JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Prerna BIHANI et Renling JIN

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Tome 18, n^{o} 2 (2006), p. 323-343.

<http://jtnb.cedram.org/item?id=JTNB_2006__18_2_323_0>

Journal de Théorie des Nombres
de Bordeaux **18** (2006), 323-343

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par Prerna BIHANI et Renling JIN

Résumé. Supposons que A soit un ensemble d'entiers non négatifs avec densité de Banach supérieure α (voir définition plus bas) et que la densité de Banach supérieure de $A + A$ soit inférieure à 2α. Nous caractérisons la structure de $A + A$ en démontrant la proposition suivante : il existe un entier positif g et un ensemble W qui est l'union des $[2\alpha g - 1]$ suites arithmétiques[1] avec la même différence g tels que $A + A \subseteq W$ et si $[a_n, b_n]$ est, pour chaque n, un intervalle d'entiers tel que $b_n - a_n \to \infty$ et la densité relative de A dans $[a_n, b_n]$ approche α, il existe alors un intervalle $[c_n, d_n] \subseteq [a_n, b_n]$ pour chaque n tel que $(d_n - c_n)/(b_n - a_n) \to 1$ et $(A + A) \cap [2c_n, 2d_n] = W \cap [2c_n, 2d_n]$.

Abstract. Suppose A is a set of non-negative integers with upper Banach density α (see definition below) and the upper Banach density of $A + A$ is less than 2α. We characterize the structure of $A+A$ by showing the following: There is a positive integer g and a set W, which is the union of $\lceil 2\alpha g - 1 \rceil$ arithmetic sequences[1] with the same difference g such that $A + A \subseteq W$ and if $[a_n, b_n]$ for each n is an interval of integers such that $b_n - a_n \to \infty$ and the relative density of A in $[a_n, b_n]$ approaches α, then there is an interval $[c_n, d_n] \subseteq [a_n, b_n]$ for each n such that $(d_n - c_n)/(b_n - a_n) \to 1$ and $(A + A) \cap [2c_n, 2d_n] = W \cap [2c_n, 2d_n]$.

1. Introduction

Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{N}$ be the set of all non-negative integers. Capital letters A, B, C, and D are always used for sets of integers and lower case letters a, b, c, d, e, g, h, etc. are always used for integers. Greek letters α, β, γ, and ϵ are reserved for standard real numbers. For any $a \leqslant b$ we write $[a, b]$ exclusively for the interval of all integers between a and

Manuscrit reçu le 22 septembre 2004.
Mots clefs. Upper Banach density, inverse problem, nonstandard analysis.
Classification math.. 11B05, 11B13, 11U10, 03H15.
The research of both authors was supported in part by the 2003 summer undergraduate research fund from the College of Charleston. The second author was also supported in part by the 2004 summer research fund from the College of Charleston. The first author is currently a graduate student. The second author is the main contributor of the paper.

[1] We call a set of the form $a + d\mathbb{N}$ an arithmetic sequence of difference d and call a set of the form $\{a, a + d, a + 2d, \ldots, a + kd\}$ an arithmetic progression of difference d. So an arithmetic progression is finite and an arithmetic sequence is infinite.

b including a and b. We denote $A \pm B$ for the set $\{a \pm b : a \in A \,\&\, b \in B\}$. We write $A \pm a$ for $A \pm \{a\}$ and $a \pm A$ for $\{a\} \pm A$. We denote kA for the set $\{ka : a \in A\}$.[1] Let $A[a,b]$ denote the set $A \cap [a,b]$ and $A(a,b)$ denote the cardinality of $A[a,b]$.

For any set A the upper Banach density of A, denoted by $BD(A)$, is defined by

$$BD(A) = \lim_{n\to\infty} \sup_{k\geqslant 0} \frac{A(k,k+n)}{n+1}.$$

Clearly $0 \leqslant BD(A) \leqslant 1$. It is easy to see that $\alpha = BD(A)$ iff α is the greatest real number satisfying that there is a sequence of intervals $\{[a_n, b_n] : n \in \mathbb{N}\}$ such that

$$\text{(I)} \qquad \lim_{n\to\infty} (b_n - a_n) = \infty \text{ and } \lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha.$$

The following is the main theorem of the paper.

Theorem 1.1. *Let A be a set of non-negative integers such that $BD(A) = \alpha$ and $BD(A+A) < 2\alpha$. Let $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a sequence of intervals satisfying (I). Then there are $g \in \mathbb{N}$, $G \subseteq [0, g-1]$, and $[c_n, d_n] \subseteq [a_n, b_n]$ for each $n \in \mathbb{N}$ such that*

(1) $|G| = m = \lceil 2\alpha g - 1 \rceil$,
(2) $\lim_{n\to\infty} \frac{d_n - c_n}{b_n - a_n} = 1$,
(3) $A + A \subseteq G + g\mathbb{N}$,
(4) $(A+A) \cap [2c_n, 2d_n] = (G + g\mathbb{N}) \cap [2c_n, 2d_n]$ *for all* $n \in \mathbb{N}$,
(5) $BD(A+A) = \frac{m}{g} \geqslant 2\alpha - \frac{1}{g}$.

Remark. (1) *Since the upper Banach density of A is achieved in a sequence of intervals $\{[a_n, b_n] : n \in \mathbb{N}\}$ satisfying (I), it is natural to characterize the structure of $(A+A) \cap [2a_n, 2b_n]$ because the part of A outside of those intervals $[a_n, b_n]$ may have nothing to do with the condition $BD(A+A) < 2BD(A)$.*

(2) *The interval $[c_n, d_n]$ in (4) of Theorem 1.1 cannot be replaced by $[a_n, b_n]$ because if we delete, for example, the first and the last k numbers for a fixed k from $A[a_n, b_n]$ for every $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \alpha$$

is still true.

(3) *Due to the asymptotic nature of the upper Banach density the asymptotic characterization of the length of $[c_n, d_n]$ in (2) of Theorem 1.1 is the best we can do.*

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[1] In some literature kA represents the k-fold sum of A. Since only the sum of two sets is considered in this paper, we would like to write $A + A$ instead of $2A$ so that the term $g\mathbb{N}$ can be reserved for the set of all multiples of g without ambiguity.

Theorem 1.1 is motivated by Kneser's Theorem for lower asymptotic density. For a set A the lower asymptotic density of A, denoted by $\underline{d}(A)$, is defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n}.$$

The lower asymptotic density is one of the densities that are of classical interest for many number theorists. Another classical density is Shnirel'man density denoted by σ (see [4] or [11] for definition). It is often the case that a theorem about the Shnirel'man density is obtained first and then a parallel theorem about the lower asymptotic density is explored. In early 1940's, H. B. Mann proved a celebrated theorem, which says that if $0 \in A \cap B$, then $\sigma(A+B) \geqslant \min\{\sigma(A)+\sigma(B), 1\}$. Is the inequality true if the Shnirel'man density is replaced by the lower asymptotic density? The answer is no and the following is a trivial counterexample. We call that two sets A and B are eventually same, denoted by $A \sim B$, if $A \smallsetminus [0,m] = B \smallsetminus [0,m]$ for some $m \in \mathbb{N}$.

Example. *Let $k, k' > 0$ and $g \geqslant k+k'$. Let $A \subseteq [0,k-1]+g\mathbb{N}$ and $B \subseteq [0,k'-1]+g\mathbb{N}$ be such that $\underline{d}(A)+\underline{d}(B) > \frac{k+k'-1}{g}$. Then $A+B \subseteq [0,k+k'-2]+g\mathbb{N}$, $A+B \sim [0,k+k'-2]+g\mathbb{N}$, and $\underline{d}(A)+\underline{d}(B)-\frac{1}{g} \leqslant \underline{d}(A+B) < \underline{d}(A)+\underline{d}(B)$.*

However, Example 1 is basically the only reason that the inequality $\underline{d}(A+B) < \underline{d}(A)+\underline{d}(B)$ can be true. More precisely Kneser proved the following theorem.

Theorem 1.2 (M. Kneser, 1953). *If $\underline{d}(A+B) < \underline{d}(A)+\underline{d}(B)$, then there are $g > 0$ and $G \subseteq [0,g-1]$ with $|G| = m$ such that*

(1) $A+B \subseteq G+g\mathbb{N}$,
(2) $A+B \sim G+g\mathbb{N}$, *and*
(3) $\underline{d}(A+B) = \frac{m}{g} \geqslant \underline{d}(A)+\underline{d}(B)-\frac{1}{g}$.

For any A and B with $\underline{d}(A+B) < \underline{d}(A)+\underline{d}(B)$ we define

$$\text{(II)} \qquad g_{A,B} = \min\{g \in \mathbb{N} : \text{There is a } G \subseteq [0,g-1] \text{ satisfying (1)–(3) of Theorem 1.2.}\}$$

The original version of Kneser's Theorem deals with the sum of multiple sets. For brevity and the purpose of this paper we stated only a version for the sum of two sets.

In [6] the second author discovered a general scheme, with the help of nonstandard analysis, how one can derive a theorem about the upper Banach density parallel to each existing theorem about the Shnirel'man density or the lower asymptotic density. The results in [6] can also be obtained using an ergodic approach in symbolic dynamics (see [7] or [3]). In

[7, Theorem 3.8] the second author derived a result about the upper Banach density, which is parallel to Kneser's Theorem. The result characterizes the structure of a very small portion of the sumset, i.e., in [7] one can only have $\lim_{n\to\infty}(d_n - c_n) = \infty$ in the place of (2) of Theorem 1.1. We believe that the general scheme in [6] and the ergodic methods in [7] are not sufficient for proving Theorem 1.1. In the following few sections we develop stronger standard lemmas and extract more strength from nonstandard methods so that we can have enough tools for proving Theorem 1.1.

2. Standard lemmas

In this section we first state some existing theorems as lemmas and then prove some new lemmas without involving nonstandard analysis.

