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On a mixed Littlewood conjecture for quadratic numbers

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par Bernard de Mathan

Résumé. Nous étudions un problème diophantien simultané relié à la conjecture de Littlewood. En utilisant des minorations connues de formes linéaires de logarithmes $p$-adiques, nous montrons qu’un résultat que nous avons précédemment obtenu, concernant les nombres quadratiques, est presque optimal.

Abstract. We study a simultaneous diophantine problem related to Littlewood’s conjecture. Using known estimates for linear forms in $p$-adic logarithms, we prove that a previous result, concerning the particular case of quadratic numbers, is close to be the best possible.

1. Introduction

In a joint paper, with O. Teulié [5], we have considered the following problem. Let $\mathcal{B} = (b_k)_{k \geq 1}$ be a sequence of integers greater than 1. Consider the sequence $(r_n)_{n \geq 0}$, where $r_0 = 1$ and $r_n = \prod_{0 < k \leq n} b_k$ for $n > 0$. For $q \in \mathbb{Z}$, set
\begin{align*}
\omega_{\mathcal{B}}(q) &= \sup\{n \in \mathbb{N} \mid q \in r_n\mathbb{Z}\} \\
|q|_{\mathcal{B}} &= \inf\{1/r_n \mid q \in r_n\mathbb{Z}\}.
\end{align*}
Notice that $|.|_{\mathcal{B}}$ is not necessarily an absolute value, but when $\mathcal{B}$ is the constant sequence $p$, where $p$ is a prime number, then $|.|_{\mathcal{B}}$ is the usual $p$-adic value.

For $x \in \mathbb{R}$, we denote by $\{x\}$ the number in $[-1/2, 1/2[$ such that $x - \{x\} \in \mathbb{Z}$. As usual, we put $\|x\| = |\{x\}|$.

Let $\alpha$ be a real number. Given a positive integer $M$, Dirichlet’s Theorem asserts that for any $n$, there exists an integer $q$, with $0 < q \leq Mr_n$, satisfying simultaneously the approximation condition $\|q\alpha\| < 1/M$ and the divisibility condition $r_n|q$, i. e. $|q|_{\mathcal{B}} \leq 1/r_n$. Indeed, it is enough to
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apply Dirichlet’s Theorem to the number \( r_n\alpha \). We thus find positive integers \( q \) with

\[
q\|q\alpha\| |q|_B < 1.
\]

By analogy with Littlewood’s conjecture, we ask whether

\[
\inf_{q \in \mathbb{N}^*} q\|q\alpha\| |q|_B = 0 \quad (1)
\]

holds. The problem is trivial for \( \alpha \) rational, and for an irrational number \( \alpha \), one can easily see [5] that condition (1) is equivalent to the following: for each \( n \in \mathbb{N} \), consider the continued fraction expansion

\[
r_n\alpha = [a_{0,n}; a_{1,n}, ..., a_{k,n} ...].
\]

We have (1) if and only if

\[
\sup_{n \geq 0, k \geq 1} a_{k,n} = +\infty.
\]

However, we shall not use this characterization here.

We do not know whether (1) is satisfied for any real number \( \alpha \). In [5], we have proved that if we assume that the sequence \( B = (b_k)_{k \geq 1} \) is bounded, (1) is true for every quadratic number \( \alpha \). More precisely:

**Theorem 1.1. (de Mathan and Teulié [5])** Suppose that the sequence \( B \) is bounded. Let \( \alpha \) be a quadratic real number. Then there exists an infinite set of integers \( q > 1 \) with

\[
\|q\alpha\| \ll 1/q \quad (2)
\]

and

\[
|q|_B \ll 1/\ln q. \quad (3)
\]

In particular, we have

\[
\liminf_{q \to +\infty} q\ln q \|q\alpha\| |q|_B < +\infty.
\]

As usual, for positive functions \( x \) and \( y \), the notation \( x \ll y \) means that there exists a positive constant \( C \) such that \( x \leq Cy \).

In our lecture at Graz, for the “Journées Arithmétiques 2003”, it was discussed whether the factor \( \ln q \) in (3) is best possible. We do not know the answer to this question, but we shall prove:
Theorem 1.2. Assume that the sequence $B$ is bounded. Let $\alpha$ be a real quadratic number, and let $S$ be a set of integers $q > 1$ with
\[ \|qa\| \ll 1/q. \] (2)
Then there exists a constant $\lambda = \lambda(S)$ such that
\[ |qB| \gg \frac{1}{(\ln q)^\lambda} \] (4)
for any $q \in S$.

