Jörn Steuding

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Résumé. On démontre que, pour tout $\theta$ réel, il existe une infinité de $s = \sigma + it$ avec $\sigma \to 1+$ et $t \to +\infty$ tel que

$$\text{Re} \left\{ \exp(i\theta) \log L(s, \chi) \right\} \geq \log \frac{\log \log \log t}{\log \log \log \log t} + O(1).$$

La démonstration est basée sur une version effective du théorème de Kronecker sur les approximations diophantiennes.

Abstract. We prove that for any real $\theta$ there are infinitely many values of $s = \sigma + it$ with $\sigma \to 1+$ and $t \to +\infty$ such that

$$\text{Re} \left\{ \exp(i\theta) \log L(s, \chi) \right\} \geq \log \frac{\log \log \log t}{\log \log \log \log t} + O(1).$$

The proof relies on an effective version of Kronecker’s approximation theorem.

1. Extremal values

Extremal values of the Riemann zeta-function in the half-plane of absolute convergence were first studied by H. Bohr and Landau [1]. Their results rely essentially on the diophantine approximation theorems of Dirichlet and Kronecker. Whereas everything easily extends to Dirichlet series with real coefficients of one sign (see [7], §9.32) the question of general Dirichlet series is more delicate. In this paper we shall establish quantitative results for Dirichlet $L$-functions.

Let $q$ be a positive integer and let $\chi$ be a Dirichlet character mod $q$. As usual, denote by $s = \sigma + it$ with $\sigma, t \in \mathbb{R}, i^2 = -1$, a complex variable. Then the Dirichlet $L$-function associated to the character $\chi$ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1},$$

where the product is taken over all primes $p$; the Dirichlet series, and so the Euler product, converge absolutely in the half-plane $\sigma > 1$. Denote by

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χ₀ the principal character mod q, i.e., χ₀(n) = 1 for all n coprime with q. Then

\[ L(s, χ₀) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right). \]

Thus we may interpret the well-known Riemann zeta-function ζ(s) as the Dirichlet L-function to the principal character χ₀ mod 1. Furthermore, it follows that L(s, χ₀) has a simple pole at s = 1 with residue 1. On the other side, any L(s, χ) with χ ≠ χ₀ is regular at s = 1 with L(1, χ) ≠ 0 (by Dirichlet’s analytic class number-formula). Since L(s, χ) is non-vanishing in σ > 1, we may define the logarithm (by choosing any one of the values of the logarithm). It is easily shown that for σ > 1

\[ \log L(s, χ) = \sum_p \sum_{k \geq 1} \frac{χ(p)^k}{kp^ks} = \sum_p \frac{χ(p)}{p^s} + O(1). \]

Obviously, | log L(s, χ)| ≤ L(σ, χ₀) for σ > 1. However

**Theorem 1.1.** For any ε > 0 and any real θ there exists a sequence of s = σ + it with σ > 1 and t → +∞ such that

\[ \text{Re}\{\exp(iθ) \log L(s, χ)\} \geq (1 - ε) \log L(σ, χ₀) + O(1). \]

In particular,

\[ \liminf_{σ > 1, t \geq 1} |L(s, χ)| = 0 \quad \text{and} \quad \limsup_{σ > 1, t \geq 1} |L(s, χ)| = \infty. \]

In spite of the non-vanishing of L(s, χ) the absolute value takes arbitrarily small values in the half-plane σ > 1!

The proof follows the ideas of H. Bohr and Landau [1] (resp. [8], §8.6) with which they obtained similar results for the Riemann zeta-function (answering a question of Hilbert). However, they argued with Dirichlet’s *homogeneous* approximation theorem for growth estimates of |ζ(s)| and with Kronecker’s *inhomogeneous* approximation theorem for its reciprocal. We will unify both approaches.