Lemma 2.1 (Birkhoff Ergodic Theorem)**.** *Suppose* (Ω, Σ, μ) *is a probability space and* T *is a measure-preserving transformation from* Ω *to* Ω*. For any function* $f \in L^1(\Omega)$*, there exists a function* $\bar{f} \in L^1(\Omega)$ *such that*

$$\mu(\{x \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m(x)) = \bar{f}(x)\}) = 1.$$

Birkhoff ergodic theorem can be found in [3, p.59] or in [12, p.30].

Lemma 2.2 (M. Kneser, 1953)**.** *Let* G *be an Abelian group and* A, B *be finite subsets of* G*. Let* $S = \{g \in G : g + A + B = A + B\}$ *be the stabilizer of* $A + B$*. If* $|A + B| < |A| + |B|$*, then* $|A + B| = |A + S| + |B + S| - |S|$*.*

Note that the stabilizer is always a subgroup of G and if $|A + B| < |A| + |B| - 1$, then the stabilizer of $A + B$ is non-trivial, i.e., $|S| > 1$. Lemma 2.2 can be found in [11, p.115].

For a positive integer g let $\mathbb{Z}/g\mathbb{Z}$ be the additive group of integers modulo g. Let $\pi_g : \mathbb{Z} \mapsto \mathbb{Z}/g\mathbb{Z}$ be the natural homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}/g\mathbb{Z}$. If $d > 0$ and $d|g$, then we write $\pi_{g,d} : \mathbb{Z}/g\mathbb{Z} \mapsto \mathbb{Z}/d\mathbb{Z}$ for the natural homomorphism from $\mathbb{Z}/g\mathbb{Z}$ onto $\mathbb{Z}/d\mathbb{Z}$. Let $\langle d\rangle_g$ be the kernel of $\pi_{g,d}$, which is a subgroup of $\mathbb{Z}/g\mathbb{Z}$. Note that every non-trivial subgroup of $\mathbb{Z}/g\mathbb{Z}$ has the form $\langle d\rangle_g$ for some proper factor d of g. We now state another version of Theorem 1.2, which is convenient to use later.

Lemma 2.3. *Suppose* $\underline{d}(A) = \underline{d}(B) = \alpha$ *and* $\underline{d}(A + B) < 2\alpha$*. Let* $g = g_{A,B}$ *be as defined in (II) and let* $G \subseteq [0, g - 1]$ *with* $|G| = m$ *be such that (1)–(3) of Theorem 1.2 are true. Then there are* $F, F' \subseteq [0, g - 1]$ *with* $|F| = |F'| = k$ *such that*

(1) $A \subseteq F + g\mathbb{N}$ *and* $B \subseteq F' + g\mathbb{N}$*,*
(2) $2k - 1 = m$*,*
(3) $\alpha > \frac{k}{g} - \frac{1}{2g}$*.*

Furthermore, for any $a \in [0, g-1] \setminus F$, there is a $b \in F'$ such that $a + b \notin G + g\mathbb{N}$.

Proof: Let $F, F' \subseteq [0, g-1]$ be the minimal sets such that $A \subseteq F + g\mathbb{N}$ and $B \subseteq F' + g\mathbb{N}$, respectively. Let $k = |F|$ and $k' = |F'|$. By the minimality of F and F' we have that $F + F' + g\mathbb{N} \sim G + g\mathbb{N}$ and hence $\pi_g[F + F'] = \pi_g[G]$.

If $k + k' \geqslant m + 2$, then $|\pi_g[F] + \pi_g[F']| < k + k' - 1$. By Lemma 2.2 the stabilizer S of $\pi_g[F] + \pi_g[F']$ is non-trivial. Let $S = \langle d \rangle_g$ for some proper factor d of g and $|S| = g/d = s > 1$. Then

$$\pi_g[A + B] = \pi_g[G] = \pi_g[F + F'] = \pi_g[G] + S.$$

This implies that

$$A + B \sim G + g\mathbb{N} \sim G + \{0, d, \ldots, (s-1)d\} + g\mathbb{N} \sim G' + d\mathbb{N}$$

where $G' \subseteq [0, d-1]$ and $\pi_d[G] = \pi_d[G']$.

We also have that

$$\underline{d}(A+B) = \frac{|G|}{g} = \frac{|\pi_g[G] + S|}{g} = \frac{|G'|}{d} \geqslant \underline{d}(A) + \underline{d}(B) - \frac{1}{g} > \underline{d}(A) + \underline{d}(B) - \frac{1}{d}.$$

Hence (1)–(3) of Theorem 1.2 are true when g is replaced by d. This contradicts the minimality of $g = g_{A,B}$.

If $k + k' \leqslant m$, then $\underline{d}(A + B) = \frac{m}{g} \geqslant \frac{k}{g} + \frac{k'}{g} \geqslant \underline{d}(A) + \underline{d}(B)$, which contradicts the assumption of the lemma.

Hence we can assume that $k + k' - 1 = m$.

If $k \neq k'$, say $k < k'$, then

$$2\alpha > \underline{d}(A + B) = \frac{m}{g} = \frac{k + k' - 1}{g} \geqslant \frac{2k}{g} \geqslant 2\underline{d}(A) = 2\alpha,$$

which is absurd. Hence we have $k = k'$, which implies $2\alpha > \underline{d}(A+B) = \frac{2k-1}{g}$ and $\alpha > \frac{k}{g} - \frac{1}{2g}$.

Let $a \in [0, g-1] \setminus F$. If $a + F' \subseteq G + g\mathbb{N}$, then $|\pi_g[F \cup \{a\}] + \pi_g[F']| = m = 2k - 1 < |\pi_g[F \cup \{a\}]| + |\pi_g[F']| - 1$. By Lemma 2.2 the stabilizer of $\pi_g[G]$ is non-trivial. This contradicts the minimality of g. □(Lemma 2.3)

Lemma 2.4. *Let $g > 0$ and $F \subseteq [0, g-1]$ with $|F| = k$. If $A \subseteq F + g\mathbb{N}$ and $\underline{d}(A) = \alpha > \frac{k}{g} - \frac{1}{2g}$, then for any $A_a = A \cap (a + g\mathbb{N})$ where $a \in F$ we have $\underline{d}(A_a) > \frac{1}{2g}$.*

Proof: Let $\bar{A} = A \smallsetminus A_a$. Then $\bar{A} \subseteq (F \smallsetminus \{a\}) + g\mathbb{N}$. Hence

$$\underline{d}(A_a) \geqslant \underline{d}(A) - \frac{k-1}{g} = \alpha - \frac{k-1}{g} > \frac{k}{g} - \frac{1}{2g} - \frac{k-1}{g} = \frac{1}{2g}.$$

□(Lemma 2.4)

Lemma 2.5. *Let $A, B \subseteq g\mathbb{N}$ be such that $\underline{d}(A) > \frac{1}{2g}$ and $\underline{d}(B) > \frac{1}{2g}$. Then $A + B \sim g\mathbb{N}$.*

Proof: Let $m_0 \in \mathbb{N}$ be such that for any $gn > m_0$, $\frac{A(0,gn)}{gn+g} > \frac{1}{2g}$ and $\frac{B(0,gn)}{gn+g} > \frac{1}{2g}$. Given any $gn > m_0$, both $A[0, gn]$ and $ng - B[0, gn]$ are subsets of $[0, gn] \cap g\mathbb{N}$. Since $|[0, gn] \cap g\mathbb{N}| = n+1$ and $A(0, gn) + B(0, gn) > n+1$, then

$$A[0, gn] \cap (gn - B[0, gn]) \neq \emptyset,$$

which implies $gn \in A[0, gn] + B[0, gn] \subseteq A + B$. Hence $A + B \sim g\mathbb{N}$. □(Lemma 2.5)

Suppose $A, B \subseteq \mathbb{N}$, $\underline{d}(A) = \underline{d}(B) = \alpha$, $g > 0$, and $F, F' \subseteq [0, g-1]$ with $|F| = |F'| = k$ such that (1) and (3) of Lemma 2.3 hold true, then $A + B \sim F + F' + g\mathbb{N}$ by Lemma 2.4 and Lemma 2.5.

Note that in Lemma 2.3, the sets F, F' and the number k are uniquely determined by $g_{A,B}$.

Lemma 2.6. *Let $\underline{d}(A) = \underline{d}(B) = \alpha$ and $\underline{d}(A+B) < 2\alpha$. Let $g = g_{A,B}$ be as defined in (II) and let $G \subseteq [0, g-1]$ with $|G| = m$ be as defined in Theorem 1.2. Let $b \in B$ and $B_b = B \cap (b + g\mathbb{N})$. Then $\underline{d}(B_b) + \frac{m}{g} \geqslant 2\alpha$.*

Proof: Let $k = \frac{m+1}{2}$. By Lemma 2.3, there are $F, F' \subseteq [0, g-1]$ such that $|F| = |F'| = k$, $A \subseteq F + g\mathbb{N}$, and $B \subseteq F' + g\mathbb{N}$. Let $\underline{d}(B_b) = \beta$. Clearly $\beta \leqslant \frac{1}{g}$. Since $(B \smallsetminus B_b)$ is a subset of the union of $k-1$ arithmetic sequences of difference g, then

$$\alpha = \underline{d}(B) = \underline{d}(B_b \cup (B \smallsetminus B_b)) \leqslant \beta + \frac{k-1}{g},$$

which implies

$$\frac{k}{g} \geqslant \alpha - \beta + \frac{1}{g}.$$

Hence we have

$$\frac{2k-1}{g} \geqslant 2\alpha - 2\beta + \frac{1}{g}.$$

This now implies that

$$\underline{d}(B_b) + \frac{m}{g} = \beta + \frac{2k-1}{g} \geqslant \beta + 2\alpha - 2\beta + \frac{1}{g} \geqslant 2\alpha - \beta + \frac{1}{g} \geqslant 2\alpha.$$

□(Lemma 2.6)

Lemma 2.7. *Let $\underline{d}(A) = \underline{d}(B) = \alpha$. Suppose there is a positive integer d such that*

(1) $|\pi_d[A]| \neq |\pi_d[B]|$ *and*
(2) $|\pi_d[A + B]| \geqslant 2\min\{|\pi_d[A]|, |\pi_d[B]|\}$.