One may expect that (4) holds for any $\lambda > 1$, but we are not able to prove this. We do not even know whether there exists a real number $\lambda$ for which (4) holds for any set $S$ of integers $q > 1$ satisfying (2). Indeed, Theorem 1.2 does not ensure that $\sup_S \lambda(S) < +\infty$.

There is some analogy between this problem, and the classical simultaneous Diophantine approximation. For instance, let us recall Peck’s Theorem. Let $n$ be an integer greater than 1, and let $\alpha_1, \ldots, \alpha_n$, be $n$ numbers in a real algebraic number field of degree $n+1$ over $\mathbb{Q}$. Then it was proved by Peck [7] that there exists an infinite set of integers $q > 1$ with
\[ \|qa_k\| \ll (\ln q)^{-1/(n-1)}q^{-1/n} \]
for $1 \leq k < n$, and
\[ \|qa_n\| \ll q^{-1/n}. \]
Assume that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$, and let $S$ be an infinite set of integers $q > 1$, with
\[ \|qa_k\| \ll q^{-1/n} \]
for each $1 \leq k \leq n$. Then we have proved in [3] that there exists a constant $\kappa = \kappa(S)$ such that
\[ \max_{1 \leq k < n} \|qa_k\| \gg (\ln q)^{-\kappa}q^{-1/n}. \]

Theorem 1.2 can be regarded as an analogue of this result with $n = 1$, and its proof is similar.

2. Proof of the result

2.1. Some rational approximations of $\alpha$.

In the quadratic field $\mathbb{Q}(\alpha)$, there exists a unit $\omega$ of infinite order. Replacing, if necessary, $\omega$ by $\omega^2$ or $1/\omega^2$, we may suppose $\omega > 1$. In his original work, Peck uses units which are “large” and whose other conjugates are “small” and close to be equal. Here, Peck’s units are just the $\omega_m$’s, with $m \in \mathbb{N}$. We shall use these units in order to describe the rational approximations of $\alpha$ which satisfy (2).
Denote by \( \sigma_0 = \text{id} \) and \( \sigma_1 = \sigma \) the automorphisms of \( \mathbb{Q}(\alpha) \). As usual, we denote by \( \text{Tr} \) the trace form \( \text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}} = \sigma_0 + \sigma_1 \). The basis \( (1, \alpha) \) of \( \mathbb{Q}(\alpha) \) admits a dual basis \( (\beta_0, \beta_1) \) for the non-degenerate \( \mathbb{Q} \)-bilinear form \( (x, y) \mapsto \text{Tr}(xy) \) on \( \mathbb{Q}(\alpha) \). That means that, if we set \( \alpha_0 = 1 \) and \( \alpha_1 = \alpha \), we have \( \text{Tr}(\alpha_k \beta_l) = \delta_{kl} \), for \( k = 0, 1 \) and \( l = 0, 1 \), where \( \delta_{ll} = 1 \), and \( \delta_{kl} = 0 \) if \( k \neq l \). Here it is easy to calculate \( \beta_0 = \frac{1}{\alpha - \sigma(\alpha)} \) and \( \beta_1 = \frac{1}{\alpha - \sigma(\alpha)} \). Hence, if we put
\[
\eta = \frac{-q \sigma(\alpha) + q'}{\alpha - \sigma(\alpha)},
\]
where \( q \) and \( q' \) are rational numbers, we have
\[
q = \text{Tr} \eta \tag{5}
\]
and
\[
q' = \text{Tr}(\alpha \eta) \tag{6}
\]
Also notice that (5) and (6) imply that
\[
q \alpha - q' = (\alpha - \sigma(\alpha)) \sigma(\eta). \tag{7}
\]

Let \( D \) be a positive integer such that \( D \alpha, \frac{D \alpha}{\alpha - \sigma(\alpha)} \), and \( \frac{D \alpha}{\alpha - \sigma(\alpha)} \) are algebraic integers.

The notation \( A \asymp B \), where \( A \) and \( B \) are positive quantities, means that \( B \ll A \ll B \).

**Lemma 2.1.** Let \( \gamma \) be a positive number in \( \mathbb{Q}(\alpha) \). Let \( \Delta \) be a positive integer such that \( \Delta \gamma \) is an algebraic integer. For each \( m \in \mathbb{N} \), define the rational number
\[
q = q(m) = \text{Tr}(\gamma \omega^m). \tag{8}
\]
Then \( \Delta q \) is a rational integer, one has \( q > 0 \) when \( m \) is large, and the integers \( D \Delta q \) satisfy (2).