**Proof.** Using (2) we have for x ≥ 2

\[ \text{Re}\{\exp(iθ) \log L(s, χ)\} \geq \sum_{p \leq x} \frac{χ₀(p)}{p^σ} \text{Re}\{\exp(iθ)χ(p)p^{-it}\} - \sum_{p > x} \frac{χ₀(p)}{p^σ} + O(1). \]
Denote by $\varphi(q)$ the number of prime residue classes mod $q$. Since the values $\chi(p)$ are $\varphi(q)$-th roots of unity if $p$ does not divide $q$, and equal to zero otherwise, there exist integers $\lambda_p$ (uniquely determined mod $\varphi(q)$) with

$$\chi(p) = \begin{cases} \exp \left( \frac{2\pi i \lambda_p}{\varphi(q)} \right) & \text{if } p \nmid q, \\ 0 & \text{if } p|q. \end{cases}$$

Hence,

$$\text{Re} \{ \exp(i\theta) \chi(p)p^{-it} \} = \cos \left( t \log p - 2\pi \frac{\lambda_p}{\varphi(q)} - \theta \right).$$

In view of the unique prime factorization of the integers the logarithms of the prime numbers are linearly independent. Thus, Kronecker’s approximation theorem (see [8], §8.3, resp. Theorem 3.2 below) implies that for any given integer $\omega$ and any $x$ there exist a real number $\tau > 0$ and integers $h_p$ such that

$$\left| \frac{\tau}{2\pi} \log p - \frac{\lambda_p}{\varphi(q)} - \frac{\theta}{2\pi} - h_p \right| < \frac{1}{\omega} \quad \text{for all } p \leq x.$$

Obviously, with $\omega \to \infty$ we get infinitely many $\tau$ with this property. It follows that

$$\cos \left( \tau \log p - 2\pi \frac{\lambda_p}{\varphi(q)} - \theta \right) \geq \cos \left( \frac{2\pi}{\omega} \right) \quad \text{for all } p \leq x,$$

provided that $\omega \geq 4$. Therefore, we deduce from (3)

$$\text{Re} \{ \exp(i\theta) \log L(\sigma + i\tau, \chi) \} \geq \cos \left( \frac{2\pi}{\omega} \right) \sum_{p \leq x} \frac{\chi_0(p)}{p^\sigma} - \sum_{p > x} \frac{\chi_0(p)}{p^\sigma} + O(1),$$

resp.

$$\text{Re} \{ \exp(i\theta) \log L(\sigma + i\tau, \chi) \} \geq \cos \left( \frac{2\pi}{\omega} \right) \log L(\sigma, \chi_0) - 2 \sum_{p > x} \frac{1}{p^\sigma} + O(1)$$

in view of (2). Obviously, the appearing series converges. Thus, sending $\omega$ and $x$ to infinity gives the inequality of Theorem 1.1. By (1) we have

$$\log L(\sigma, \chi_0) = \log \left( \frac{1}{\sigma - 1} + O(1) \right) = \log \frac{1}{\sigma - 1} + o(1)$$

for $\sigma \to 1+$. Therefore, with $\theta = 0$, resp. $\theta = \pi$, and $\sigma \to 1+$ the further assertions of the theorem follow.

The same method applies to other Dirichlet series as well. For example, one can show that the Lerch zeta-function is unbounded in the half-plane.
of absolute convergence:

$$\limsup_{\sigma>1,t\geq1} \sum_{n=0}^{\infty} \frac{\exp(2\pi i \lambda n)}{(n+\alpha)^s} = +\infty$$

if $\alpha > 0$ is transcendental; note that in the case of transcendental $\alpha$ the Lerch zeta-function has zeros in $\sigma > 1$ (see [3] and [4]).

In view of Theorem 1.1 we have to ask for quantitative estimates. Let $\pi(x)$ count the prime numbers $p \leq x$. By partial summation,

$$\sum_{x<p\leq y} \frac{1}{p^s} \sim \frac{\pi(y)}{y^s} - \frac{\pi(x)}{x^s} + \sigma \int_x^y \frac{\pi(u)}{u^s+1} du.$$ 

The prime number theorem implies for $x \geq 2$

$$\sum_{x<p\leq y} \frac{1}{p^s} \sim \left( \frac{y^{1-s}}{\log y} - \frac{x^{1-s}}{\log x} \right) + \sigma \int_x^y \frac{du}{u^s \log u}.$$ 

By the second mean-value theorem,

$$\int_x^y \frac{du}{u^s \log u} \log u = \frac{1}{\log \xi} \int_x^y \frac{du}{u^s} = \frac{x^{1-s} - y^{1-s}}{(\sigma - 1) \log \xi}$$

for some $\xi \in (x, y)$. Thus, substituting $\xi$ by $x$ and sending $y \to \infty$, we obtain the estimate

$$\sum_{x<p\leq y} \frac{1}{p^s} \leq (1 + o(1)) \frac{x^{1-s}}{(\sigma - 1) \log x}$$

as $x \to \infty$. This gives in (6)