Then $\underline{d}(A + B) \geqslant 2\alpha$.

Proof: Suppose d is the least positive integer satisfying the conditions of the lemma. Let $k = |\pi_d[A]|$ and $k' = |\pi_d[B]|$. Let $\pi_d[A] = \{u_1, u_2, \ldots, u_k\}$ and $\pi_d[B] = \{v_1, v_2, \ldots, v_{k'}\}$. For each $i = 1, 2, \ldots, k$ let $a_i = \min(A \cap \pi_d^{-1}(u_i))$ and for each $j = 1, 2, \ldots, k'$ let $b_j = \min(B \cap \pi_d^{-1}(v_j))$. Without loss of generality we assume $k < k'$.

Given any $\epsilon > 0$, we want to show that $\underline{d}(A + B) \geqslant 2\alpha - \epsilon$.

Let $t = \max(\{a_i : i = 1, \ldots, k\} \cup \{b_j : j = 1, \ldots, k'\})$. Let $m_0 > t$ be such that for every $m > m_0$,

$$\frac{A(0, m)}{m + 1 + t} > \alpha - \frac{\epsilon}{2} \text{ and } \frac{B(0, m)}{m + 1 + t} > \alpha - \frac{\epsilon}{2}.$$

Fix an $m > m_0$. For each $i = 1, \ldots, k$ let $A_i = A[0, m] \cap \pi_d^{-1}(u_i)$. Without loss of generality we can assume that $|A_1| \geqslant |A_2| \geqslant \cdots \geqslant |A_k|$. For each $i = 1, \ldots, k$ let $U_i = \{u_1, \ldots, u_i\}$.

Case 2.7.1 There is an $r \in [1, k]$ such that $|U_r + \pi_d[B]| \leqslant 2r - 1$.

We want to show that Case 2.7.1 is impossible. Let r be the least number satisfying the case. Since $|U_r + \pi_d[B]| \leqslant 2r - 1 < r + k' - 1$, by Lemma 2.2 the stabilizer S of $U_r + \pi_d[B]$ is non-trivial and

$$|U_r + \pi_d[B]| = |U_r + S| + |\pi_d[B] + S| - |S|.$$

Let $d' < d$ and $d'|d$ be such that $S = \langle d' \rangle_d$. Then $|S| = s = d/d' > 1$.

Suppose $|\pi_d[B] + S| > |U_r + S|$. Since $|U_r + S|$ and $|\pi_d[B] + S|$ are all multiples of s, then $|\pi_d[B] + S| - |S| \geqslant |U_r + S|$. Hence we have

$$|U_r + \pi_d[B]| \geqslant 2|U_r + S| \geqslant 2r,$$

which contradicts the assumption of the case.

From the arguments above we can assume that $|\pi_d[B] + S| \leqslant |U_r + S|$. Now we have

$$|\pi_{d'}[B]| = |\pi_d[B] + S|/s \leqslant |U_r + S|/s = |\pi_{d,d'}[U_r]|.$$

Suppose for each $u \in \{u_{r+1}, \ldots, u_k\}$ we have $u + \pi_d[B] \subseteq U_r + \pi_d[B]$. Then

$$|\pi_d[A + B]| = |U_r + \pi_d[B]| \leqslant 2r - 1 < 2k,$$

which contradicts (2) of the lemma. Hence there is a $u \in \{u_{r+1}, \ldots, u_k\}$ and there is a $v \in \pi_d[B]$ such that $u + v \notin U_r + \pi_d[B] = U_r + \pi_d[B] + S$.

This implies $\pi_{d,d'}(u+v) \notin \pi_{d,d'}[U_r] + \pi_{d'}[B]$ and $\pi_{d,d'}(u) \notin \pi_{d,d'}[U_r]$. Hence $|\pi_{d'}[A]| > |\pi_{d,d'}[U_r]| \geqslant |\pi_{d'}[B]|$ and

$$\begin{aligned}&|\pi_{d'}[A+B]| \\ &\quad \geqslant |\pi_{d,d'}[U_r \cup \{u\}] + \pi_{d'}[B]| \geqslant 1 + |\pi_{d,d'}[U_r] + \pi_{d'}[B]| \\ &\quad = |\pi_{d,d'}[U_r]| + |\pi_{d'}[B]| \geqslant 2|\pi_{d'}[B]|.\end{aligned}$$

Hence $d' < d$ is a positive integer, which satisfies (1) and (2) of the lemma. This contradicts the minimality of d. □(Case 2.7.1)

Case 2.7.2 For every $r \in [0,k]$, $|U_r + \pi_d[B]| \geqslant 2r$.

By the assumption of the case, we can select, inductively on i, the elements $w_{i,j} \in U_i$ and $z_{i,j} \in \pi_d[B]$ for $j = 1,2$ such that $\{w_{i,j} + z_{i,j} : i = 1,\ldots,k \,\&\, j = 1,2\}$ in $\mathbb{Z}/d\mathbb{Z}$ has the cardinality $2k$. For each $w_{i,j}$ and $z_{i,j}$ we have

$$|\pi_d^{-1}(w_{i,j} + z_{i,j}) \cap (A+B)[0,m+t]| \geqslant |A_i|$$

by the enumeration of U_k, the definition of t, and the choices of $w_{i,j}$'s. Hence

$$(A+B)(0,m+t) \geqslant \sum_{i=1}^{k} 2|A_i| = 2A(0,m).$$

This implies

$$\frac{(A+B)(0,m+t)}{m+1+t} \geqslant \frac{2A(0,m)}{m+1+t} \geqslant 2(\alpha - \frac{\epsilon}{2}) = 2\alpha - \epsilon.$$

□(Case 2.7.2)

Since Case 2.7.1 is impossible, then by Case 2.7.2 we have $\underline{d}(A+B) \geqslant 2\alpha - \epsilon$. Now the lemma follows because $\epsilon > 0$ is arbitrary. □(Lemma 2.7)

The following lemma might be considered as trivial.

Lemma 2.8. *Let g, g' be positive and $d = \gcd(g,g')$. Let $G \subseteq [0,g-1]$ and $G' \subseteq [0,g'-1]$ be such that $X = G + g\mathbb{Z} = G' + g'\mathbb{Z}$. Then there is an $F \subseteq [0,d-1]$ such that $X = F + d\mathbb{Z}$.*

Proof: Let $F \subseteq [0,d-1]$ be minimal such that $X \subseteq F + d\mathbb{Z}$. It suffices to show $F + d\mathbb{Z} \subseteq X$.

Let $s,t \in \mathbb{Z}$ be such that $d = sg + tg'$. Given any $c \in F$ and $n \in \mathbb{Z}$, by the minimality of F we can find $e \in G$ and $k \in \mathbb{Z}$ such that $c + kd = e$. Hence $c + nd = e + (n-k)d = e + (n-k)sg + (n-k)tg'$. Clearly $e + (n-k)sg \in G + g\mathbb{Z}$. Hence $e + (n-k)sg \in G' + g'\mathbb{Z}$. This implies $e + (n-k)sg + (n-k)tg' \in G' + g'\mathbb{Z}$. Therefore, $c + nd \in X$. □(Lemma 2.8)

Corollary 2.1. *Let g, g' be positive and $d = \gcd(g,g')$. Let $G \subseteq [0,g-1]$ and $G' \subseteq [0,g'-1]$. If $A \sim G + g\mathbb{N}$ and $A \sim G' + g'\mathbb{N}$, then there is an $F \subseteq [0,d-1]$ such that $A \sim F + d\mathbb{N}$.*

Proof: Since $G + g\mathbb{N} \sim G' + g'\mathbb{N}$, then $G + g\mathbb{Z} = G' + g'\mathbb{Z}$. Hence by Lemma 2.8 we have $G + g\mathbb{Z} = G' + g'\mathbb{Z} = F + d\mathbb{Z}$ for some $F \subseteq [0, d-1]$. This implies $A \sim F + d\mathbb{N}$. □(Corollary 2.1)

We now prove a lemma, which may be interesting for its own sake.

Lemma 2.9. *Suppose $\underline{d}(A) = \underline{d}(B) = \alpha$, $\underline{d}(A + A) < 2\alpha$, and $\underline{d}(A + B) < 2\alpha$. Let $g_0 = g_{A,A}$ and $g_1 = g_{A,B}$ be as defined in (II). Then $g_0 = g_1$.*

Proof: We proceed the proof in two claims. We show $g_0|g_1$ in the first claim and show $g_0 = g_1$ in the second claim.

Claim 2.9.1 $g_0|g_1$.

Proof of Claim 2.9.1: By Lemma 2.3 we have $k = |\pi_{g_1}[A]| = |\pi_{g_1}[B]|$ and $\alpha > \frac{k}{g_1} - \frac{1}{2g_1}$. By Lemma 2.4 and Lemma 2.5 we can find a $G \subseteq [0, g_1-1]$ such that $A+A \sim G+g_1\mathbb{N}$. On the other hand there is a $G' \subseteq [0, g_0-1]$ such that $A+A \sim G'+g_0\mathbb{N}$ by the definition of g_0. Hence there is a $G'' \subseteq [0, d-1]$ where $d = \gcd(g_0, g_1)$ such that $A + A \sim G'' + d\mathbb{N}$ by Corollary 2.1. By the minimality of g_0, $d = g_0$. This ends the proof. □(Claim 2.9.1)

Claim 2.9.2 $g_0 = g_1$.