**Proof.** Also define
\[
q' = q'(m) = \text{Tr}(\alpha \gamma \omega^m).
\]
As \( \Delta \gamma \omega^m \) and \( D \Delta \alpha \gamma \omega^m \) are algebraic integers, \( \Delta q \) and \( D \Delta q' \) are rational integers. As \( \sigma(\omega) = 1/\omega \), we have \( q = \gamma \omega^m + \sigma(\gamma) \omega^{-m} \), hence \( q > 0 \) as soon as \( \omega^{2m} > -\sigma(\gamma)/\gamma \), and then
\[
q \asymp \omega^m. \tag{9}
\]
From (7), we get
\[
q \alpha - q' = (\alpha - \sigma(\alpha)) \sigma(\gamma) \omega^{-m}, \text{ hence}
\]
\[
|q \alpha - q'| \asymp \omega^{-m}. \tag{10}
\]
As \( D \Delta q \) and \( D \Delta q' \) are integers, it follows from (10) that for large \( m \) we have \( \|D \Delta q \alpha\| = D \Delta |q \alpha - q'| \), and by (9) and (10), the integers \( D \Delta q \) satisfy (2).
Conversely:

Lemma 2.2. Let $S$ be a set of positive integers $q$ satisfying (2). Then there exists a finite set $\Gamma$ of numbers $\gamma \in \mathbb{Q}(\alpha)$, $\gamma \neq 0$, such that for any $q \in S$, there exist $\gamma \in \Gamma$ and $m \in \mathbb{N}$ such that

$$q = \text{Tr}(\gamma \omega^m). \quad (8)$$

Proof. For $q \in S$, let $m(q) = m$ be the positive integer such that $\omega^{m-1} \leq q < \omega^m$. We thus have $\omega^m \asymp q$. Let $q'$ be the rational integer such that \{q\alpha\} = q\alpha - q'. Set

$$\gamma = \frac{-q\sigma(\alpha) + q'}{\alpha - \sigma(\alpha)} \omega^{-m}.$$ 

First, notice that $D\gamma$ is an algebraic integer. From (5), we get (8). Writing

$$\gamma \omega^m = q - \frac{q\alpha - q'}{\alpha - \sigma(\alpha)}$$

we see that $\gamma > 0$ when $q$ is large, and $\gamma \omega^m \asymp q$. As we have $\omega^m \asymp q$, we thus get $\gamma \asymp 1$. We also have

$$\sigma(\gamma) = \frac{q\alpha - q'}{\alpha - \sigma(\alpha)} \omega^m,$$

hence, by (2), $|\sigma(\gamma)| \ll \omega^m / q$, and thus, $|\sigma(\gamma)| \ll 1$. Then, as $D\gamma$ is an algebraic integer in $\mathbb{Q}(\alpha)$, and $\max(|\gamma|, |\sigma(\gamma)|) \ll 1$, the set of the $\gamma$'s is finite.

2.2. End of proof.

Denote by $P$ the set of all prime numbers dividing one of the $b_k$. Since we assume that the sequence $(b_k)$ is bounded, this set is finite. For $p \in P$, we extend the $p$-adic absolute value to $\mathbb{Q}(\alpha)$. The completion of this field is $\mathbb{Q}_p(\alpha)$. As above, let $\omega$ be a unit in $\mathbb{Q}(\alpha)$ with $\omega > 1$. Note that $|\omega|_p = 1$. The ball \{ $x \in \mathbb{Q}_p(\alpha); |x - 1|_p < p^{-1/(p-1)}$ \} is a subgroup of finite index in the multiplicative group \{ $x \in \mathbb{Q}_p(\alpha); |x|_p = 1$ \}. Hence, replacing $\omega$ by $\omega^n$, where $n$ is a suitable positive integer, we may also suppose that $|\omega - 1|_p < p^{-1/(p-1)}$ for every $p \in P$.

