Stark’s conjecture in multi-quadratic extensions, revisited


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1. The elements of Stark’s refined abelian conjecture

Units. Let:

- \( L/F \) be an abelian extension of number fields in which a distinguished (finite or infinite) prime of \( F \) denoted by \( \nu \) splits completely. These and all number fields will be assumed to lie in a fixed algebraic closure \( \mathbb{Q} \) of the field \( \mathbb{Q} \) of rational numbers.
- \( | |_{\mathfrak{n}} \) be the normalized absolute value at a fixed prime \( \mathfrak{n} \) of \( L \) above \( \nu \).
- \( w_L \) be the order of the group \( \mu_L \) of roots of unity in \( L \).
- \( U_L^{(\nu)} \) be the group of elements of \( L \) having absolute value equal to 1 at each (finite or infinite) absolute value of \( L \) except for those associated with primes above \( \nu \), in other words, those which are conjugates of \( | |_{\mathfrak{n}} \). We sometimes refer to \( U_L^{(\nu)} \) as the \( \nu \)-units of \( L \).
Let:
- $G$ be the abelian Galois group of the extension $L/F$.
- $\hat{G}$ be the character group of $G$.
- $S$ be a fixed finite set of primes of $F$ of cardinality $|S| \geq 3$, and assume that $S$ contains $\mathfrak{p}$, all finite primes which ramify in $L/F$, and all infinite primes. The Stark conjecture we are concerned with must be formulated differently when $|S| = 2$, and is known to be true in this case by [4] and [5].
- $S^0 = S - \{\mathfrak{p}\}$.
- $S_{\text{fin}}$ be the set of finite primes in $S$.
- $\p$ run through the finite primes of $F$ not in $S$.
- $a$ run through integral ideals of $F$, prime to the elements of $S$.
- $N_a$ denote the absolute norm of the ideal $a$.
- $\sigma_a \in G$ be the well-defined automorphism attached to $a$ via the Artin map.

For each $\chi \in \hat{G}$, we have the Artin $L$-function with Euler factors at the primes in $S$ removed:

$$L_S(s, \chi) = \sum_{\text{integral } a \text{ s.t. } (a, S) = 1} \frac{\chi(\sigma_a)}{N_a^s} = \prod_{\text{prime } p \notin S} \left(1 - \frac{\chi(p)}{N_p^s}\right)^{-1}.$$  

It is known that $L_S(s, \chi)$ has an analytic continuation and a functional equation relating it to $L_S(1 - s, \overline{\chi})$. The order of its zero at $s = 0$ is

$$\tau_S(\chi) = \begin{cases} |S| - 1 & \text{if } |\{q \in S : q \text{ splits completely in the field fixed by the kernel of } \chi\}| \\ \{q \in S : q \text{ splits completely in the field fixed by the kernel of } \chi\} & \text{if not } \end{cases}$$

depending on whether or not $\chi$ is the trivial character $\chi_0$. See [5] for further background and references.

**The conjecture.** We first single out the key equality in Stark's refined abelian conjecture for first derivatives of $L$-functions which posits the existence of a special $\mathfrak{p}$-unit $\epsilon$ serving as an "$L$-function evaluator."

**Conjecture** $\text{St}'(L/F, S)$. There exists an element (often called a "Stark unit") $\epsilon \in U_L^{(\mathfrak{p})}$ such that

$$L_S'(0, \chi) = -\frac{1}{w_L} \sum_{\sigma \in G} \chi(\sigma) \log(|\epsilon^{\sigma}|_\mathfrak{p}) \quad \text{for all } \chi \in \hat{G}.$$  

**Remark 1.** The conditions on $\epsilon$ specify all of its absolute values and thus determine $\epsilon$ up to a root of unity in $L$. This ambiguity still remains when
we impose Stark's additional condition below. Nevertheless, we sometimes refer to $\varepsilon$ as "the" Stark unit.

The full Stark conjecture in this setting (cf. [4], [5]) says more.

**Conjecture** $\text{St}(L/F,S)$. $\text{St}'(L/F,S)$ holds, and furthermore $L(e^{1/w_L})/F$ is an abelian Galois extension.

2. Statements of the results

We assume from now on that there are at least 2 infinite primes $\infty_1, \infty_2$ in $S$. Otherwise $\text{St}(L/F,S)$ is known to be true by [4] (see also [5,IV.3.9]). We may then assume that $\infty_2 \neq v$. Also assume from now on that $G = \text{Gal}(L/F)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m$ for some positive integer $m$. We then call $L/F$ a multiquadratic extension of rank $m$.

Our aim in this paper is to prove the following theorems.

**Theorem 1.** Let $S_{\text{fin}} \subset S$ consist of the finite primes in $S$, and let $r_F(S)$ denote the 2-rank of the $S_{\text{fin}}$-class group of $F$. If $|S| > m + 1 - r_F(S)$, then $\text{St}(L/F,S)$ holds for the multiquadratic extension $L/F$.

**Theorem 2.** $\text{St}'(L/F,S)$ holds for the multiquadratic extension $L/F$, hence for an arbitrary multiquadratic extension.

**Theorem 3.** $\text{St}(L/F, S)$ holds for the multiquadratic extension $L/F$ if $v$ is a real infinite prime or a finite prime, except possibly when $L$ is the maximal multiquadratic extension of $F$ which is unramified outside of $S$ and in which $v$ splits completely.