(8) $\text{Re} \{\exp(i\theta) \log L(\sigma + i\tau, \chi)\} \geq \cos \left( \frac{2\pi}{\omega} \right) \log L(\sigma, \chi_0) - (2 + o(1)) \frac{x^{1-s}}{(\sigma - 1) \log x} + O(1).$

Substituting (7) in formula (8) yields

$\text{Re} \{\exp(i\theta) \log L(\sigma + i\tau, \chi)\}$

$$\geq (1 + O(\omega^{-2})) \log \frac{1}{\sigma - 1} - (2 + o(1)) \frac{x^{1-s}}{(\sigma - 1) \log x} + O(1).$$

Let

$$x = \exp \left( \frac{1}{\sigma - 1} \frac{1}{\sigma - 1} \right),$$

then $x$ tends to infinity as $\sigma \to 1+$. We obtain for $x$ sufficiently large

(9) $\text{Re} \{\exp(i\theta) \log L(\sigma + i\tau, \chi)\} \geq (1 + O(\omega^{-2})) \log \frac{\log x}{\log \log x} + O(1).$

The question is how the quantities $\omega, x$ and $\tau$ depend on each other.
2. Effective approximation

H. Bohr and Landau [2] (resp. [8], §8.8) proved the existence of a $\tau$ with $0 \leq \tau \leq \exp(N^6)$ such that
\[
\cos(\tau \log p_\nu) < -1 + \frac{1}{N} \quad \text{for} \quad \nu = 1, \ldots, N,
\]
where $p_\nu$ denotes the $\nu$-th prime number. This can be seen as a first effective version of Kronecker’s approximation theorem, with a bound for $\tau$ (similar to the one in Dirichlet’s approximation theorem). In view of (5) this yields, in addition with the easier case of bounding $|\zeta(s)|$ from below, the existence of infinite sequences $s_\pm = \sigma_\pm + it_\pm$ with $\sigma_\pm \to 1+$ and $t_\pm \to +\infty$ for which
\[
|\zeta(s_+)| \geq A \log \log t_+ \quad \text{and} \quad \left| \frac{1}{\zeta(s_-)} \right| \geq A \log \log t_-,
\]
where $A > 0$ is an absolute constant. However, for Dirichlet $L$-functions we need a more general effective version of Kronecker’s approximation theorem. Using the idea of Bohr and Landau in addition with Baker’s estimate for linear forms, Rieger [6] proved the remarkable

**Theorem 2.1.** Let $\nu, N \in \mathbb{N}, b \in \mathbb{Z}, 1 \leq \omega, U \in \mathbb{R}$. Let $p_1 < \ldots < p_N$ be prime numbers (not necessarily consecutive) and $u_\nu \in \mathbb{Z}, \quad 0 < |u_\nu| \leq U, \quad \beta_\nu \in \mathbb{R} \quad \text{for} \quad \nu = 1, \ldots, N.$

Then there exist $h_\nu \in \mathbb{Z}, 0 \leq \nu \leq N$, and an effectively computable number $C = C(N, p_N) > 0$, depending on $N$ and $p_N$ only, with
\[
\left| h_0 \frac{u_\nu}{v} \log p_\nu - \beta_\nu - h_\nu \right| < \frac{1}{\omega} \quad \text{for} \quad \nu = 1, \ldots, N,
\]
and $b \leq h_0 \leq b + (2U\nu\omega)^C$.

We need $C$ explicitly. Therefore we shall give a sketch of Rieger’s proof and add in the crucial step a result on an explicit lower bound for linear forms in logarithms due to Waldschmidt [9].