We shall use the $p$-adic logarithm function, which is defined on the multiplicative group $\{ x \in \mathbb{C}_p; |x - 1|_p < 1 \} \subset \mathbb{C}_p$ by

$$\log x = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(x - 1)^n}{n}.$$ 

This function satisfies

$$\log xy = \log x + \log y,$$
and, for $|x - 1|_p < p^{-1/(p-1)}$, $|\log x|_p = |x - 1|_p$. Hence, for $|x - 1|_p < p^{-1/(p-1)}$ and $|y - 1|_p < p^{-1/(p-1)}$, we have

$$|\log x - \log y|_p = |\log \frac{x}{y}|_p = \frac{|x - 1|_p}{|y|_p}.$$  \hfill (11)

We prove:

**Lemma 2.3.** Let $p$ be a number of $P$. Let $\gamma$ be a positive number of $\mathbb{Q}(\alpha)$. For $m \in \mathbb{N}$, set

$$q = q(m) = \text{Tr}(\gamma \omega^m).$$  \hfill (8)

Then, if

$$\left| \frac{\sigma(\gamma)}{\gamma} + 1 \right|_p \geq p^{-1/(p-1)},$$

we have

$$|q|_p \asymp 1$$

for large $m$; if

$$\left| \frac{\sigma(\gamma)}{\gamma} + 1 \right|_p < p^{-1/(p-1)},$$

then

$$|q|_p \asymp 2m \log \omega - \log(-\sigma(\gamma)/\gamma)|_p.$$  \hfill (12)

**Proof.** Recall that $q > 0$ when $m$ is large (Lemma 2.1). From the definition, we get for each $p \in P$, $|q|_p = |\gamma \omega^m + \sigma(\gamma) \omega^{-m}|_p = |\gamma|_p |\omega^{2m} - \delta|_p$, where $\delta = -\sigma(\gamma)/\gamma$. If $|\delta - 1|_p \geq p^{-1/(p-1)}$, we have $|\omega^{2m} - \delta|_p \geq p^{-1/(p-1)}$, since $|\omega - 1|_p < p^{-1/(p-1)}$ and $|\omega^{2m} - 1|_p < p^{-1/(p-1)}$. Then we get

$$|q|_p \asymp 1.$$  

If $|\delta - 1|_p < p^{-1/(p-1)}$, then, by (11), we write $|\omega^{2m} - \delta|_p = |2m \log \omega - \log \delta|_p$, and we obtain (12).

Accordingly, in order to achieve the proof of the result, we shall use known lower bounds for linear forms in $p$-adic logarithms. For instance, it follows from [8] that:

**Lemma 2.4. (K. Yu [8])** Let $x$ and $y$ be algebraic numbers in $\mathbb{C}_p$, with $|x - 1|_p < p^{-1/(p-1)}$ and $|y - 1|_p < p^{-1/(p-1)}$. Then there exists a real constant $\kappa$ such that for any pair $(k, \ell)$ of rational integers with $k \log x + \ell \log y \neq 0$, one has

$$|k \log x + \ell \log y|_p \gg (\max(|k|, |\ell|))^{-\kappa}.$$  

Note that this result is trivial, with $\kappa = 1$, if $\log x$ and $\log y$ are not linearly independent over $\mathbb{Q}$, and $\log x \neq 0$, i.e., $x \neq 1$. Indeed, if $a \log x = b \log y$, where $a$ and $b$ are rational integers with $b \neq 0$, then we write $|k \log x + \ell \log y|_p = \frac{1}{|b|_p} |bk + a\ell|_p |x - 1|_p$. Hence we get $|k \log x + \ell \log y|_p \gg |bk + a\ell|_p \gg \max(|k|, |\ell|)$, when $k \log x + \ell \log y \neq 0$. 

We can then achieve the proof of Theorem 1.2. Applying Lemma 2.2, we can suppose that the set $\Gamma$ contains a unique element $\gamma > 0$, i.e., for any $q \in \mathcal{S}$, there exists $m \in \mathbb{N}$ such that we have (8). It follows from Lemma 2.3 and 2.4 that there exists a constant $\kappa$ such that $|q|_p \gg m^{-\kappa}$ (one may take $\kappa = 0$ if $|\sigma(\gamma)/\gamma + 1|_p \geq p^{-1/(p-1)}$). As $q \asymp \omega^m$, hence $m \asymp \ln q$, we get $|q|_p \gg (\ln q)^{-\kappa}$. Now set $\kappa = \kappa_p$ (the constant $\kappa_p$ may depend upon $p \in P$). Note that $|q|_B \geq \prod_{p \in P} |q|_p$. Indeed, putting $|q|_B = 1/r_n$, we have $q \in r_n\mathbb{Z}$, hence $|q|_p \leq |r_n|_p$ and $\prod_{p \in P} |q|_p \leq \prod_{p \in P} |r_n|_p = 1/r_n$. We thus get (4) with $\lambda = \sum_{p \in P} \kappa_p$, and Theorem 1.2 is proved.