**Theorem 4.** $\text{St}(L/F, S)$ holds for the multiquadratic extension $L/F$ when the rank of this extension is $m = 2$, i.e. $L/F$ is biquadratic. Thus it holds for an arbitrary biquadratic extension.

**Remark 2.** $\text{St}(L/F, S)$ was proved for the multiquadratic extension $L/F$ in [2] and [3] under the assumption that either $|S| > m + 1$, or that no prime above 2 (i.e. no dyadic prime) is ramified in $L/F$.

3. The relative quadratic case

Assume $K/F$ is a relative quadratic extension. This section summarizes some basic results from [3] and [5] on $\text{St}(K/F,S)$.

We set:

- $\text{Gal}(K/F) = \langle \tau \rangle$ of order 2.
- $\eta_K = 1$ if $S$ contains two split primes of $K/F$.
- $\eta_K$ generator of the infinite cyclic group $U_K^{(0)}/\mu_K$ with $|\eta_K|_v < 1$, otherwise. Note that since $\infty_2$ does not split in this case, we may also describe $U_K^{(0)}$ as the $S$-units $u$ of $K$ such that $u^{1+\tau} = 1$. If $w$ is a real infinite prime, choose $\eta_K$ to be positive in the embedding induced by
m. Since $\eta_K^{1+\tau} = 1$, this then implies that $\eta_K$ is positive at both of the primes above v.

- $\text{Cl}_F(S) = \text{Cl}_F(S_{\text{fin}}) = S_{\text{fin}}$-ideal class group of $F$, the quotient of the ideal class group $\text{Cl}(F)$ of $F$ by the subgroup generated by the ideal classes of the primes in $S_{\text{fin}}$.
- $S_K =$ the set of primes of $K$ lying above those in $S$.
- $\text{Cl}_K(S) = S_K$-ideal class group of $K$.
- $H_K = H_K(S) = \text{Cl}_K(S)/\iota(\text{Cl}_F(S))$, the cokernel of the map $\iota$ induced by extension of ideals.
- $M_K = M_K(S) = |H_K|$, the order of this group.

**Theorem** (Stark-Tate, cf. [Ta, IV.5.4]). $\text{St}(K/F,S)$ holds with Stark unit

$$\varepsilon_K = \eta_K^{M_K \cdot 2^{[S]} - 3},$$

and $K(\eta_K^{1/w_K})/F$ is abelian.

**Remark 3.** The extra factor $e^+$ in [5, IV.5.4] equals 1 when $\eta_K \neq 1$ as this implies that the infinite prime $\infty_2$ of $F$ does not split in $K$.

4. Passage to the multiquadratic case via $L$-function properties

We have assumed that $L/F$ is multiquadratic with the distinguished prime $v$ of $F$ splitting completely in $L$, and that $\infty_2 \neq v$ is an infinite prime of $F$. From now on, we also assume that:

- $\infty_2$ does not split completely in $L/F$. (Otherwise $\text{St}(L/F)$ is trivially true with $\varepsilon = 1$.)
- Let:
  - $\tau =$ complex conjugation at $\infty_2$ in $L/F$.
  - $K_i$ for $i = 1, 2, \ldots, 2^{(m-1)}$ be the relative quadratic extensions of $F$ in $L$ which are not fixed by $\tau$. (These generate $L$.)
  - $\eta_i = \eta_{K_i}$.
  - $M_i = M_{K_i}$.
  - $w_i = w_{K_i}$.

**Proposition 1.** If

$$\varepsilon = \prod_{i=1}^{2^{m-1}} \eta_i^{M_i \cdot 2^{[S]} - m - 2(w_L/w_i)}$$

lies in $L$, then it is the Stark unit $\varepsilon_L$ satisfying $\text{St}'(L/F,S)$.

**Proof.** (This is a straightforward adaptation of the proof of Theorem 2.6 of [3].) Clearly $\varepsilon \in U_L^{(v)}$ because each $\eta_i \in U_{K_i}^{(v)} \subset U_L^{(v)}$. In particular, $\varepsilon^{1+\tau} = 1$ because this represents the absolute value of $\varepsilon$ above $\infty_2$. We now show that $\varepsilon$ is an $L$-function evaluator.
Fix an arbitrary character \( \chi \in \hat{G} \). If \( \chi(\tau) = 1 \), then \( r_S(\chi) > 1 \) by the formula for this quantity, and therefore \( L'_S(\chi, 0) = 0 \). At the same time,

\[-\frac{1}{w_L} \sum_{\sigma \in G} \chi(\sigma) \log(|\varepsilon^\sigma|_\mathfrak{m}) = -\frac{1}{2} \frac{1}{w_L} \sum_{\sigma \in G} \chi(\sigma) \log(|\varepsilon^{1+\tau}\sigma|_\mathfrak{m}) = 0,\]

by the observation in the last paragraph. So \( \varepsilon \) is an \( L \)-function evaluator for this type of \( \chi \).