Let $\mathbb{K}$ be a number field of degree $D$ over $\mathbb{Q}$ and denote by $L_{\mathbb{K}}$ the set of logarithms of the elements of $\mathbb{K} \setminus \{0\}$, i.e.,
\[
L_{\mathbb{K}} = \{ \ell \in \mathbb{C} : \exp(\ell) \in \mathbb{K} \}.
\]
If $a$ is an algebraic number with minimal polynomial $P(X)$ over $\mathbb{Z}$, then define the absolute logarithmic height of $a$ by
\[
h(a) = \frac{1}{D} \int_0^1 \log |P(\exp(2\pi i \phi))| d\phi;
\]
note that $h(a) = \log a$ for integers $a \geq 2$. Waldschmidt proved
Theorem 2.2. Let $\ell_{\nu} \in L_{\mathbb{K}}$ and $\beta_{\nu} \in \mathbb{Q}$ for $\nu = 1, \ldots, N$, not all equal zero. Define $a_{\nu} = \exp(\ell_{\nu})$ for $\nu = 1, \ldots, N$ and
\[
\Lambda = \beta_{0} + \beta_{1} \log a_{1} + \ldots + \beta_{N} \log a_{N}.
\]
Let $E, W$ and $V_{\nu}, 1 \leq \nu \leq N$, be positive real numbers, satisfying
\[
W \geq \max_{1 \leq \nu \leq N} \{h(\beta_{\nu})\},
\]
\[
\frac{1}{D} \leq V_{1} \leq \ldots \leq V_{N},
\]
\[
V_{\nu} \geq \max \left\{h(a_{\nu}), \frac{|\log a_{\nu}|}{D} \right\} \quad \text{for } \nu = 1, \ldots, N
\]
and
\[
1 < E \leq \min \left\{\exp(V_{1}), \min_{1 \leq \nu \leq N} \left\{\frac{4DV_{\nu}}{|\log a_{\nu}|} \right\} \right\}.
\]
Finally, define $V_{\nu}^{+} = \max\{V_{\nu}, 1\}$ for $\nu = 1$ and $\nu = N - 1$, with $V_{1}^{+} = 1$ in the case $N = 1$. If $\Lambda \neq 0$, then
\[
|\Lambda| > \exp \left( - c(N)DN^{2}(W + \log(EDV_{N}^{+})) \log(EDV_{N-1}^{+}) \times \right.
\]
\[
\times (\log E)^{-N-1} \prod_{\nu=1}^{N} V_{\nu} \right)\]
with $c(N) \leq 2^{8N+51}N^{2N}$.

This leads to

Theorem 2.3. With the notation of Theorem 2.1 and under its assumptions there exists an integer $h_{0}$ such that (11) holds and
\[
b \leq h_{0} \leq b + 2 + ((3\omega U(N + 2) \log p_{N})^{4} + 2)^{N+2} \times \]
\[
\times \exp \left( 2^{8N+51}N^{2N}(1 + 2 \log p_{N})(1 + \log p_{N-1}) \prod_{\nu=2}^{N} \log p_{\nu} \right) ;
\]
if $p_{N}$ is the $N$-th prime number, then, for any $\epsilon > 0$ and $N$ sufficiently large,
\[
b \leq h_{0} \leq b + (\omega U)^{(4+\epsilon)N} \exp \left( N^{(2+\epsilon)N} \right) .
\]

Proof. For $t \in \mathbb{R}$ define
\[
f(t) = 1 + \exp(t) + \sum_{\nu=1}^{N} \exp \left( 2\pi i \left( t \frac{u_{\nu}}{v} \log p_{\nu} - \beta_{\nu} \right) \right) .
\]
With \( \gamma_{-1} := 0, \beta_{-1} := 0, \gamma_0 := 1, \beta_0 := 0 \) and \( \gamma_\nu := \frac{u_\nu}{v} \log p_\nu, 1 \leq \nu \leq N \), we have

\[
(14) \quad f(t) = \sum_{\nu=-1}^{N} \exp(2\pi i (t \gamma_\nu - \beta_\nu)).
\]

By the multinomial theorem,

\[
f(t)^k = \sum_{\nu_0 \geq 0}^{\nu_1} \ldots \sum_{\nu_N \geq 0}^{\nu_N} \frac{k!}{j_0! \cdots j_N!} \exp \left( 2\pi i \sum_{\nu=-1}^{N} j_\nu (t \gamma_\nu - \beta_\nu) \right) \cdot \prod_{\nu=1}^{N} \left( j_\nu - j'_\nu \right)^{\gamma_\nu}.
\]