2.3. A remark.

Note that one may also use Lemma 2.3 for solving the opposite problem. For simplicity, consider the case where $|.|_B$ is the $p$-adic value for a prime number $p$. If we take a positive number $\gamma \in \mathbb{Q}(\alpha)$ such that $\sigma(\gamma) = -\gamma$, for instance, $\gamma = \alpha - \sigma(\alpha)$ (one may replace $\alpha$ by $-\alpha$, and so, we can suppose $\alpha - \sigma(\alpha) > 0$), then we have $\log(-\sigma(\gamma)/\gamma) = 0$, and by (12), we get $|\text{Tr}(\gamma \omega^m)|_p \leq |m|_p$. By Lemma 2.1, there exists a positive integer $A$ such that for every large $m$, the numbers $q = q(m) = A\text{Tr}(\gamma \omega^m)$ are positive integers satisfying (2). For $m = p^s$ with $s \in \mathbb{N}$, we get $|m|_p = 1/m$, hence $|q|_p \propto 1/m$. Since $m \asymp \ln q$, we have thus proved that there exists an infinite set of integers $q > 1$ satisfying (2) and (3) (which is Theorem 1.1). In this way we obtain integers $q > 1$ satisfying (2) and such that $|q|_p \asymp 1/\ln q$.

One can ask whether there exists an infinite set of integers $q > 1$ satisfying (2), with

$$\inf |q|_p \ln q = 0.$$  \hspace{1cm} (3’)

Given a positive decreasing sequence $(\epsilon_m)$ with $\sum_{m=0}^{+\infty} \epsilon_m = +\infty$, a $p$-adic version [4] of Khintchine’s Theorem ensures that for almost all $x \in \mathbb{Z}_p$, there exist infinitely many positive integers $m$ such that $|x - m|_p \leq \epsilon_m$. One often considers as reasonable the hypothesis that a given “special” irrational number $x \in \mathbb{Z}_p$ satisfies this condition, with $\epsilon_m = 1/(m \ln m)$ for $m > 1$ (which is false if $x \in \mathbb{Z}_p \cap \mathbb{Q}$, since in this case, we have $|x - m|_p \gg 1/m$ for large $m$). Let us prove that we can choose $\gamma > 0$ in $\mathbb{Q}(\alpha)$, with $|\sigma(\gamma)/\gamma + 1|_p < |\omega - 1|_p$, such that $\frac{\log(-\sigma(\gamma)/\gamma)}{\log \omega}$ is an irrational number in $\mathbb{Z}_p$. In order to make this obvious, we prove:

Lemma 2.5. There exists $\xi \in \mathbb{Q}(\alpha)$ such that $\xi$ is not a unit, $N_{\mathbb{Q}(\alpha):\mathbb{Q}} \xi = 1$, and $|\xi|_p = 1$.

Proof. The number $\omega$ is a root of the equation $\omega^2 - S \omega + 1 = 0$, where $S$ is a rational integer, $S = \text{Tr} \omega$. The number $\xi$ must be a root of an equation $\xi^2 - t\xi + 1 = 0$, where $t$ is a rational number for which there exists a positive
rational number $\rho$ such that $t^2 - 4 = \rho^2(S^2 - 4)$. Such pairs $(t, \rho)$ can be expressed by using a rational parameter $\theta$:

$$t = \frac{2(S^2 - 4)t^2 + 2}{(S^2 - 4)t^2 - 1} = 2 + \frac{4}{(S^2 - 4)t^2 - 1}$$

$$\rho = \frac{4\theta}{(S^2 - 4)t^2 - 1}.$$ 

Let us show that we can choose $\theta \in \mathbb{Q}^*$ such that $t \notin \mathbb{Z}$ and $|t|_p \leq 1$. It is enough to take $\theta = p$. As we have $S^2 > 4$, hence $S^2 \geq 9$ and $(S^2 - 4)p^2 - 1 > 4$, $t$ cannot be an integer for this choice of $\theta$. But we have $|t|_p \leq 1$, since $|(S^2 - 4)p^2 - 1|_p = 1$. Then there exists a number $\xi \in \mathbb{Q}(\alpha)$ such that $\xi^2 - t\xi + 1 = 0$, and $\xi$ is neither a rational number, since $\rho > 0$, nor an algebraic integer, since $t \notin \mathbb{Z}$. Then we have $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\xi) = 1$, and $|\xi|_p = 1$ because either condition $|\xi|_p < 1$ or $|\xi|_p > 1$ would imply $|t|_p = |\xi + \xi^{-1}|_p > 1$.