Now suppose that \( \chi(\tau) = -1 \). The fixed field of the kernel of \( \chi \) must then be one of the \( K_i \) for some \( i = i(\chi) \). Letting \( G_i = \text{Gal}(L/K_i) \), we observe that

\[
\sum_{\sigma \in G_i} \chi(\sigma) = \begin{cases} 
2^{m-1}, & \text{if } i = i(\chi) \\
0, & \text{otherwise.}
\end{cases}
\]

We use the definition of \( \varepsilon \), the fact that \( \chi(\tau) = -1 \), and the fact that \( G_i \) fixes \( \eta_i \), along with the evaluation of the last sum and finally the Stark-Tate theorem for relative quadratic extensions and the inflation property of Artin \( L \)-functions to see that

\[
-\frac{1}{w_L} \sum_{\sigma \in G} \chi(\sigma) \log(|\varepsilon^\sigma|_\mathfrak{m}) = -\frac{1}{w_L} \sum_{\sigma \in G} \frac{1}{2^{m-1}} \sum_{i=1}^{2^{m-1}} \chi(\sigma)(w_L/w_i) \log(|\eta_i^{M_i|S|-3}\sigma|_\mathfrak{m})
\]

\[
= \sum_{i=1}^{2^{m-1}} \frac{-1}{w_i} \frac{1}{2^{m-1}} \sum_{\sigma \in G} \chi(\sigma) \log(|\eta_i^{M_i|S|-3}\sigma|_\mathfrak{m})
\]

\[
= \sum_{i=1}^{2^{m-1}} \frac{-1}{w_i} \frac{1}{2^{m-1}} \sum_{\sigma \in G_i} \chi(\sigma) \log(|\eta_i^{M_i|S|-3(1-\tau)}\sigma|_\mathfrak{m})
\]

\[
= \sum_{i=1}^{2^{m-1}} \frac{-1}{w_i} \frac{1}{2^{m-1}} \sum_{\sigma \in G_i} \chi(\sigma) \log(|\eta_i^{M_i|S|-3(1-\tau)}|_\mathfrak{m})
\]

\[
= \frac{-1}{w_i(\chi)} \log(|\eta_i^{M_i(\chi)|S|-3}(1-\tau)|_\mathfrak{m}) = L'_S(0, \chi).
\]

So \( \varepsilon \) is an \( L \)-function evaluator for this type of \( \chi \) as well, and the proof is complete.

5. Class field theory

Recall that \( H_K = \text{Cl}_K(S)/\iota(\text{Cl}_F(S)) \).
Proposition 2. Let \( K \) be any of the \( K_i \) for which \( \eta_{i} \neq 1 \).
Then \( \text{rank}_2(H_K) \geq \text{rank}_2(\text{Cl}_F(S)) = r_F(S) \), with equality holding if \( |S| = 3 \).

Proof. We will show that the norm map induces a surjective homomorphism \( H_K/H_K^2 \rightarrow \text{Cl}_F(S)/\text{Cl}_F(S)^2 \) which is an isomorphism when \( |S| = 3 \).

The assumption that \( \eta_{i} \neq 1 \) implies that \( \omega_2 \) and the other primes of \( S^0 \) do not split in \( K/F \). Thus the complex conjugation \( \tau \) at \( \omega_2 \) restricts to a generator of \( \text{Gal}(K/F) \).

Let \( I_K \) denote the group of fractional ideals of \( K \), \( P_K \) denote the subgroup of principal fractional ideals, and \( \mathcal{I}_F \) denote the subgroup of fractional ideals of \( K \) which are extended from fractional ideals of \( F \). Also let \( S_{\text{fin}}(K) \) denote the set of ideals of \( K \) which lie above those in \( S_{\text{fin}} \). This contains all of the ideals of \( K \) which are ramified over \( F \). From the factorization of ideals into primes, we see that \( \mathcal{I}_F(S_{\text{fin}}(K)) \), where \( J_K \) denotes the group generated by the prime ideals of \( K \) which are inert over \( F \). Then

\[
H_K/H_K^2 \cong I_K/P_K I_K^2 \mathcal{I}_F(S_{\text{fin}}(K)) = I_K/P_K I_K^2 I_K^{1+\tau} J_K(S_{\text{fin}}(K))
\]

Under the Artin map of class field theory, \( I_K/P_K (\cong \text{Cl}_K) \) corresponds to the maximal unramified abelian extension (Hilbert Class Field) \( \mathcal{H} \) of \( K \), in the sense that this map induces an isomorphism of \( I_K/P_K \) with \( \text{Gal}(\mathcal{H}/K) \). From this it is clear that \( I_K/P_K I_K^2 \) corresponds to the maximal abelian unramified elementary 2-extension of \( K \) in the same way. Similarly, \( I_K/P_K I_K^2 I_K^{1-\tau} \) corresponds to the maximal abelian unramified elementary 2-extension \( \mathcal{F} \) of \( K \) having the property that \( \tau \) acts (by conjugation) trivially on \( \text{Gal}(\mathcal{F}/K) \). By maximality, \( \mathcal{F}/F \) is Galois.

Let \( G_{\mathcal{F}} = \text{Gal}(\mathcal{F}/F) \) and \( N_{\mathcal{F}} = \text{Gal}(\mathcal{F}/K) \). Then \( N_{\mathcal{F}} \) is normal of index 2 in \( G_{\mathcal{F}} \), and \( \tau \) acts trivially on \( N_{\mathcal{F}} \). So \( N_{\mathcal{F}} \) and any lift of \( \tau \) commute with \( N_{\mathcal{F}} \), which suffices to show that \( N_{\mathcal{F}} \) lies in the center of \( G_{\mathcal{F}} \). Now \( G_{\mathcal{F}}/N_{\mathcal{F}} \) is cyclic of order 2, so that \( G_{\mathcal{F}} \) acts trivially on \( N_{\mathcal{F}} \). Hence in fact \( \mathcal{F}/F \) is an abelian extension.