Proof of Claim 2.9.2: Suppose $|\pi_{g_0}[A]| = k$ and $|\pi_{g_0}[B]| = k'$. By Lemma 2.3, we have $\alpha > \frac{k}{g_0} - \frac{1}{2g_0}$. If $k' \leqslant k$, then $\alpha > \frac{k'}{g_0} - \frac{1}{2g_0}$. For each $v \in \pi_{g_0}[B]$ let $B_v = B \cap \pi_{g_0}^{-1}(v)$. Then $\underline{d}(B_v) \geqslant \underline{d}(B) - \frac{k'-1}{g_0} > \frac{1}{2g_0}$ by Lemma 2.4. Hence by Lemma 2.5 there is a $G \subseteq [0, g_0 - 1]$ such that $A + B \sim G + g_0\mathbb{N}$. By the minimality of g_1 we have $g_0 = g_1$.

So we can assume $k < k'$. If $|\pi_{g_0}[A + B]| \geqslant 2k$, then by Lemma 2.7 we have $\underline{d}(A + B) \geqslant 2\alpha$, a contradiction.

Hence we can assume $|\pi_{g_0}[A + B]| \leqslant 2k - 1 < k + k' - 1$. By Lemma 2.2 the stabilizer $S \subseteq \mathbb{Z}/g_0\mathbb{Z}$ of $\pi_{g_0}[A] + \pi_{g_0}[B]$ is non-trivial and

$$|\pi_{g_0}[A + B]|| = |\pi_{g_0}[A] + S| + |\pi_{g_0}[B] + S| - |S|.$$

Let $d|g_0$ and $d < g_0$ be such that $S = \langle d\rangle_{g_0}$ and let $s = |S| = g_0/d$.

If $|\pi_d[A]| \neq |\pi_d[B]|$, say $|\pi_d[A]| < |\pi_d[B]|$, then

$$\begin{aligned}&|\pi_d[A + B]|\\ &\quad = |\pi_{g_0}[A + B]|/s = |\pi_{g_0}[A] + S|/s + |\pi_{g_0}[B] + S|/s - |S|/s\\ &\quad = |\pi_d[A]| + |\pi_d[B]| - 1 \geqslant 2|\pi_d[A]|.\end{aligned}$$

By Lemma 2.7 we have $\underline{d}(A + B) \geqslant 2\alpha$, which contradicts the assumption of the lemma.

So we can assume $|\pi_d[A]| = |\pi_d[B]|$ or equivalently $|\pi_{g_0}[A] + S| = |\pi_{g_0}[B] + S|$. This implies $|\pi_{g_0}[A + B]| = 2|\pi_{g_0}[A] + S| - s$. On the other hand, we have $|\pi_{g_0}[A + B]| \leqslant 2k - 1$. Hence $|\pi_{g_0}[A] + S| \leqslant k + \frac{s-1}{2}$. This implies that for each $u \in \pi_{g_0}[A]$ we have

$$|\pi_{g_0}[A] \cap (u+S)| \geqslant k - (|\pi_{g_0}[A]+S| - |S|) \geqslant -\frac{s-1}{2} + s = \frac{s+1}{2}.$$

Hence each coset of S is either disjoint from $\pi_{g_0}[A]$ or contains more than $s/2$ elements from $\pi_{g_0}[A]$. So $\pi_{g_0}[A+A]$ is the union of S cosets, i.e., there exists an $F \subseteq [0, d-1]$ such that $\pi_{g_0}[F] + S = \pi_{g_0}[A+A]$. This implies

$$A + A \sim G + g_0\mathbb{N} = \pi_{g_0}^{-1}[\pi_{g_0}[A+A]] = \pi_{g_0}^{-1}[\pi_{g_0}[F] + S] = F + d\mathbb{N},$$

which contradicts the minimality of g_0. □(Lemma 2.9)

3. Nonstandard Analysis

In this section, we introduce needed notation, basic lemmas, and principles in nonstandard analysis. The reader is recommended to consult [6] for the detailed proofs of the lemmas and principles if they are omitted here. The reader who is familiar with nonstandard analysis should skip this section. The detailed introduction for nonstandard analysis can also be found in [5, 10] or many other nonstandard analysis books. [5] is written for the reader who has no prior knowledge in mathematical logic.

Let $(\mathbb{R}; +, \cdot, \leqslant, 0, 1)$ be the (standard) real ordered field. We often write $\mathbb{R}$ for this field as well as its base set. Let $\wp(\mathbb{R})$ be the collection of all subsets of $\mathbb{R}$. We call the structure $\mathbb{V} = (\mathbb{R} \cup \wp(\mathbb{R}); +, \cdot, \leqslant, 0, 1, \in, |\cdot|, \mathcal{F})$ the standard model, where $\in$ is the membership relation between $\mathbb{R}$ and $\wp(\mathbb{R})$, $|\cdot|$ is the cardinality function from the collection $Fin(\mathbb{R})$ of all finite subsets A of $\mathbb{R}$ to $\mathbb{N}$ such that $|A|$ is the number of elements in A, and $\mathcal{F}$ is a set of n_f-ary functions or partial functions f from $\mathbb{R}^{n_f}$ to $\mathbb{R}$. For example, a sequence of real numbers $\langle x_n : n \in \mathbb{N}\rangle$ can be viewed as a partial function from $\mathbb{R}$ to $\mathbb{R}$. We will also write $\mathbb{V}$ for the set $\mathbb{R} \cup \wp(\mathbb{R})$. We use an ultrapower construction to construct a nonstandard model ${}^*\mathbb{V}$ as the following. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ (see [6] for definition) and let $\mathbb{V}^{\mathbb{N}}$ be the set of all sequences $\langle x_n : n \in \mathbb{N}\rangle$ in $\mathbb{V}$. Two sequences $\langle x_n\rangle$ and $\langle y_n\rangle$ are equivalent (modulo $\mathcal{U}$) if $\{n \in \mathbb{N} : x_n = y_n\} \in \mathcal{U}$. For a sequence $\langle x_n\rangle$ in $\mathbb{V}$ let $[\langle x_n\rangle]$ denote the equivalence class containing $\langle x_n\rangle$. Let $\mathbb{V}^{\mathbb{N}}/\mathcal{U}$ be the set of all equivalence classes of the sequences in $\mathbb{V}^{\mathbb{N}}$. One can embed $\mathbb{V}$ into $\mathbb{V}^{\mathbb{N}}/\mathcal{U}$ by identifying each $x \in \mathbb{V}$ with ${}^*x = [\langle x\rangle]$, the equivalence class of the constant sequence $\langle x\rangle$. We consider $\mathbb{R}$ as a subset of $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$ and denote ${}^*X$, where $X \subseteq \mathbb{R}$, for the version of X in $\mathbb{V}^{\mathbb{N}}/\mathcal{U}$, i.e., ${}^*X = [\langle X\rangle]$. A number is called standard if it is in $\mathbb{R}$ and a set $Y \subseteq {}^*\mathbb{R}$ is called standard if there is an $X \subseteq \mathbb{R}$ with $Y = {}^*X$. There is a natural way to extend relations and functions on $\mathbb{V}$ such as $+, \cdot, \leqslant, \in$, and $|\cdot|$ to $\mathbb{V}^{\mathbb{N}}/\mathcal{U}$. The nonstandard model ${}^*\mathbb{V}$ for the purpose of this paper is the set $\mathbb{V}^{\mathbb{N}}/\mathcal{U}$ together with the extensions of all relevant relations and functions such as $+, \cdot, \leqslant, \in, |\cdot|$, and each function in $\mathcal{F}$ to $\mathbb{V}^{\mathbb{N}}/\mathcal{U}$.[2] The details of the construction can be found in [6].

In ${}^*\mathbb{V}$ there are non-zero infinitesimals, which are numbers whose absolute values are less than all standard positive real numbers. There are many positive integers in ${}^*\mathbb{N}$ but not in $\mathbb{N}$. All integers in ${}^*\mathbb{N} \setminus \mathbb{N}$ are called hyperfinite integers that are *finite from nonstandard point of view but greater than every standard integer. We reserve the letters H, K, N for hyperfinite integers.

For two real numbers $x, y \in {}^*\mathbb{R}$, x and y are called infinitesimally close, denoted by $x \approx y$, if $x - y$ is an infinitesimal. We write $x \ll y$ ($x \gg y$) if $x < y$ ($x > y$) and $x \not\approx y$. By the least upper bound axiom one can show that for any real number $r \in {}^*\mathbb{R}$ with $\alpha < r < \beta$ for some $\alpha, \beta \in \mathbb{R}$ there is a unique standard real number γ such that $r \approx \gamma$. The number γ is called the standard part of r denoted by $st(r) = \gamma$ where st is called the standard part map.

A subset Y of ${}^*\mathbb{R}$ is called internal if $Y = [\langle X_n \rangle]$ where X_n is a subset of $\mathbb{R}$ for each n. So a standard set is an internal set but an internal set may not be a standard set. A subset of ${}^*\mathbb{R}$, which is not internal, is called an external set. For example $\mathbb{N}$ and $\mathbb{R}$ are external sets. An internal set $[\langle X_n \rangle]$ is called hyperfinite if there exists a sequence of finite sets $\langle X_n \rangle$ such that $X = [\langle X_n \rangle]$, hence $|X| = [\langle |X_n| \rangle]$ is a hyperfinite integer. Any hyperfinite set $[\langle X_n \rangle]$ has a greatest element $[\langle \max X_n \rangle]$ and a least element $[\langle \min X_n \rangle]$.

By a formula in this paper we mean a first-order formula (see [6] for definition).

The following lemma [6, Proposition 2] is called the transfer principle.