Replacing $\xi$ by $\xi^n$, where $n$ is a suitable positive integer, we thus may find a $\xi$ satisfying Lemma 2.5, with moreover $|\xi - 1|_p < |\omega - 1|_p$. Then we have $|\log \xi|_p < |\log \omega|_p$. Further let us prove that $\frac{\log \xi}{\log \omega} \in \mathbb{Q}_p$. Indeed that is trivial if $\alpha \in \mathbb{Q}_p$, since in this case $\xi$ and $\omega$ lie in $\mathbb{Q}_p$, hence so do $\log \xi$ and $\log \omega$. If $\mathbb{Q}_p(\alpha)$ has degree 2 over $\mathbb{Q}_p$, then $\log \xi$ and $\log \omega$ lie in $\mathbb{Q}_p(\alpha)$. But $\sigma$ can be extended into a continuous $\mathbb{Q}_p$-automorphism of $\mathbb{Q}_p(\alpha)$, and we get $\sigma\left(\frac{\log \xi}{\log \omega}\right) = \frac{\log \sigma(\xi)}{\log \sigma(\omega)} = -\frac{\log \xi}{\log \omega} = -\frac{\log \xi}{\log \omega}$, since $\xi \sigma(\xi) = \omega \sigma(\omega) = 1$.

That proves that $\frac{\log \xi}{\log \omega} \in \mathbb{Q}_p$, and since $|\log \xi|_p < |\log \omega|_p$, we conclude that $\frac{\log \xi}{2\log \omega} \in \mathbb{Z}_p$. Lastly, $\frac{\log \xi}{\log \omega}$ is not a rational number, since $\xi$ is not a unit. Now, by Hilbert’s Theorem, there exists $\gamma \in \mathbb{Q}(\alpha)$, with $\gamma > 0$, such that $\xi = -\sigma(\gamma)/\gamma$. We thus have found $\gamma > 0$ in $\mathbb{Q}(\alpha)$, such that $\left|\frac{\sigma(\gamma)}{\gamma} + 1\right|_p < p^{-1/(p-1)}$ and $\left|\log\left(-\frac{\sigma(\gamma)}{\gamma}\right)\right|_p \ll m/\log m$, and, by (12), we could obtain an infinite set of integers $q > 1$, $q = A\text{Tr}(\gamma \omega^m)$ where $A$ is a positive integer, satisfying (2) and such that $|q|_p \ll \frac{1}{\ln q \ln \ln q}$. In particular, (3’) would be satisfied.

3. Conclusion

For a sequence $\mathcal{B}$ bounded, the Roth-Ridout Theorem [6] allows us to see that for any irrational algebraic real number $\alpha$, thus in particular for $\alpha$ quadratic, we have:

$$\inf_{q > 0} q^{1+\epsilon} \|q\alpha\| q|\mathcal{B} > 0$$
Of course, our method is far from enabling us to prove that there exists a real constant $\lambda$ such that
\[ \inf_{q>1} q(\ln q)^\lambda \|q\alpha\| \|q\|_B > 0. \]

We can only study the approximations with $q\|q\alpha\| \ll 1$. It seems difficult to study approximations in the “orthogonal direction” $q\|q\|_B \ll 1$, with for instance, $q = p^n$, for a prime number $p$. For such approximations, it is not known whether $\inf_{n\in\mathbb{N}} \|p^n\alpha\| = 0$ holds, neither if there exists $\lambda$ such that $\inf_{n>0} n^\lambda \|p^n\alpha\| > 0$. It is very difficult to obtain more precise results than the Roth-Ridout Theorem (see [1]).

Even for rational approximations satisfying (2), we are not able to prove that the constants $\lambda(S)$ are bounded. This is related to Lemma 2.4. It would be necessary to prove that there exists a real constant $\kappa$ for which this Lemma holds for $x = \omega$ and for any $y \in \mathbb{Q}(\alpha)$ with $|y - 1|_p < p^{-1/(p-1)}$ and $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(y) = 1$. There exist many effective estimates of $|k \log x + \ell \log y|_p$ (see for instance [2] and [8]), but they do not provide the needed result. It seems difficult to take the particular conditions required into account.

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