We now know that \( I_K/P_K I_K^2 I_K^{1-\tau} \) corresponds to the maximal unramified elementary 2-extension \( \mathcal{F} \) of \( K \) which is abelian over \( F \). So the extension \( I_K/P_K I_K^2 I_K^{1-\tau} \langle S_{\text{fin}}(K) \rangle \) corresponds to the maximal such extension \( \mathcal{L}_K \) of \( K \) in which all primes of \( S_{\text{fin}}(K) \) split completely.

For each finite or infinite prime \( p \in S^0 \), let \( D_p \) denote its decomposition group in the abelian extension \( \mathcal{L}_K/F \). Under our assumptions, such a prime \( p \) does not split in the quadratic extension \( K/F \), while the prime \( \wp \) above it in \( K \) splits completely in \( \mathcal{L}_K/K \). Thus \( D_p = \langle \tau_p \rangle \) has order 2. Let \( D \) be the subgroup of \( \text{Gal}(\mathcal{L}_K/F) \) generated by all the \( D_p \) for finite and infinite primes \( p \in S^0 \) except \( \omega_2 \). Hence \( D \) is an elementary abelian 2-group with
2-rank \( \text{rank}_2(D) \leq |S| - 2 \). Let \( L'_p \) be the fixed field of \( D \). Since \( L'_p \subset L_K \), no primes ramify in \( L'_p/K \). Also, only primes in \( S^0 \) can ramify in \( K/F \). So only primes in \( S^0 \) can ramify in \( L'_p/F \). The definition of \( L'_p \) requires that the primes in \( S^0 \) other than \( \infty_2 \) split completely in \( L'_p/F \). Hence \( L'_p/F \) can ramify only at \( \infty_2 \). Thus \( L'_p \) is contained in the ray class field modulo \( \infty_2 \) for \( F \). But the ray classes modulo \( \infty_2 \) are the same as the ray classes modulo 1, due to the presence of the unit \(-1\). Thus \( L'_p/F \) is unramified at \( \infty_2 \) as well, and is therefore everywhere unramified, with all primes in \( S \) splitting completely.

The fact that \( \infty_2 \) splits in \( L'_p/F \) but not in the quadratic extension \( K/F \) implies that \( L'_p \cap K = F \). Thus the elementary abelian 2-group \( N_L = \text{Gal}(L_K/K) \) and the elementary abelian 2-group \( D = \text{Gal}(L_K/L'_p) \) generate the abelian group \( \text{Gal}(L_K/F) \), which is therefore also an elementary abelian 2-group. Hence if \( q \) is any prime of \( F \) which is inert in \( K \), we may consider the Frobenius of the extended prime \( \Omega \) of \( K \) in the extension \( L_K/K \) and use the properties of the Frobenius in relative extensions (see [1, III.2.4]): \( \sigma(\Omega, L_K/K) = \sigma(q, L_K/F)^2 = 1 \). This shows that \( J_K \) has trivial image in \( \text{Gal}(L_K/K) \) under the Artin map.

We return now to the isomorphism from \( I_K/P_KI_K^{-1}(S_{\text{fin}}(K)) \) to \( N_L = \text{Gal}(L_K/K) \) which is induced by the Artin map as described above. Since the image of \( J_K \) lies in the kernel of this isomorphism, we conclude that

\[
(1) \quad H_K/H_K^2 \cong I_K/P_KI_K^{-1}(S_{\text{fin}}(K))J_K \cong \text{Gal}(L_K/K) = N_L
\]

Now we observe that \( L'_p \) has an intrinsic definition in terms of \( F \). Since \( L'_p/F \) is an unramified elementary abelian 2-extension in which all primes of \( S \) split completely, it is contained in the maximal such extension, which we denote by \( L_F \). Then \( L_F \cdot K \) is an unramified elementary abelian 2-extension of \( K \) in which all primes of \( S_{\text{fin}}(K) \) (indeed \( S(K) \), as unramified is the same as split for the infinite primes) split completely, and is abelian over \( F \). But \( L_K \) was defined to be the maximal such extension, so \( L_K \subset L_F \). As all primes in \( S \) split completely in \( L_F/F \), \( L_F \) must be fixed by the decomposition groups generating \( D \). This means that \( L_F \subset L'_p \). We conclude that \( L_F = L'_p \).

Finally define \( D_0 = \text{Gal}(L_K/(K \cdot L_F)) \), which has index 2 in \( D = \text{Gal}(L_K/L_F) \). Thus \( D_0 \) is an elementary abelian 2-group with \( \text{rank}_2(D_0) \leq |S| - 3 \). Then we have an exact sequence:

\[
(2) \quad 1 \to D_0 \to N_L \to \text{Gal}(L'_p/F) \to 1
\]

This simply comes from the natural restriction map identifying \( N_L/D_0 \cong \text{Gal}(K \cdot L_F)/K \) with \( \text{Gal}(L_F/F) \).