Lemma 3.1. *Let $\varphi(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n)$ be a formula such that*

$$\alpha_1, \ldots, \alpha_m \in \mathbb{R} \text{ and } X_1, \ldots, X_n \subseteq \mathbb{R}$$

are the only constants in φ and φ contains no free variables. Then

$$\varphi(\alpha_1, \ldots, \alpha_m, X_1, \ldots, X_n) \text{ is true in } \mathbb{V}$$

if and only if

$$\varphi(\alpha_1, \ldots, \alpha_m, {}^*X_1, \ldots, {}^*X_n) \text{ is true in } {}^*\mathbb{V}.$$

The next lemma can be found in [6, Proposition 2]

Lemma 3.2. *Let $\varphi(x, v_1, \ldots, v_k)$ be a formula such that $v_1, \ldots, v_k \in {}^*\mathbb{V}$ and x is the only free variable. Then the set*

$$\{r \in {}^*\mathbb{R} : \varphi(r, v_1, \ldots, v_k) \text{ is true in } {}^*\mathbb{V}\}$$

is internal.

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[2] We consider that the standard model $\mathbb{V}$ contains only $\mathbb{R}$ and its power set $\wp(\mathbb{R})$ because we want to introduce nonstandard model as simple as possible for the reader who does not have prior knowledge of nonstandard analysis.

The next lemma [6, Proposition 4] is called countable saturation.

Lemma 3.3. *Suppose* $\{X^{(k)} : k \in \mathbb{N}\}$ *is a collection of non-empty internal subsets of* ${}^*\mathbb{R}$ *such that* $X^{(1)} \supseteq X^{(2)} \supseteq \ldots \supseteq X^{(k)} \supseteq \ldots$. *Then there is a* $v \in {}^*\mathbb{R}$ *such that* $v \in X^{(k)}$ *for every* $k \in \mathbb{N}$.

Note that if H is hyperfinite, then there are always hyperfinite integers in $[0, H] \setminus (\mathbb{N} \cup (H - \mathbb{N}))$. To see this let $A_n = [n, H - n]$. Then $A_n \supseteq A_{n+1}$ for every $n \in \mathbb{N}$. Hence there exists N, which belongs to A_n for every $n \in \mathbb{N}$. It is easy to check that we have $N + \mathbb{Z} \subseteq [0, H]$.

The next lemma [6, Lemma 1] establishes a nonstandard equivalence of $BD(A) \geqslant \alpha$.

Lemma 3.4. *Given* α, *for any set* $A \subseteq \mathbb{N}$, $BD(A) \geqslant \alpha$ *iff there is an infinitesimal* $\iota \geqslant 0$ *and an interval* $I = [n, n + H] \subseteq {}^*\mathbb{N}$ *of hyperfinite length such that*

$$\frac{{}^*A(n, n + H)}{H + 1} \geqslant \alpha - \iota.$$

Loeb spaces: Given a hyperfinite integer H, $\Omega = [0, H - 1]$ is a hyperfinite set. Let $A \subseteq \Omega$ be an internal set. Then $|A|$ is an integer between 0 and H. Hence $|A|/H$ is a real number in ${}^*\mathbb{R}$ between 0 and 1. Let Σ_0 be the collection of all internal subsets of Ω and let $\mu(A) = st(|A|/H)$ for every $A \in \Sigma_0$. Then (Ω, Σ_0, μ) is a finitely-additive probability space from the standard point of view. For any subset S of Ω, internal or external, define

$$\bar{\mu}(S) = \inf\{\mu(A) : A \in \Sigma_0 \text{ and } A \supseteq S\},$$
$$\underline{\mu}(S) = \sup\{\mu(A) : A \in \Sigma_0 \text{ and } A \subseteq S\}, \text{ and}$$
$$\Sigma = \{S \subseteq \Omega : \bar{\mu}(S) = \underline{\mu}(S)\}.$$

It is easy to see that $\Sigma_0 \subseteq \Sigma$. For each $S \in \Sigma$, define $\mu_L(S) = \bar{\mu}(S) = \underline{\mu}(S)$. Then (Ω, Σ, μ_L) is a standard, countably-additive, atomless, complete probability space, which is called a hyperfinite Loeb space generated by a normalized uniform counting measure $|\cdot|/H$. Let's call it simply a Loeb space on Ω. Note that the Loeb space construction can be carried out on any hyperfinite set. Note also that the verification of the countable-additivity requires using countable saturation.

The following notation is non-traditional and will be frequently used in the proof of the main theorems of the paper.

Let $A \subseteq {}^*\mathbb{Z}$ and $x \in {}^*\mathbb{Z}$. Define

$$A_{x+\mathbb{N}} = x + (A - x) \cap \mathbb{N},$$
$$A_{x-\mathbb{N}} = x - (x - A) \cap \mathbb{N},$$
$$A_{x+\mathbb{Z}} = x + (A - x) \cap \mathbb{Z},$$

$$\underline{d}_{x+\mathbb{N}}(A) = \underline{d}((A-x)\cap\mathbb{N}), \text{ and}$$
$$\underline{d}_{x-\mathbb{N}}(A) = \underline{d}((x-A)\cap\mathbb{N}).$$

In words, $A_{x+\mathbb{N}}$ is the part of A in $x+\mathbb{N}$ where $x+\mathbb{N}$ is a copy of $\mathbb{N}$ lying upward from x. Likewise, $A_{x-\mathbb{N}}$ is the part of A in $x-\mathbb{N}$ and $A_{x+\mathbb{Z}}$ is the part of A in $x+\mathbb{Z}$.

The following lemma is a variation of [6, Lemma 2]. Although the style of the lemma is slightly different from [6, Lemma 2], the ideas of the proofs are similar.

Lemma 3.5. *Suppose $A \subseteq \mathbb{N}$, $BD(A) = \alpha$, and $\{[a_n, b_n] : n \in \mathbb{N}\}$ is a sequence of intervals of standard integers satisfying (I). Let N be any hyperfinite integer. Then for almost all $x \in [a_N, b_N]$ in terms of the Loeb measure μ_L on $[a_N, b_N]$, we have $\underline{d}_{x+\mathbb{N}}({}^*A) = \underline{d}_{x-\mathbb{N}}({}^*A) = \alpha$. On the other hand, if $A \subseteq \mathbb{N}$ and there is an $x \in {}^*\mathbb{N}$ such that $\underline{d}_{x+\mathbb{N}}({}^*A) \geqslant \alpha$ or $\underline{d}_{x-\mathbb{N}}({}^*A) \geqslant \alpha$, then $BD(A) \geqslant \alpha$.*

Proof Suppose $BD(A) = \alpha$ and $\{[a_n, b_n] : n \in \mathbb{N}\}$ is the sequence satisfying (I). By Lemma 3.1

$$\frac{{}^*A(a_N, b_N)}{b_N - a_N + 1} \approx \alpha.$$

Hence the Loeb measure of the set ${}^*A[a_N, b_N]$ in $[a_N, b_N]$ is α. Let T be the map from $[a_N, b_N]$ to $[a_N, b_N]$ such that $T(b_N) = a_N$ and $T(x) = x+1$ for every $x \in [a_N, b_N - 1]$. Let T' be the map from $[a_N, b_N]$ to $[a_N, b_N]$ such that $T'(a_N) = b_N$ and $T'(x) = x-1$ for every $x \in [a_N+1, b_N]$. Then T and T' are Loeb measure-preserving transformation. Let f be the characteristic function of the set ${}^*A[a_N, b_N]$. By Lemma 2.1 there is an L^1 function $\bar{f}$ such that for almost all $x \in [a_N, b_N]$,

$$\text{(III)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m(x)) = \bar{f}(x).$$

Since the integration over $[a_N, b_N]$ of the left-hand side of (III) is α, then $\int_{[a_N,b_N]} \bar{f} d\mu_L = \alpha$. We need to show that $\bar{f}(x) = \alpha$ almost surely. Note that the set $\cap_{n=0}^{\infty}[a_N, b_N - n]$ has Loeb measure 1.

Suppose there is an $x \in \cap_{n=0}^{\infty}[a_N, b_N - n]$ such that the left-hand side of (III) is $\beta > \alpha$. Then $\frac{{}^*A(x,x+n)}{n+1} > \frac{\beta+\alpha}{2}$ for sufficiently large $n \in \mathbb{N}$. Let D be the set of all $n \in {}^*\mathbb{N}$ such that $\frac{{}^*A(x,x+n)}{n+1} > \frac{\beta+\alpha}{2}$. By Lemma 3.2, D is an internal subset of ${}^*\mathbb{N}$, which contains all sufficiently large n in $\mathbb{N}$. Then D must contain a hyperfinite integer N'. Hence $\frac{{}^*A(x,x+N')}{N'+1} > \frac{\beta+\alpha}{2}$. By Lemma 3.4, $BD(A) \geqslant \frac{\beta+\alpha}{2} > \alpha$. This contradicts $BD(A) = \alpha$.

Since $\bar{f}(x) \leqslant \alpha$ for almost all $x \in [a_N, b_N]$ and $\int_{[a_N,b_N]} \bar{f}(x) d\mu_L = \alpha$, then $\bar{f} = \alpha$ almost surely. Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(x+m) = \alpha$$

implies $\underline{d}_{x+\mathbb{N}}({}^*A) = \alpha$, then for almost all $x \in [a_N, b_N]$, we have $\underline{d}_{x+\mathbb{N}}({}^*A) = \alpha$.

If T is replaced by T' in the arguments above, then $\underline{d}_{x-\mathbb{N}}({}^*A) = \alpha$ for almost all $x \in [a_N, b_N]$. Hence for almost all $x \in [a_N, b_N]$ we have $\underline{d}_{x+\mathbb{N}}({}^*A) = \underline{d}_{x-\mathbb{N}}({}^*A) = \alpha$.

Suppose $\underline{d}_{x+\mathbb{N}}({}^*A) \geqslant \alpha$ ($\underline{d}_{x-\mathbb{N}}({}^*A) \geqslant \alpha$). Then there is a hyperfinite integer N such that $\frac{{}^*A(x,x+N)}{N+1} \approx \alpha$ or $> \alpha$ ($\frac{{}^*A(x-N,x)}{N+1} \approx \alpha$ or $> \alpha$). By Lemma 3.4 we have $BD(A) \geqslant \alpha$. □(Lemma 3.5)

4. Proof of theorem 1.1

In this section we state Theorem 4.1, which can be viewed as a non-standard version of Theorem 1.1. We first prove Theorem 1.1 assuming Theorem 4.1 and then prove Theorem 4.1.