Hence, for \( 0 < B \in \mathbb{R} \) and \( k \in \mathbb{N} \)

\[
J := \int_{b}^{b+B} |f(t)|^{2k} \, dt
\]

\[
= \sum_{\nu_0 \geq 0}^{\nu_1} \ldots \sum_{\nu_N \geq 0}^{\nu_N} \frac{k!}{j_0! \cdots j_N!} \exp \left( 2\pi i \sum_{\nu=-1}^{N} j_\nu (t \gamma_\nu - \beta_\nu) \right) \cdot \prod_{\nu=1}^{N} \left( j_\nu - j'_\nu \right)^{\gamma_\nu} \int_{b}^{b+B} \left( \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \gamma_\nu t - \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \beta_\nu \right) \, dt.
\]

By the theorem of Lindemmann

\[
\sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \gamma_\nu
\]

vanishes if and only if \( j_\nu = j'_\nu \) for \( \nu = -1, 0, \ldots, N \). Thus, integration gives

\[
\int_{b}^{b+B} \exp \left( 2\pi i \left( \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \gamma_\nu t - \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \beta_\nu \right) \right) \, dt = B
\]

if \( j_\nu = j'_\nu, \nu = -1, 0, \ldots, N \), and

\[
\left| \int_{b}^{b+B} \exp \left( 2\pi i \left( \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \gamma_\nu t - \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \beta_\nu \right) \right) \, dt \right|
\]

\[
\leq \frac{1}{\pi} \left| \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \gamma_\nu \right|^{-1}
\]

if \( j_\nu \neq j'_\nu \) for some \( \nu \in \{-1, 0, \ldots, N\} \). In the latter case there exists by Baker’s estimate for linear forms an effectively computable constant \( A \) such that

\[
\left| \sum_{\nu=-1}^{N} (j_\nu - j'_\nu) \gamma_\nu \right|^{-1} < A.
\]
Setting $\beta_0 = j_0 - j_0', \beta_\nu = \frac{u_\nu}{v}(j_\nu - j_\nu')$ and $a_\nu = p_\nu$ for $\nu = 1, \ldots, N$, we have, with the notation of Theorem 2.2,

$$\Lambda = \sum_{\nu=-1}^{N} (j_\nu - j_\nu') \gamma_\nu.$$ 

We may take $E = 1, W = \log p_N, V_1 = 1$ and $V_\nu = \log p_\nu$ for $\nu = 2, \ldots, N$. If $N \geq 2$, Theorem 2.2 gives

$$|\Lambda| > \exp \left(-2^{8N+51} N^{2N} (1 + 2 \log p_N)\left(1 + \log p_{N-1}\right) \prod_{\nu=2}^{N} \log p_\nu \right).$$

Thus we may take

\begin{equation}
A = \exp \left(2^{8N+51} N^{2N} (1 + 2 \log p_N)\left(1 + \log p_{N-1}\right) \prod_{\nu=2}^{N} \log p_\nu \right).
\end{equation}

Hence, we obtain

\begin{equation}
J \geq B \sum_{\sum_{j_{-1}+\ldots+j_N = k} j_{-1}! \cdots j_N!} \left( \frac{k!}{j_{-1}! \cdots j_N!} \right)^2 - \frac{A}{\pi} \sum_{\sum_{j_{-1}+\ldots+j_N = k} j_{-1}! \cdots j_N!} \frac{k!}{j_{-1}! \cdots j_N!} \sum_{\sum_{j_{-1}+\ldots+j_N = k} j_{-1}! \cdots j_N!} \frac{k!}{j_{-1}'! \cdots j_N'!}.
\end{equation}

Since

$$\sum_{\sum_{j_{-1}+\ldots+j_N = k}} 1 \leq (k + 1)^{N+2},$$

application of the Cauchy Schwarz-inequality to the first multiple sum and of the multinomial theorem to the second multiple sum on the right hand side of (16) yields

$$J \geq \left( \frac{B}{(k + 1)^{N+2}} - \frac{A}{\pi} \right) \left( \sum_{\sum_{j_{-1}+\ldots+j_N = k}} \frac{k!}{j_{-1}! \cdots j_N!} \right)^2 \geq \left( \frac{B}{(k + 1)^{N+2}} - \frac{A}{\pi} \right) (N + 2)^{2k}.$$ 