Interpreting \( \text{Gal}(L_F/F) \) via the class field theory of \( F \), we have

\[
(3) \quad \text{Gal}(L_F/F) \cong I_F/P_FI_F^{-1}(S_{\text{fin}}) \cong \text{Cl}_F(S_{\text{fin}})/\text{Cl}_F(S_{\text{fin}})^2
\]
Thus in terms of class groups (using (1) and (3)), the exact sequence (2) becomes

\[ 1 \to C_0 \to H_K/H_K^2 \to \text{Cl}_F(S_{\text{fin}})/\text{Cl}_F(S_{\text{fin}})^2 \to 1, \]

where the kernel \( C_0 \) is an elementary abelian 2-group of rank \( \leq |S| - 3 \) and the map on the right is induced by the norm map on ideals. The conclusion of the theorem follows. \( \square \)

**Corollary 1.** The integer \( 2^{r_F(S)} \) divides \( M_i \) when \( \eta_i \neq 1 \).

**Proof.** This is clear since \( r_F(S) = \text{rank}_2(\text{Cl}_F(S_{\text{fin}})) \leq \text{rank}_2(H_K) \) for \( K = K_i \). \( \square \)

### 6. Proof of Theorem 1

The assumption is that \( |S| \geq m + 2 - r_F(S) \). In view of Proposition 1, we consider

\[ \varepsilon = \prod \eta_i^{M_i - 2S| - m - 2(w_L/w_i)}, \]

where the product may clearly be taken over \( i \) for which \( \eta_i \neq 1 \). For such \( i \), the expression \( e_i = M_i \cdot 2^{|S| - m - 2} \) is an integer multiple of \( 2^{r_F(S)} \cdot 2^{|S| - m - 2} \), by Corollary 1, and this in turn is integral by assumption. Thus \( e_i \) is integral and so

\[ \varepsilon = \prod \eta_i^{e_i / w_L}, \]

does in fact lie in \( L \), since each \( \eta_i \) lies in \( K_i \subset L \). Then \( \text{St'}(L/F, S) \) holds by Proposition 1. Furthermore

\[ \varepsilon^{1 / w_L} = \prod \eta_i^{e_i / w_i}, \]

and each \( \eta_i^{1 / w_i} \) lies in an abelian extension of \( F \), by the Stark-Tate theorem. As the composite of abelian extensions is abelian, we conclude that \( \varepsilon^{1 / w_L} \) lies in an abelian extension of \( F \). This completes the proof of Theorem 1.

### 7. Kummer theory

Let:

- \( m_S = \prod_{p \in S_{\text{fin}}} p \)
- \( \mathcal{L} = L_S \) be the composite of all quadratic extensions of \( F \) in \( \overline{\mathbb{Q}} \) with relative discriminant dividing \( 4m_S \).
- \( \mathcal{O}_F \) be the ring of integers of \( F \).
- \( \mathcal{O}_F(S_{\text{fin}}) \) be the ring of \( S_{\text{fin}} \)-integers of \( \mathcal{O}_F \).

**Lemma 1.** Suppose \([K : F] = 2\). Then \( K = F(\sqrt{\gamma}) \) for some \( \gamma \in F \) which generates a fractional ideal of \( F \) of the form \( (\gamma) = a^2b \) with \( b \) supported in \( S_{\text{fin}} \) if and only if the relative discriminant \( \delta(K/F) \) of \( K \) over \( F \) divides \( 4m_S \). In particular, if \( K/F \) is unramified outside \( S \), then \( \delta(K/F) | 4m_S \).
Proof. First suppose that $K = F(\sqrt{\gamma})$ with $(\gamma) = a^2b$ and $b$ supported in $\mathcal{S}_{\text{fin}}$. The relative discriminant may be computed locally, so we reduce to the case of an extension of local fields $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ by passing to the completions at a fixed arbitrary prime $p$ of $F$ and a prime $\mathfrak{B}$ over $p$ in $K$. That is, the $p$-part $(\delta(K/F))_p$ of the relative discriminant of $K/F$ equals the relative discriminant of $K_{\mathfrak{p}}/F_{\mathfrak{p}}$, and it suffices to show that this divides $4ms$ for each $p$. Let $\pi$ be a uniformizing parameter for the ring of integers $\mathcal{O}_p$ of $F_p$. Then $\gamma = u\pi^{2e+a}$ where $u$ is a unit of $\mathcal{O}_p$ and $a$ equals 0 or 1. So $K_{\mathfrak{p}} = F_p(\sqrt{\gamma}) = F_p(\sqrt{u\pi^a})$. We treat the two possibilities for $a$ individually.

When $a = 0$ we have Then the relative discriminant $\delta(K_{\mathfrak{p}}/F_{\mathfrak{p}})$ divides the discriminant of the polynomial $x^2 - u$ which is $(4u) = (4)$, and this clearly divides $4ms$.

When $a = 1$, it evidently must be the case that $p$ divides $b$, and therefore $p$ divides $m_\mathfrak{S}$. We have $K_{\mathfrak{p}} = F_p(\sqrt{u\pi}) = F_p(\sqrt{\pi'})$, where $\pi'$ is another uniformizing parameter. Thus $K_{\mathfrak{p}}/F_p$ is an Eisenstein extension for which it is known that $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_p(\sqrt{\pi'})$. Therefore $\delta(K_{\mathfrak{p}}/F_{\mathfrak{p}})$ equals the discriminant of $x^2 - \pi'$, namely $(4\pi') = 4p$. Again this divides $4ms$, as $p$ divides $m_\mathfrak{S}$. This completes the first half of the proof.

