\ell(p) \geq C(E)^{1/2} \log p \log \log p$$

holds for all prime numbers $p$ of good reduction, where the constant $C(E) > 0$ depends only on the curve $E$.

Duke [7], has recently shown, unconditionally for elliptic curves with complex multiplication, and under the Extended Riemann Hypothesis for elliptic curves without complex multiplication, that for any function $f(x)$ that tends to infinity as $x$ tends to infinity, the lower bound $\ell(p) \geq p/f(p)$ holds for almost all primes $p$. However, for elliptic curves without complex multiplication, the only unconditional result available is also in [7], and asserts that the weaker inequality $\ell(p) \geq p^{3/4}/\log p$ holds for almost all primes $p$. It has also been shown in [11], that, under the Extended Riemann Hypothesis, for any curve $E$ over $\mathbb{Q}$,

(1.3) \[ \liminf_{p \to \infty} \frac{\ell(p)}{p^{7/8} \log p} < \infty \]

where $p$ runs through prime numbers. This bound rests on an explicit form of the Chebotarev Density Theorem. Accordingly, unconditional results of [9] lead to an unconditional, albeit much weaker, upper bound on $\ell(p)$.

In extension fields of $\mathbb{F}_q$, with $E$ defined over $\mathbb{F}_q$, stronger lower bounds on $\ell(q^n)$ can be obtained. For example, it has recently been shown in [10] that for any $\varepsilon > 0$, the inequality $\ell(q^n) \leq q^{n(1-\varepsilon)}$ holds only for finitely many values of $n$. In particular, this means that no result of the same strength as (1.3) is possible for elliptic curves in extension fields. Accordingly, here we obtain a much more modest bound which asserts that for some positive constant $\eta > 0$ depending only on $q$,

(1.4) \[ \liminf_{n \to \infty} \frac{\ell(q^n)}{q^n \exp \left(-n^{\eta/\log \log n}\right)} < \infty. \]

The question of cyclicity, that is, whether $\ell(q^n) = \#E(\mathbb{F}_{q^n})$, has also been addressed in the literature. For curves in extension fields, this question has
been satisfactorily answered by Vlăduţ [16]. In the situation where \( E \) is defined over \( \mathbb{Q} \), the question about the cyclicity of the reduction \( E(\mathbb{F}_p) \) when \( p \) runs over the primes appears to be much harder (see [4, 5, 6] for recent advances and surveys of other related results). In particular, this problem is closely related to the famous \textit{Lang–Trotter conjecture}.

Finally, one can also study an apparently easier question about the distribution of \( \ell(q) \) “on average” over various families of elliptic curves defined over \( \mathbb{F}_q \) (see [12, 15]).

Throughout this paper, all the explicit and implied constants in the symbol ‘\( O \)’ may depend only on \( q \). For a positive real number \( z > 0 \), we write \( \log z \) for the maximum between 1 and the natural logarithm of \( z \).

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2. The field of definition of torsion points

Let \( \overline{\mathbb{F}}_q \) be the algebraic closure of \( \mathbb{F}_q \). Given an elliptic curve \( E \) over \( \mathbb{F}_q \), the points \( P \in E(\overline{\mathbb{F}}_q) \) with \( kP = O \) for some fixed integer \( k \geq 1 \), form a group, which is called the \( k \)-torsion group and denoted by \( E[k] \). If \( \gcd(k, q) = 1 \), then

\[
E[k] \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}.
\]

Henceforth, we assume that \( \gcd(k, q) = 1 \), so that (2.1) holds. Let \( \mathbb{K}_k \) be the field of definition of \( E[k] \), that is the field generated by the coordinates of all the \( k \)-torsion points, and let \( d(k) \) denote the degree of \( \mathbb{K}_k \) over \( \mathbb{F}_q \). Then \( \mathbb{K}_k \) is a Galois extension of \( \mathbb{F}_q \). Let \( G_k \) denote the Galois group of this extension. Having chosen generators \( P_1, P_2 \) for the \( k \)-torsion group, one gets a representation of \( G_k \) as a subgroup of \( \text{GL}_2(\mathbb{Z}/k\mathbb{Z}) \): any element of \( G_k \) maps each \( P_i \) to a \( (\mathbb{Z}/k\mathbb{Z}) \)-linear combination of \( P_1 \) and \( P_2 \) for \( i = 1, 2 \).

Although the following statement does not seem to appear in the literature, it is based on an approach which is not new. For example, for the \( \text{PGL}_2 \) analogue, see Proposition VII.2 of [2].

**Lemma 2.1.** Let \( t = q+1-\#E(\mathbb{F}_q) \). If \( r \) is a prime with \( \gcd(r, q(t^2-4q)) = 1 \) and such that \( t^2-4q \) is a quadratic residue modulo \( r \), then \( d(r) \mid (r-1) \).

**Proof.** Since \( r \) does not divide \( q \), \( E[r] \cong \mathbb{F}_r \times \mathbb{F}_r \), and the above Galois representation exhibits \( G_r \) as a subgroup of \( \text{GL}_2(\mathbb{F}_r) \). Since \( \mathbb{F}_q \) is a finite
field, $G_r$ is cyclic, generated by the Frobenius map $\tau(\vartheta) = \vartheta^q$. Let $A \in \text{GL}_2(F_r)$ correspond to $\tau$. Now $d(r)$ is the order of $A$ in $\text{GL}_2(F_r)$.