Theorem 4.1. *Let B be an internal subset of an interval $[0, H]$ of hyperfinite length. Suppose*

(a) *for any $x \in [0, H]$, $\underline{d}_{x+\mathbb{N}}(B) \leqslant \alpha$ and $\underline{d}_{x-\mathbb{N}}(B) \leqslant \alpha$,*
(b) *there is an $S \subseteq [0, H]$ with Loeb measure one such that for any $x \in S$, $\underline{d}_{x+\mathbb{N}}(B) = \underline{d}_{x-\mathbb{N}}(B) = \alpha$,*
(c) *for any $x \in [0, 2H]$, $\underline{d}_{x+\mathbb{N}}(B + B) < 2\alpha$ and $\underline{d}_{x-\mathbb{N}}(B + B) < 2\alpha$.*

Then there is a $g \in \mathbb{N}$ such that for any $x \in S$, $g = g_{C,C}$ as defined in (II) where $C = (B - x) \cap \mathbb{N}$, and there are $G \subseteq [0, g - 1]$ and $[c, d] \subseteq [0, H]$ such that

(1) $\frac{d-c}{H+1} \approx 1$,
(2) $B + B \subseteq G + g\,{}^*\mathbb{N}$,
(3) $(B + B) \cap [2c, 2d] = (G + g\,{}^*\mathbb{N}) \cap [2c, 2d]$,
(4) $\frac{|G|}{g} \geqslant 2\alpha - \frac{1}{g}$.

Proof of Theorem 1.1 assuming Theorem 4.1: Let $A \subseteq \mathbb{N}$ be such that $BD(A) = \alpha$ and $BD(A+A) < 2\alpha$. Let $\{[a_n, b_n] : n \in \mathbb{N}\}$ be a sequence of intervals satisfying (I). Let N be an arbitrary hyperfinite integer. There is a subset $S \subseteq [a_N, b_N]$ of Loeb measure one (on $[a_N, b_N]$) such that for any $x \in S$ we have $\underline{d}_{x+\mathbb{N}}({}^*A) = \underline{d}_{x-\mathbb{N}}({}^*A) = \alpha$ by Lemma 3.5. For any $x \in [a_N, b_N]$ we have $\underline{d}_{x+\mathbb{N}}({}^*A) \leqslant \alpha$ and $\underline{d}_{x-\mathbb{N}}({}^*A) \leqslant \alpha$, and for any $x \in [2a_N, 2b_N]$ we have $\underline{d}_{x+\mathbb{N}}({}^*A + {}^*A) < 2\alpha$ and $\underline{d}_{x-\mathbb{N}}({}^*A + {}^*A) < 2\alpha$, again by Lemma 3.5. Applying Theorem 4.1 with $[0, H] = [a_N, b_N] - a_N$ and $B = {}^*A[a_N, b_N] - a_N$ we can find g_N, G_N, $[c_N, d_N]$ such that

(1) $\frac{d_N - c_N}{b_N - a_N} \approx 1$,
(2) ${}^*A[a_N, b_N] + {}^*A[a_N, b_N] \subseteq G_N + g_N\, {}^*\mathbb{N}$,
(3) $({}^*A[a_N, b_N] + {}^*A[a_N, b_N]) \cap [2c_N, 2d_N] = (G_N + g_N\, {}^*\mathbb{N}) \cap [2c_N, 2d_N]$,
(4) $\frac{|G_N|}{g_N} \geqslant 2\alpha - \frac{1}{g_N}$.

We show first that g_N and $|G_N|$ do not depend on N and then show that G_N does not depend on N.

Let N' be another hyperfinite integer. Let $x \in [a_N, b_N]$ and $y \in [a_{N'}, b_{N'}]$ be such that $\underline{d}_{x+\mathbb{N}}({}^*A[a_N, b_N]) = \underline{d}_{y+\mathbb{N}}({}^*A[a_{N'}, b_{N'}]) = \alpha$. Then

$$\underline{d}_{x+y+\mathbb{N}}({}^*A_{x+\mathbb{N}} + {}^*A_{y+\mathbb{N}}) \leqslant \underline{d}_{x+y+\mathbb{N}}({}^*A + {}^*A) < 2\alpha$$

by Lemma 3.5. Let $g' = g_{C,D}$ be as defined in (II) where

$$(C, D) = (({}^*A - x) \cap \mathbb{N}, ({}^*A - y) \cap \mathbb{N}).$$

Then by Lemma 2.9 we have $g_N = g' = g_{N'}$. Clearly $m = |G_N| = \lceil 2\alpha g - 1 \rceil$ depends only on α and g.

We now show that G_N does not depend on N. Given a hyperfinite integer N, let $F_N \subseteq [0, g-1]$ be minimal such that ${}^*A[a_N, b_N] \subseteq F_N + g\, {}^*\mathbb{N}$. Note that $F_N + F_N = G$ in $\mathbb{Z}/g\mathbb{Z}$. By Lemma 2.2 and the minimality of $g = g_{C,C}$ as defined in Theorem 4.1 we have $|F_N| = \frac{m+1}{g}$.

Claim 1.1.1 ${}^*A \subseteq F_N + g\, {}^*\mathbb{N}$.

Proof of Claim 1.1.1: Suppose not and let $z \in {}^*A \setminus (F_N + g\, {}^*\mathbb{N})$. By Lemma 3.1 we can assume $z \in A \setminus (F_N + g\mathbb{N})$. Choose another hyperfinite integer N' such that $b_{N'} - a_{N'} > 2b_N$. Such N' exists by Lemma 3.1. Let $x \in [a_N, b_N]$ be such that $x + \mathbb{N} \subseteq [a_N, b_N]$ and $\underline{d}_{x+\mathbb{N}}({}^*A) = \alpha$. Note that ${}^*A_{x+\mathbb{N}} \subseteq F_N + g\, {}^*\mathbb{N}$. Since $(z + b_{N'}) - (x + a_{N'}) \geqslant \frac{b_{N'} - a_{N'}}{2}$, then by Lemma 3.5 one can find $y_1, y_2 \in [a_{N'}, b_{N'}]$ such that $\underline{d}_{y_1+\mathbb{N}}({}^*A) = \underline{d}_{y_2+\mathbb{N}}({}^*A) = \alpha$ and $x + y_1 = z + y_2 = u$. Since $\underline{d}_{u+\mathbb{N}}({}^*A_{x+\mathbb{N}} + {}^*A_{y_1+\mathbb{N}}) < 2\alpha$, then there is a $G' \subseteq [0, g-1]$ such that ${}^*A_{x+\mathbb{N}} + {}^*A_{y_1+\mathbb{N}} \sim x + y_1 + G' + g\mathbb{N}$. Let $F, F' \subseteq [0, g-1]$ be minimal such that ${}^*A_{x+\mathbb{N}} \subseteq F + g\, {}^*\mathbb{N}$ and ${}^*A_{y_1+\mathbb{N}} \subseteq F' + g\, {}^*\mathbb{N}$. By Lemma 2.3 we have $|F| = |F'| = \frac{m+1}{g}$. Hence $F = F_N$. Since ${}^*A_{y_1+\mathbb{Z}} \subseteq {}^*A[a_{N'}, b_{N'}] \subseteq F_{N'} + g\, {}^*\mathbb{N}$, then $F' = F_{N'}$. Since ${}^*A_{y_2+\mathbb{N}} \subseteq F_{N'} + g\, {}^*\mathbb{N}$, then ${}^*A_{y_2+\mathbb{N}} \subseteq F' + g\, {}^*\mathbb{N}$. Note that $z \notin F + g\, {}^*\mathbb{N}$. Hence there is a $v \in {}^*A_{y_2+\mathbb{N}}$ such that $z + v \notin F + F' + g\, {}^*\mathbb{N}$ by Lemma 2.3. Now we have

$$z + {}^*A_{y_2+\mathbb{N}} \subseteq x + y_1 + \mathbb{N} \text{ and } z + {}^*A_{y_2+\mathbb{N}} \cap (v + g\, {}^*\mathbb{N}) \nsubseteq {}^*A_{x+\mathbb{N}} + {}^*A_{y_1+\mathbb{N}}.$$

Since

$$\underline{d}_{x+y_1+\mathbb{N}}({}^*A + {}^*A) \geqslant \underline{d}_{x+y_1+\mathbb{N}}({}^*A_{x+\mathbb{N}} + {}^*A_{y_1+\mathbb{N}}) + \underline{d}_{z+y_2+\mathbb{N}}(z + ({}^*A_{y_2+\mathbb{N}} \cap (v + g\, {}^*\mathbb{N}))),$$

then by Lemma 2.6 we have $\underline{d}_{x+y_1+\mathbb{N}}({}^*A + {}^*A) \geqslant 2\alpha$, which implies $BD(A+A) \geqslant 2\alpha$ by 3.5, a contradiction to the assumption $BD(A+A) < 2\alpha$. □(Claim 1.1.1)

By Claim 1.1.1 we conclude that ${}^*A \subseteq F_N + g\,{}^*\mathbb{N}$ for any hyperfinite integer N. Hence there is an $F \subseteq [0, g-1]$ such that $F_N = F$ for any hyperfinite integer N by the minimality of F_N. Let $G \subseteq [0, g-1]$ be such that $\pi_g[F+F] = \pi_g[G]$. Then $G_N = G$ for any hyperfinite integer N and $m = |G| = \lceil 2\alpha g - 1\rceil$. Clearly ${}^*A + {}^*A \subseteq G + g\,{}^*\mathbb{N}$ and hence $A+A \subseteq G + g\mathbb{N}$.