Next assume that the relative discriminant $\delta(K/F)$ of $K$ over $F$ divides $4m_\mathfrak{S}$. Since $K/F$ is a relative quadratic extension, we know that $K = F(\sqrt{\gamma})$ for some $\gamma \in F$. Write $(\gamma) = a^2b$, and $b$ a square free fractional ideal. If a prime $p$ appears in the factorization of $b$, let $\mathfrak{B}$ be a prime above $p$ in $K$. Then we are in the situation appearing in the first half of the proof where $a = 1$ and $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ is an Eisenstein extension. In this case we saw that $\delta(K_{\mathfrak{p}}/F_{\mathfrak{p}}) = 4p$. We are assuming that this divides $4m_\mathfrak{S}$, so may clearly conclude that $p$ divides $m_\mathfrak{S}$ and thus $\mathfrak{p}$ is in $\mathcal{S}_{\text{fin}}$. This shows that $b$ is supported in $\mathcal{S}_{\text{fin}}$, and concludes the proof. \qed

Proposition 3. The field $L = L_\mathcal{S}$ contains $L(\{\sqrt{\eta_i} : i = 1 \ldots, 2^{m-1}\})$.

Proof. We show that $L$ contains $L(\{\sqrt{\eta_i} : i = 1 \ldots, 2^{m-1}\})$ by showing that $L$ contains $L$ and each $\sqrt{\eta_i}$. First, each $K_i/F$ is a quadratic extension, so $K_i = F(\sqrt{\eta_i})$. We may write $(\eta_i) = a^2b$ with $b$ square free. Then $K_i$ is ramified at the divisors of $b$, by Kummer theory. Since $K_i/F$ is unramified outside $\mathcal{S}$, we conclude that $b$ is supported in $\mathcal{S}_{\text{fin}}$. It now follows from the Lemma that $K_i$ is a quadratic extension of $F$ with relative discriminant dividing $4m_\mathfrak{S}$. But $L$ was defined to be the composite of all such extensions. Thus $L$ contains the composite of all the $K_i$, which is $L$, as we observed in the beginning of section III.

Having shown that $L$ contains $L$, we proceed to show that $L$ contains each $\sqrt{\eta_i}$. This is trivial if $\eta_i = 1$, so we may assume that we are not in this situation. Then the image of $\eta_i$ generates the infinite cyclic group $U_{K_i}^{(b)}/\mu_{K_i}$. Thus $\eta_i$ is not a square in $K_i$, and $\eta_i$ does not lie in $F$. So
$F(\sqrt{\eta_i})$ is an extension of degree 4 over $F$. We know that $\eta_i = 1/\eta_i$, so the conjugates of $\sqrt{\eta_i}$ over $F$ are $\pm \sqrt{\eta_i}$ and $\pm 1/\sqrt{\eta_i}$. Thus $F(\sqrt{\eta_i})/F$ is a Galois extension of degree 4. It is in fact the composite of the relative quadratic extension $K_i = F(\eta_i)$ in which $\infty_2$ ramifies, and the relative quadratic extension $K_i' = F(\sqrt{\eta_i} + 1/\sqrt{\eta_i})$ in which $\infty_2$ splits. We have already seen that $K_i$ lies in $\mathcal{L}$, so we now show that $K_i'$ lies in $\mathcal{L}$. This will imply that the composite $F(\sqrt{\eta_i})$ lies in $\mathcal{L}$, as desired.

Above we saw that $\delta(K_i/F)\mid 4m_S$. In fact, $\delta(K_i/F)\mid 4m_S\nu$, since this extension is unramified at $v$. Since $\eta_i$ lies in $U_{K_i}$, the Lemma yields $\delta(K_i(\sqrt{\eta_i})/K_i)\mid 4\nu$. Hence

$$\delta(K_i(\sqrt{\eta_i})/F) = (N_{K_i/F}\delta(K_i(\sqrt{\eta_i})/K_i))\delta(K_i/F)^2,$$

which divides $16\nu^2\delta(K_i/F)^2$. Similarly, $\delta(K_i'/F)^2$ divides $\delta(K_i(\sqrt{\eta_i})/F)$, and thus divides $16\nu^2\delta(K_i/F)^2$. We conclude that $\delta(K_i'/F)\mid 4\nu\delta(K_i/F)$.

We examine this divisibility statement one prime at a time and show that it implies $\delta(K_i'/F)\mid 4m_S$ for each $p$. Observe that $K_i/F$ is unramified outside of $S$, so $\delta(K_i/F)\mid p = 1$ for $p$ not dividing $m_S$. Consequently $\delta(K_i/F)\mid 4\nu$ which divides $4m_S$ in this case.