If $A$ is a scalar multiple of the identity matrix, then it has order dividing $r - 1$. Otherwise, the characteristic polynomial of $A$ equals its minimal polynomial. Since the relation $\tau^2 - t\tau + q = 0$ holds in the endomorphism ring, we have $A^2 - tA + qI = 0$ over $F_r$, and this must be the minimal polynomial of $A$. Since $t^2 - 4q$ is a quadratic residue in $F_r$, $A$ has two distinct eigenvalues in $F_r$, from which the result follows immediately. $\square$

We remark that without the condition that $t^2 - 4q$ is a quadratic residue modulo $r$, similar arguments imply that the relation $d(r) \mid (r^2 - 1)$ holds for any prime $r$ with $\gcd(r, q(t^2 - 4q)) = 1$.

3. Main result

Lemma 2.1 immediately implies that $\ell(q^n) = O(q^n n^{-1})$ infinitely often (namely for each $n = d(r)$, where $r$ is a prime with $\gcd(r, q(t^2 - 4q)) = 1$ and such that $t^2 - 4q$ is a quadratic residue modulo $r$). Here, we prove a much stronger bound.

**Theorem 3.1.** There exists a positive constant $\eta > 0$ such that for infinitely many pairs of positive integers $(m, n)$ we have $E[m] \subseteq E(F_{q^n})$ and

$$m \geq \exp\left(\frac{n^\eta}{\log \log n}\right).$$

**Proof.** We let $\Delta = 4(t^2 - 4q)$ and we show that there exists a constant $\kappa > 0$ such for any sufficiently large $x$ there exists a set of primes $R$ such that each $r \in R$ has the properties that

$$\gcd(r, q) = 1 \quad \text{and} \quad r \equiv 1 \pmod{\Delta},$$

and also that

$$\#R \geq \exp(\kappa \log x / \log \log x) \quad \text{and} \quad \text{lcm}\{r - 1 \mid r \in R\} \leq x^2.$$

We follow closely the proof of Proposition 10 of [1]. However, we replace the condition of $r - 1$ being squarefree by the conditions (3.1). Namely, let $k_0$ be the integer of Proposition 8 of [1]. Assuming that $x$ is sufficiently large, as in Proposition 10 of [1], we define $k_1$ as the product of all primes up $0.5\delta \log x$ for a sufficiently small positive constant $\delta$. We now put $k_2 = k_1 / \gcd(k_1, \Delta)$ and finally $k = k_1 / P(\gcd(k_0, k_2))$. It is clear that $k_0 \nmid \Delta k$ (note that we have not imposed the squarefreeness condition, and thus we do not need the condition $k_0 \nmid 2\Delta k$ to hold, as in [1]). For each $d \mid k$, we denote by $A_d$ the number pairs $(m, r)$ consisting of a positive integer $m \leq x$ and a prime $r \leq x$, with

$$\gcd(r, q) = 1 \quad \text{and} \quad \gcd(m, k) = k/d,$$
and which satisfy the system of congruences
\[ m(r - 1) \equiv 0 \pmod{k} \quad \text{and} \quad r \equiv 1 \pmod{\text{lcm}(\Delta, d)}. \]
As in [1], we derive that for some constant \( C > 0 \), the inequality
\[ A_d \geq C \frac{x^2 \varphi(d)}{dk \log x} \]
holds uniformly in \( d \), where \( \varphi(d) \) is the Euler function. Repeating the same steps as in the proof of Proposition 10 of [1], we obtain the desired set \( \mathcal{R} \) satisfying (3.1) and (3.2). It is clear that \( t^2 - 4q \) is a quadratic residue modulo every \( r \in \mathcal{R} \) and thus, by Lemma 2.1, the relation \( d(r) \mid (r - 1) \) holds for all \( r \in \mathcal{R} \).

We now define
\[ m = \prod_{r \in \mathcal{R}} r \quad \text{and} \quad n = \text{lcm}\{r - 1 \mid r \in \mathcal{R}\}. \]
Since, \( E[r] \subseteq E(\mathbb{F}_{q^n}) \) holds for every \( r \in \mathcal{R} \), it follows that \( E[m] \subseteq E(\mathbb{F}_{q^n}) \).

We now derive, from (3.2), that \( n \leq x^2 \), and using the Prime Number Theorem, we get
\[ m \geq \exp\left(((1 + o(1))\#\mathcal{R}) \geq \exp\left((1 + o(1)) \exp(\kappa \log x / \log \log x)\right)\right), \]
which finishes the proof.

It is now clear that Theorem 3.1 implies relation (1.4).

4. Applications to Lucas sequences

Let \( u_n = (\alpha^n - \beta^n)/(\alpha - \beta) \) be a Lucas sequence, where \( \alpha \) and \( \beta \) are roots of the characteristic polynomial \( f(X) = X^2 + AX + B \in \mathbb{Z}[X] \). Then the arguments of the proof of Theorem 3.1 show that there are many primes \( r \) such that \( A^2 - 4B \) is a quadratic residue modulo \( r \) and the least common multiple of all the \( r - 1 \) is small. In a quantitative form this implies that, for infinitely many positive integers \( n \),
\[ \omega(u_n) \geq n^{\eta/\log \log n} \]
for some positive constant \( \eta > 0 \), where \( \omega(u) \) is the number of distinct prime divisors of an integer \( u \geq 2 \).

Moreover, given \( s \geq 2 \) Lucas sequences \( u_{i,n}, i = 1, \ldots, s \), one can use the same arguments to show that, for infinitely many positive integers \( n \),
\[ \omega(\gcd(u_{1,n}, \ldots, u_{s,n})) \geq n^{\eta/\log \log n}. \]

This generalises and refines a remark made in [3]. In particular, we see that for any integers \( a > b > 1 \), the result of [1] immediately implies that
\[ \gcd(a^n - 1, b^n - 1) \geq \exp\left(n^{\eta/\log \log n}\right) \]
inexhaustibly often (which shows that the upper bound of [3] is rather tight).

References


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