Note that $I = \{[a_n, b_n] : n \in {}^*\mathbb{N}\}$ is an internal sequence of intervals[3]. Given $k \in \mathbb{N}$ let $\varphi_k(x, g, G, {}^*A, I)$ be the formula saying that x is a positive integer and there is a $[c_x, d_x] \subseteq [a_x, b_x]$ with $\frac{d_x - c_x}{b_x - a_x} > 1 - \frac{1}{k}$ such that

$$({}^*A + {}^*A) \cap [2c_x, 2d_x] = (G + g\,{}^*\mathbb{N}) \cap [2c_x, 2d_x].$$

We will omit the parameters of the formula φ_k and simply write $\varphi_k(x)$ instead. Let X_k be the set of all positive integers $x \in {}^*\mathbb{N}$ such that $\varphi_k(x)$ is true in ${}^*\mathbb{V}$. Then X_k is internal by Lemma 3.2 and contains all hyperfinite integers. Let

$$n_k = \min\{x \in {}^*\mathbb{N} : {}^*\mathbb{N} \smallsetminus [0, x-1] \subseteq X_k\}.$$

Then $n_k \in \mathbb{N}$. One can now choose a strictly increasing sequence $n'_k \geqslant n_k$ in $\mathbb{N}$. For each $n \in [n'_k, n'_{k+1} - 1]$ for $k > 0$ choose $[c_n, d_n]$ guaranteed by the truth of $\varphi_k(n)$ in ${}^*\mathbb{V}$. For $n < n'_1$ choose $[c_n, d_n] = \emptyset$. Then $\lim_{n\to\infty} \frac{d_n - c_n}{b_n - a_n} = 1$.

Finally

$$\frac{m}{g} = BD(G + g\mathbb{N}) \geqslant BD(A+A) \geqslant \lim_{n\to\infty} \frac{|(A+A) \cap [2c_n, 2d_n]|}{2d_n - 2c_n + 1} = \frac{m}{g}$$

implies $BD(A+A) = \frac{m}{g} \geqslant 2\alpha - \frac{1}{g}$. □(Theorem 1.1)

Proof of Theorem 4.1: Without loss of generality let

$$S = \{x \in [0, H] : \underline{d}_{x+\mathbb{N}}(B) = \underline{d}_{x-\mathbb{N}}(B) = \alpha\}.$$

Given $x, y \in S$, let $g_{x,y} = g_{C,D}$ be as defined in (II) where

$$(C, D) = ((B - x) \cap \mathbb{N}, (B - y) \cap \mathbb{N})$$

and let $h_{x,y} = g_{C,D}$ where

$$(C, D) = ((x - B) \cap \mathbb{N}, (y - B) \cap \mathbb{N}).$$

Note that for any $k \in \mathbb{Z}$, $x \in S$ iff $x + k \in S$. We also have $g_{x,y} = g_{x+k,y} = g_{x,y+k}$ and $h_{x,y} = h_{x+k,y} = h_{x,y+k}$. For any $C, D \subseteq x + \mathbb{N}$ we write $C \sim D$ if $C - x \sim D - x$. For any $C, D \subseteq x - \mathbb{N}$ we write $C \sim D$ if $x - C \sim x - D$.

Claim 4.1.1 For any $x, y \in S$, $g_{x,x} = g_{x,y}$ and $h_{x,x} = h_{x,y}$.
Proof of Claim 4.1.1: The claim follows from Lemma 2.9. □(Claim 4.1.1)

[3] I can be viewed as a pair of sequences in $\mathcal{F}$ as we define the nonstandard universe.

By Claim 4.1.1 we can fix g and h such that $g = g_{x,y}$ and $h = h_{x,y}$ for any $x, y \in S$.

Claim 4.1.2 $g = h$.
Proof of Claim 4.1.2: Assume the claim is not true. By symmetry we can assume $h < g$. Let $x \in S$ and let $G \subseteq [0, h-1]$ with $|G| = m$ be such that

$$B_{x-\mathbb{N}} + B_{x-\mathbb{N}} \sim 2x + G - h\mathbb{N}.$$

Let $F \subseteq [0, h-1]$ with $|F| = k$ be minimal such that $B_{x-\mathbb{N}} \subseteq x + F - h\mathbb{N}$. Then $m = 2k - 1$. Note that $\alpha > \frac{k}{h} - \frac{1}{2h}$.

Since $\underline{d}_{x-\mathbb{N}}(B) = \alpha$, then there is a hyperfinite integer N such that $\frac{B(x-N,x)}{N+1} \approx \alpha$ and $B[x-N, x] \subseteq x + F - h\,{}^*\mathbb{N}$ by Lemma 3.1 and Lemma 3.2. By the proof of Lemma 3.5 there is a $y \in [x-N, x]$ such that $y + \mathbb{Z} \subseteq [x-N, x]$ and $\underline{d}_{y+\mathbb{N}}(B) = \underline{d}_{y-\mathbb{N}}(B) = \alpha$. So $y \in S$. Since $B_{y+\mathbb{N}} \subseteq x + F - h\,{}^*\mathbb{N}$, then for any $z \in B_{y+\mathbb{N}}$ and $B_z = B_{y+\mathbb{N}} \cap (z + h\,{}^*\mathbb{Z})$, we have $\underline{d}_{y+\mathbb{N}}(B_z) > \frac{1}{2h}$ by Lemma 2.4. This implies that there is $G' \subseteq [0, h-1]$ such that $B_{y+\mathbb{N}} + B_{y+\mathbb{N}} \sim G' + h\mathbb{N}$. Hence $g = g_{y,y} \leqslant h$ by Claim 4.1.1 and by the definition of g. This contradicts the assumption that $h < g$.

□(Claim 4.1.2)

Claim 4.1.3 For any $x, y \in S$, there is $G \subseteq [0, g-1]$ such that $|G| = \lceil 2\alpha g - 1 \rceil$ and

$$B_{x+\mathbb{Z}} + B_{y+\mathbb{Z}} = (x+y) + G + g\mathbb{Z}.$$

Proof of Claim 4.1.3: Let $G, G' \subseteq [0, g-1]$ be such that

$$\begin{aligned} B_{x+\mathbb{N}} + B_{y+\mathbb{N}} &\subseteq (x+y) + G + g\mathbb{N}, \\ B_{x+\mathbb{N}} + B_{y+\mathbb{N}} &\sim (x+y) + G + g\mathbb{N}, \\ B_{x-\mathbb{N}} + B_{y-\mathbb{N}} &\subseteq (x+y) + G' - g\mathbb{N}, \text{ and} \\ B_{x-\mathbb{N}} + B_{y-\mathbb{N}} &\sim (x+y) + G' - g\mathbb{N}. \end{aligned}$$

Note that $|G| = |G'| = \lceil 2\alpha g - 1 \rceil$.

Suppose $G \neq G'$. There are $b \in B_{x-\mathbb{N}}$ and $c \in B_{y-\mathbb{N}}$ such that

$$b + c \notin (x+y) + G + g\mathbb{Z}.$$

Clearly $b, c \in S$. Let $G'' \subseteq [0, g-1]$ be such that

$$B_{b+\mathbb{N}} + B_{c+\mathbb{N}} \sim (b+c) + G'' + g\mathbb{N}.$$

Then $|G''| \geqslant m + 1$, which contradicts that $|G''| = \lceil 2\alpha g - 1 \rceil$. This shows $G = G'$.

Suppose

$$B_{x+\mathbb{Z}} + B_{y+\mathbb{Z}} \neq (x+y) + G + g\mathbb{Z}$$

and let

$$u \in ((x+y) + G + g\mathbb{Z}) \setminus (B_{x+\mathbb{Z}} + B_{y+\mathbb{Z}}).$$

Let $b \in B_{x+\mathbb{Z}}$ and $c \in B_{y+\mathbb{Z}}$ be such that $b+c \equiv u \pmod g$. Let $n \in \mathbb{Z}$ be such that $b+c+ng = u$. Let $B_b^- = B_{b-\mathbb{N}} \cap (b+g\mathbb{Z})$ and $B_{c+ng}^+ = B_{c+ng+\mathbb{N}} \cap (c+ng+g\mathbb{Z})$. Then

$$\underline{d}_{b-\mathbb{N}}(B_b^-) > \frac{1}{2g} \text{ and } \underline{d}_{c+ng+\mathbb{N}}(B_{c+ng}^+) > \frac{1}{2g}.$$

Hence we have $B_{c+ng}^+ \cap (b+c+ng-B_b^-) \neq \emptyset$. This implies

$$u = b+c+ng \in B_b^- + B_{c+ng}^+ \subseteq B_{x+\mathbb{Z}} + B_{y+\mathbb{Z}},$$

which contradicts the choice of u. □(Claim 4.1.3)

Claim 4.1.4 Let $x \in S$ be such that $0 \ll \frac{x}{H} \ll 1$ and let $G \subseteq [0, g-1]$ be such that

$$B_{x+\mathbb{Z}} + B_{x+\mathbb{Z}} = (G + g\,{}^*\mathbb{N}) \cap (2x+\mathbb{Z}).$$

Then there is $[c,d] \subseteq [0,H]$ with $\frac{c}{H} \approx 0$ and $\frac{d}{H} \approx 1$ such that

$$(G + g\,{}^*\mathbb{N}) \cap [2c, 2d] \subseteq (B+B) \cap [2c, 2d].$$

Proof of Claim 4.1.4: Let c and d be defined by

$$c = \min\{y \in [0,x] : (G + g\,{}^*\mathbb{N}) \cap [2y, 2x] \subseteq (B+B) \cap [2y, 2x]\}$$
$$d = \max\{y \in [x,H] : (G + g\,{}^*\mathbb{N}) \cap [2x, 2y] \subseteq (B+B) \cap [2x, 2y]\}.$$

The number c and d are well defined by Claim 4.1.3. It suffices to show that $\frac{c}{H} \approx 0$ and $\frac{d}{H} \approx 1$. We show $\frac{d}{H} \approx 1$ first. The statement $\frac{c}{H} \approx 0$ then follows from symmetry.