Now for $p$ dividing $m_S$, the lemma applied to $K_i'/F$ implies that $\delta(K_i'/F)\mid 4p$, which in turn divides $4m_S$. This shows that $\delta(K_i'/F)$ divides $4m_S$. Hence $K_i'$ lies in $\mathcal{L}$, by its very definition. \qed

**Proposition 4.** \(\mathcal{L}_S : F = 2^{|p(S)|+|S|}\)

**Proof.** Let \(\{a_i : i = 1, \ldots, t\}\) be a minimal set of generators for the 2-torsion subgroup $\text{Cl}_F(S_{\text{fin}})[2]$ of the $S_{\text{fin}}$-class group $\text{Cl}_F(S_{\text{fin}})$. So $t = \text{rank}_2(\text{Cl}_F(S_{\text{fin}})) = r_F(S)$. We view $\text{Cl}_F(S_{\text{fin}})$ as the group of invertible ideals modulo principal fractional ideals of $\mathcal{O}_F(S_{\text{fin}})$, the ring of elements of $F$ which are integral at all finite primes not in $S_{\text{fin}}$. Using Chebotarev's density theorem, we choose the representatives $a_i$ to be prime ideals of $\mathcal{O}_F(S_{\text{fin}})$. The units of this ring are denoted $U_F(S_{\text{fin}})$ and called the $S_{\text{fin}}$-units. Now $a_i^2 = a_i \mathcal{O}_F(S_{\text{fin}})$ for some $a_i$. Let $A = \langle\{a_i : i = 1, \ldots, t\}\rangle U_F(S_{\text{fin}})$.

We begin by noting that $A \cong \langle\{a_i\}\rangle \times U_F(S_{\text{fin}})$. For a non-trivial element of $\langle\{a_i\}\rangle$ generates an ideal which is a non-trivial product of the prime ideals $a_i$, while each element of $U_F(S_{\text{fin}})$ generates the unit ideal. Thus by the Dirichlet-Chevalley-Hasse unit theorem,

$$\text{rank}_2(A) = \text{rank}_2(\langle\{a_i\}\rangle) + \text{rank}_2(U_F(S_{\text{fin}})) = t + |S| = r_F(S) + |S|.$$

We will establish a one-to-one correspondence between the non-trivial elements of $A/A^2$ and the relative quadratic extensions $K/F$ contained in $\mathcal{L}$. This implies that $\text{rank}_2(\text{Gal}(\mathcal{L}/F)) = \text{rank}_2(A/A^2)$, which combined with the displayed equality yields the statement of the proposition.

Now observe that $A \cap (F^\times)^2 = A^2$ as follows. If $\gamma^2 \in (F^\times)^2$ lies in $A$, then $\gamma^2 = \prod_i a_i^{2\alpha_i} u$, for some $u \in U_F(S_{\text{fin}})$. Hence $\gamma^2 \mathcal{O}_F(S_{\text{fin}}) = \prod_i a_i^{2\alpha_i}$.
and therefore \( \gamma \mathcal{O}_F(S_{\text{fin}}) = \prod_i a_i^{c_i} \). The fact that this is a principal ideal generated by the \( a_i \) implies by their definition that all of the exponents are even, \( c_i = 2b_i \). We now have \( \gamma^2 = \prod_i \alpha_i^{2b_i} u \), and this shows that \( u = v^2 \) is a square. Clearly \( v \in U_F(S_{\text{fin}}) \), so \( \gamma = \prod_i \alpha_i^{b_i} v \), after choosing the correct sign for \( v \). From this we see that \( \gamma^2 \in A^2 \), which was to be proved.

Given a \( \gamma \) representing a non-trivial class in \( A/A^2 \), this will correspond to the field \( K = F(\sqrt{\gamma}) \). According to the last paragraph, \( K \) will in fact be a relative quadratic extension of \( F \). We check that \( K \) lies in \( \mathcal{L} \) by showing that the relative discriminant \( \delta(K/F) \) divides \( 4m_S \). The fact that \( \gamma \in A \) means that \( \gamma = u \prod \alpha_i^{e_i} \) for some integers \( e_i \) and some \( u \in U_F(S_{\text{fin}}) \). Then the principal \( \mathcal{O}_F \)-ideal generated by \( \gamma \) is \( \gamma = \prod \alpha_i^{2e_i}(v) = a^2 b \), where \( b = (v) \), and \( \alpha_i \) is the (prime) ideal of \( \mathcal{O}_F \) supported outside of \( S_{\text{fin}} \) such that \( \alpha_i \mathcal{O}_F(S_{\text{fin}}) = a_i \). Since \( b = (v) \) is supported in \( S_{\text{fin}} \), Lemma 1 allows us to conclude that \( \delta(K/F) \) divides \( 4m_S \), as desired.

Conversely, given a relative quadratic extension \( K/F \) contained in \( \mathcal{L} \), we will produce the corresponding \( \gamma \in A \). First we note that the relative discriminant of a relative quadratic extension is equal to the finite part of its conductor, by the conductor-discriminant theorem. Thus every relative quadratic extension of \( F \) with discriminant dividing \( 4m_S \) is contained in the ray class field of \( F \) with conductor equal to \( 4m_S \) multiplied by all of the infinite primes. Hence the field \( \mathcal{L} \) generated by all of these relative quadratic extensions is also contained in this ray class field. Then any quadratic extension of \( F \) contained in \( \mathcal{L} \) will have conductor dividing the product of \( 4m_S \) with all of the infinite primes, so that its discriminant also divides \( 4m_S \). We can conclude that the discriminant of our given \( K \) divides \( 4m_S \). Lemma 1 now implies that \( K = F(\sqrt{\gamma}) \), where \( (\gamma) = a^2 b \) and \( b \) is supported in \( S_{\text{fin}} \). Hence \( \gamma \mathcal{O}_F(S_{\text{fin}}) = (a \mathcal{O}_F(S_{\text{fin}}))^2 \), so that \( a \mathcal{O}_F(S_{\text{fin}}) \) represents an element of \( \text{Cl}_F(S_{\text{fin}})[2] \). But this group is generated by the images of the \( a_i \). Thus \( a \mathcal{O}_F(S_{\text{fin}}) = \prod a_i^{c_i} \beta \mathcal{O}_F(S_{\text{fin}}) \) for some \( \beta \in F \). Then \( \gamma \mathcal{O}_F(S_{\text{fin}}) = \prod a_i^{2c_i} \beta^2 \mathcal{O}_F(S_{\text{fin}}) = \prod a_i^{c_i} \beta \mathcal{O}_F(S_{\text{fin}}) \). Let \( \gamma = \gamma' / \beta^2 \). We clearly have \( K = F(\sqrt{\gamma'}) = F(\sqrt{\gamma}) \), while \( \gamma \mathcal{O}_F(S_{\text{fin}}) = (\prod a_i^{c_i})^2 \mathcal{O}_F(S_{\text{fin}}) = (\prod a_i^{c_i}) \mathcal{O}_F(S_{\text{fin}}) \). Thus \( \gamma = u \prod a_i^{c_i} \) for some \( u \in U_F(S_{\text{fin}}) \) and therefore \( \gamma \in A \).