Setting $B = A(k + 1)^{N+2}$ and with $\tau \in [b, b + B]$ defined by

$$|f(\tau)| = \max_{t \in [b, b + B]} |f(t)|,$$

we obtain

$$\frac{B(N + 2)^{2k}}{2(k + 1)^{N+2}} \leq J \leq B|f(\tau)|^{2k}.$$
This gives

\[(17) \quad |f(\tau)| > N + 2 - 2\mu, \quad \text{where} \quad \mu := \frac{(N + 2)^2 \log k}{3k}; \]

note that \(\mu < 1\) for \(k \geq 11\). By definition

\[f(t) = 1 + \exp(2\pi i (t\gamma_\nu - \beta_\nu)) + \sum_{m=0}^{N} \exp(2\pi i (t\gamma_m - \beta_m)).\]

Therefore, using the triangle inequality,

\[|f(t)| \leq N + |1 + \exp(2\pi i (t\gamma_\nu - \beta_\nu))| \quad \text{for} \quad \nu = 0, \ldots, N,\]

and arbitrary \(t \in \mathbb{R}\). Thus, in view of (17)

\[|1 + \exp(2\pi i (t\gamma_\nu - \beta_\nu))| > 2 - 2\mu \quad \text{for} \quad \nu = 0, \ldots, N.\]

If \(h_\nu\) denotes the nearest integer to \(t\gamma_\nu - \beta_\nu\), then

\[|t\gamma_\nu - \beta_\nu - h_\nu| < \frac{\sqrt{\mu}}{2} \quad \text{for} \quad \nu = 0, \ldots, N.\]

For \(\nu = 0\) this implies \(|t - h_0| < \sqrt{\mu}\). Replacing \(\tau\) by \(h_0\) yields

\[|h_0\gamma_\nu - \beta_\nu h_\nu| < \sqrt{\mu} \left(1 + \max_{\nu=1,\ldots,N} |\gamma_\nu| \right) \quad \text{for} \quad \nu = 1, \ldots, N.\]

Putting \(k = \left[(3wU(N + 2) \log p_N)^4\right] + 1\) we get

\[b - 1 \leq h_0 \leq b + 1 + B = b + 1 + A\left((3wU(N + 2) \log p_N)^4\right) + 2)^{N+2}.\]

Substituting (15) and replacing \(b - 1\) by \(b\), the assertion of Theorem 2.1 follows with the estimate (12) of Theorem 2.3; (13) can be proved by standard estimates.

\[3. \quad \text{Quantitative results}\]

We continue with inequality (9). Let \(p_N\) be the \(N\)-th prime. Then, using Theorem 2.3 with \(N = \pi(x), \nu = u_\nu = 1,\) and

\[\beta_\nu = \frac{\lambda_{p_\nu}}{\varphi(q)} + \frac{\theta}{2\pi} \quad \text{for} \quad \nu = 1, \ldots, N,\]

yields the existence of \(\tau = 2\pi h_0\) with

\[(18) \quad b \leq \frac{\tau}{2\pi} \leq b + \omega^{(4+\epsilon)N} \exp(N^{(2+\epsilon)N})\]

such that (4) holds, as \(N\) and \(x\) tend to infinity. We choose \(\omega = \log \log x,\) then the prime number theorem and (18) imply

\[\log x = \log N + O(\log \log N), \quad \log N \geq \log \log \log \tau + O(\log \log \log \log \tau).\]

Substituting this in (9) we obtain
Theorem 3.1. For any real $\theta$ there are infinitely many values of $s = \sigma + it$ with $\sigma \to 1+$ and $t \to +\infty$ such that
\[ \Re \{ \exp(i\theta) \log L(s, \chi) \} \geq \log \frac{\log \log t}{\log \log \log \log t} + O(1). \]

Using the Phragmén-Lindelöf principle, it is even possible to get quantitative estimates on the abscissa of absolute convergence. We write $f(x) = \Omega(g(x))$ with a positive function $g(x)$ if
\[ \liminf_{x \to \infty} \frac{|f(x)|}{g(x)} > 0; \]
hence, $f(x) = \Omega(g(x))$ is the negation of $f(x) = o(g(x))$. Then, by the same reasoning as in [8], §8.4, we deduce
\[ L(1 + it, \chi) = \Omega \left( \frac{\log \log \log t}{\log \log \log \log t} \right), \]
and
\[ \frac{1}{L(1 + it, \chi)} = \Omega \left( \frac{\log \log \log t}{\log \log \log \log t} \right). \]