Suppose $\frac{d}{H} \ll 1$. Then either $2d+1$ or $2d+2$ belongs to the set $(G + g\,{}^*\mathbb{N}) \setminus (B+B)$. Suppose

$$2d+1 \in (G + g\,{}^*\mathbb{N}) \setminus (B+B).$$

Let $u = \min\{H-d, d\}$. Then $\frac{u}{H} \gg 0$ and $[d-u+1, d+u] \subseteq [0,H]$. Let

$$S' = S \cap [d-u+1, d+u].$$

Then $2d+1-S' \subseteq [d-u+1, d+u]$. Since S has Loeb measure one, then $S' \cap (2d+1-S') \neq \emptyset$, which implies that there are $y, z \in S'$ such that $2d+1 = y+z$. Let $G' \subseteq [0, g-1]$ be such that

$$B_{y+\mathbb{Z}} + B_{z+\mathbb{Z}} = (G' + g\,{}^*\mathbb{N}) \cap (y+z+\mathbb{Z}).$$

Then we have $|G| = |G'|$ by Claim 4.1.3. Note that $\underline{d}_{x+y-1-\mathbb{N}}(B+B) < 2\alpha$ and

$$(G + g\,{}^*\mathbb{N}) \cap (y+z-1-\mathbb{N}) \subseteq B+B$$

by the maximality of d. Hence $G \subseteq G'$, which implies $G = G'$ because they have the same cardinality . So we have

$$\begin{aligned} 2d+1 &\in (G + g\,{}^*\mathbb{N}) \cap (2d+1+\mathbb{Z}) \\ &= (G' + g\,{}^*\mathbb{N}) \cap (x+y+\mathbb{Z}) = B_{y+\mathbb{Z}} + B_{z+\mathbb{Z}} \subseteq (B+B). \end{aligned}$$

This contradicts the assumption that $2d+1 \notin (B+B)$. If $2d+1 \in B+B$ and

$$2d+2 \in (G + g\,{}^*\mathbb{N}) \smallsetminus (B+B),$$

then one needs only to replace $[d-u+1, d+u]$ by $[d-u, d+u]$ in the arguments above. $\square$(Claim 4.1.4)

By Claim 4.1.4 we can now fix $G \subseteq [0, g-1]$ and $[c,d]$ such that the conclusions of Claim 4.1.4 are true. Let $x \in S$ be such that $x + \mathbb{Z} \subseteq [c,d]$. Note that $({}^*A + {}^*A) \cap (2x + \mathbb{Z}) = (G + g\,{}^*\mathbb{N}) \cap (2x + \mathbb{Z})$. Let $F \subseteq [0, g-1]$ be minimal such that $B[c,d] \subseteq F + g\,{}^*\mathbb{N}$. Note that $F + F = G$ in $\mathbb{Z}/g\mathbb{Z}$. Since $g = g_{x,x}$, then by Lemma 2.2 and minimality of $g_{x,x}$ it is impossible to have $2|F| - 1 > |G|$ because otherwise G would have a non-trivial stabilizer S in $\mathbb{Z}/g\mathbb{Z}$, which contradicts the minimality of $g_{x,x}$. On the other hand, since ${}^*A_{x+\mathbb{Z}} \subseteq F + g\,{}^*\mathbb{N}$, then by Lemma 2.3 we have $2|F| - 1 = |G|$.

Claim 4.1.5 $B + B \subseteq G + g\,{}^*\mathbb{N}$.

Proof of Claim 4.1.5: Suppose the claim is not true and let

$$a \in (B+B) \smallsetminus (G + g\,{}^*\mathbb{N}).$$

Then there are $u, v \in B$ such that $u + v = a$. Without loss of generality we can assume $u \notin F + g\,{}^*\mathbb{N}$. By Lemma 2.3 we have $u + F + g\,{}^*\mathbb{N} \nsubseteq G + g\,{}^*\mathbb{N}$. Choose $z \in S$ such that $u + z + \mathbb{Z} \subseteq [2c, 2d]$. Without loss of generality we can assume that $z \in B_{z+\mathbb{Z}}$ and $u + z \notin G + g\,{}^*\mathbb{N}$ because otherwise z can be replaced by $z + n$ for some $n \in \mathbb{Z}$. Note that $z + n$ is also in S. Let $B_z^+ = B \cap (z + g\mathbb{N})$. Then $(u + B_z^+) \cap (G + g\,{}^*\mathbb{N}) = \emptyset$. Since

$$(u + B_z^+) \cup ((G + g\,{}^*\mathbb{N}) \cap (u + z + \mathbb{Z})) \subseteq (B+B) \cap (u + z + \mathbb{Z}),$$

then

$$\underline{d}_{u+z+\mathbb{N}}(B+B) \geqslant \frac{m}{g} + \underline{d}_{z+\mathbb{N}}(B_z^+) \geqslant 2\alpha$$

by Lemma 2.6, which contradicts (c) of Theorem 4.1. $\square$(Claim 4.1.5)

Clearly, the conclusion (3) of the theorem follows from Claim 4.1.4 and Claim 4.1.5. $\square$(Theorem 4.1)

5. Comments and questions

In Theorem 1.1 we can replace $A + A$ by $A + B$ as long as $BD(A) = BD(B) = \alpha$, $BD(A+B) < 2\alpha$, and the sequence of intervals $\{[a_n.b_n] : n \in \mathbb{N}\}$ satisfies

$$\lim_{n\to\infty}(b_n - a_n) = \infty \text{ and } \lim_{n\to\infty} \frac{A(a_n, b_n)}{b_n - a_n + 1} = \lim_{n\to\infty} \frac{B(a_n, b_n)}{b_n - a_n + 1} = \alpha.$$

However, the change will make the theorem more tedious and less natural. Without the last condition above, the problem becomes complicated. The following is a general question about the sum of multiple sets.

Question 5.1. *Let $A_i \subseteq \mathbb{N}$ for $i = 1, 2, \ldots, k$ and*

$$BD(\sum_{i=1}^{k} A_i) < \sum_{i=1}^{k} BD(A_i).$$

Let $\{[a_n^{(i)}, b_n^{(i)}] : n \in \mathbb{N}\}$ be such that

$$\lim_{n\to\infty}(b_n^{(i)} - a_n^{(i)}) = \infty \text{ and } \lim_{n\to\infty} \frac{A_i(a_n^{(i)}, b_n^{(i)})}{b_n^{(i)} - a_n^{(i)} + 1} = BD(A_i)$$

for $i = 1, 2, \ldots, k$. How well can we characterize the structure of $\sum_{i=1}^{k} A_i$ in

$$\bigcup_{n=1}^{\infty}[\sum_{i=1}^{k} a_n^{(i)}, \sum_{i=1}^{k} b_n^{(i)}]?$$

The following question for finite Abelian groups and the results of the paper seem to have some similar flavor.

Question 5.2. *Let $G = \mathbb{Z}/g\mathbb{Z}$. Let $F_0, F_1 \subseteq G$ be such that $|F_0| = |F_1| = k$, $|F_0 + F_0| = 2k - 1$, and $|F_0 + F_1| = 2k - 1$. Can we conclude that there is an $h \in G$ such that $F_1 = h + F_0$?*

Inverse problems for the addition of two sets $A+A$ concern the structure of A when the size of $A+A$ is relatively small. G. A. Freiman proved a series of important theorems on the inverse problems for a finite set A (cf.[2, 11]).

For an infinite set $A \subseteq \mathbb{N}$, we use densities to measure the size of A. Lemma 2.3 can be viewed as a theorem for the inverse problems for infinite sets, which says roughly that if $\underline{d}(A+A) < 2\underline{d}(A)$ (i.e., the lower asymptotic density of $A + A$ is small), then $A \subseteq F + g\mathbb{N}$ and $\underline{d}(A) > \frac{|F|}{g} - \frac{1}{2g}$ (i.e., A is a large subset of the union of arithmetic sequences with a common difference g). In [8, 9] a theorem is proven for the inverse problem about upper asymptotic density. We can also state a corollary of Theorem 1.1 as a result for the inverse problem about the upper Banach density.

Corollary 5.1. *Let A be a set of non-negative integers such that $BD(A) = \alpha$ and $BD(A+A) < 2\alpha$. Then there are $g \in \mathbb{N}$ and $F \subseteq [0, g-1]$ such that $A \subseteq F + g\mathbb{N}$ and $\frac{|F|}{g} - \frac{1}{2g} < \alpha \leqslant \frac{|F|}{g}$.*

There is a generalization of Theorem 1.2 for $A + A$ by Y. Bilu [1]. It is interesting to see whether we can also derive a theorem on the inverse problems about the upper Banach density parallel to Bilu's result. In [1] a condition $\mathrm{lac}(A) < \Lambda$ is imposed on A. Note that in the first part of Lemma 3.5 we actually have $\underline{d}_{x+\mathbb{N}}({}^*A) = \underline{d}_{x-\mathbb{N}}({}^*A) = \bar{d}_{x+\mathbb{N}}({}^*A) = \bar{d}_{x-\mathbb{N}}({}^*A)$. Since $\underline{d}(A) = \bar{d}(A)$ implies $\mathrm{lac}(A) = 1$, then the condition $\mathrm{lac}(({}^*A - x) \cap \mathbb{N}) \leqslant \Lambda$ is automatically satisfied when we apply some existing theorems about the lower asymptotic density to the set ${}^*A_{x+\mathbb{N}}$ in Lemma 3.5.

Question 5.3. *Suppose $BD(A + A) \leqslant cBD(A)$ for some $c \geqslant 2$. What should be the structure of A?*

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Prerna Bihani
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556, U.S.A.
E-mail: pbihani@nd.edu

Renling Jin
Department of Mathematics
College of Charleston
Charleston, SC 29424, U.S.A.
E-mail: jinr@cofc.edu