However, the method of Ramachandra [5] yields better results. As for the Riemann zeta-function (10) it can be shown that
\[ L(1 + it, \chi) = \Omega(\log \log t), \quad \text{and} \quad \frac{1}{L(1 + it, \chi)} = \Omega(\log \log t), \]
and further that, assuming Riemann’s hypothesis, this is the right order (similar to [8], §14.8). Hence, it is natural to expect that also in the half-plane of absolute convergence for Dirichlet $L$-functions similar growth estimates as for the Riemann zeta-function (10) should hold. We give a heuristic argument. Weyl improved Kronecker’s approximation theorem by

Theorem 3.2. Let $a_1, \ldots, a_N \in \mathbb{R}$ be linearly independent over the field of rational numbers, and let $\gamma$ be a subregion of the $N$-dimensional unit cube with Jordan volume $\Gamma$. Then
\[ \lim_{T \to \infty} \frac{1}{T} \text{meas} \{ \tau \in (0, T) : (a_1 t, \ldots, a_N t) \in \gamma \text{ mod } 1 \} = \Gamma. \]

Since the limit does not depend on translations of the set $\gamma$, we do not expect any deep influence of the inhomogeneous part to our approximation problem (4) (though it is a question of the speed of convergence). Thus, we may conjecture that we can find a suitable $\tau \leq \exp(N^c)$ with some positive constant $c$ instead of (13), as in Dirichlet’s homogeneous approximation theorem. This would lead to estimates similar to (10).
We conclude with some observations on the density of extremal values of \( \log L(s, \chi) \). First of all note that if

\[
|L(1 + i\tau, \chi)|^{\pm 1} \geq f(T)
\]

holds for a subset of values \( \tau \in [T, 2T] \) of measure \( \mu T \), where \( f(T) \) is any function which tends with \( T \) to infinity, then

\[
\int_T^{2T} |L(1 + it, \chi)|^{\pm 2} dt \geq \mu T f(T)^2.
\]

In view of well-known mean-value formulae we have \( \mu = 0 \), which implies

\[
\lim_{T \to \infty} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : |L(\sigma + i\tau)|^{\pm 1} \geq f(T) \} = 0.
\]

This shows that the set on which extremal values are taken is rather thin.

The situation is different for fixed \( \sigma > 1 \). Let \( Q \) be the smallest prime \( p \) for which \( \chi_0(p) \neq 0 \). Then

\[
|\log L(s, \chi)| \leq \log L(\sigma, \chi_0) = Q^{-\sigma} \left( 1 + O \left( \left( \frac{Q}{Q+1} \right)^{\sigma} \right) \right);
\]

note that the right hand side tends to 0+ as \( \sigma \to +\infty \), and that \( Q \leq q + 1 \).

**Theorem 3.3.** Let \( 0 < \delta < \frac{1}{2} \). Then, for arbitrary \( \theta \) and fixed \( \sigma > 1 \),

\[
\liminf_{M \to \infty} \frac{1}{M^2} \{ m \leq M : (1 - \delta) \log L(\sigma, \chi_0) - \Re \{ \exp(i\theta) \log L(\sigma + 2\pi im, \chi) \} \}
\geq Q^{-2\sigma} \left( 1 + \frac{24}{\sigma} \right) \geq \delta^{2Q^2+8} (2Q)^{-8Q^2-32} \exp \left( -2^{3Q^2+51} Q^{4Q^2+2} \right).
\]

**Proof.** We omit the details. First, we may replace (2) by

\[
\left| \log L(s, \chi) - \sum_p \frac{\chi(p)}{p^s} \right| \leq \sum_{p, k \geq 2} \frac{\chi_0(p)}{kp^k}\sigma .
\]

This gives with regard to (8)

\[
\Re \{ \exp(i\theta) \log L(\sigma + 2\pi im, \chi) \} \geq (1 - \delta) \log L(\sigma, \chi_0) - 2 \frac{x^{1-\sigma}}{\sigma - 1} - \frac{2Q^{2-2\sigma}}{2\sigma(\sigma - 1)}
\]

for some integer \( h_0 = m \), satisfying (12), where \( N = \pi(x) \) and \( \cos \frac{2\pi}{\sigma} = 1 - \delta \).

Putting \( x = Q^2 \), proves (after some simple computation) the theorem. \( \square \)

For example, if \( \chi \) is a character with odd modulus \( q \), then the quantity of Theorem 3.3 is bounded below by

\[
\geq \frac{\delta^{16}}{2^{128} \exp(2^{81})}.
\]
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References