**Corollary 2.** 1. We have \( [\mathcal{L} : L] = 2^{r_F(S) + |\mathcal{S}| - m} \).

2. Let \( \zeta_i \) be a generator of \( \mu_{K_i} \). When \( \eta_i \) is not equal to 1, the exponent \( M_i \cdot 2^{|\mathcal{S}| - m - 2}(w_L/w_i) \) is in \( \frac{1}{2} \mathbb{Z} \). If it is not in \( \mathbb{Z} \), then either \( L = \mathcal{L} \) or \( [\mathcal{L} : L] = 2 \) and \( \sqrt{\zeta_i} \notin L \).

**Proof.** 1. From the fact that \( [L : F] = 2^m \) and Propositions 3 and 4, we conclude that \( [\mathcal{L} : L] = 2^{r_F(S) + |\mathcal{S}| - m} \), and thus this rational number is in fact an integer.
2. Now we can see that $2^p(F(S)+|S|−m−2) = [L : L]/4$ lies in $1/2\mathbb{Z}$. By Corollary 1, it follows that $M_i \cdot 2^{|S|−m−2}$ is in $1/4\mathbb{Z}$, and that if it does not lie in $1/2\mathbb{Z}$, we must have $L = \mathcal{L}$. In this case, note that the ambiguity up to a root of unity in the choice of $\eta_i$ allows us to conclude from Proposition 3 that $L = \mathcal{L}$ contains both $\sqrt[n_i]{\eta_i}$ and $\sqrt[n_i]{\eta_i}$ and therefore $\sqrt[n_i]{\eta_i} \in L$. Thus $w_i/L/w_i$ is even and $M_i \cdot 2^{|S|−m−2}(w_i/L/w_i)$ lies in $1/2\mathbb{Z}$. Finally, since $[L : L]$ is a power of 2, the only other situation in which $2^p(F(S)+|S|−m−2) = [L : L]/4$ is not integral clearly occurs when $[L : L] = 2$ and it is half-integral. Then $M_i \cdot 2^{|S|−m−2}$ is in $1/2\mathbb{Z}$, so $M_i \cdot 2^{|S|−m−2}(w_i/L/w_i)$ is integral unless $w_i/L/w_i$ is odd, i.e. $\sqrt[n_i]{\eta_i} \notin L$.

8. Proofs of Theorems 2 and 3

Under our standing assumptions that $L/F$ is multiquadratic, and that in order to avoid special cases of the conjecture which have already been proved, $S$ contains at least two infinite primes and one other finite or infinite prime, we now have:

- $\epsilon = \prod \eta_i^{M_i \cdot 2^{|S|−m−2}(w_i/L/w_i)}$ by Proposition 1.
- The exponent $M_i \cdot 2^{|S|−m−2}(w_i/L/w_i)$ is either integral or half-integral when $\eta_i \neq 1$, by Corollary 2.
- If it is half-integral for some $i$, then either $L = L$ or we have both $[L : L] = 2$ and $\sqrt[n_i]{\eta_i} \notin L$, also by Corollary 2.

If $L = L$, then $\sqrt[n_i]{\eta_i} \in L$ for all $i$, since $\sqrt[n_i]{\eta_i} \in L$, by Proposition 3. If $[L : L] = 2$ and $\sqrt[n_i]{\eta_i} \notin L$ for some $i$, notice that both $\sqrt[n_i]{\eta_i}$ and $\sqrt[n_i]{\eta_i}$ lie in $L$ by Proposition 3 again and the ambiguity in $\eta_i$. If neither of them lie in $L$, then $L(\sqrt[n_i]{\eta_i}) = L(\sqrt[n_i]{\eta_i})$. This implies that $\sqrt[n_i]{\eta_i} \in L$, which is not the case. Hence either $\sqrt[n_i]{\eta_i} \in L$ or $\sqrt[n_i]{\eta_i} \in L$. By renaming we may again assume $\sqrt[n_i]{\eta_i} \in L$.

Thus in all cases, Theorem 2 follows from Proposition 1.

Turning to the proof of Theorem 3, we now assume that $v$ is either real or finite. When $L$ is not the maximal multiquadratic extension $M$ of $F$ which is unramified outside of $S$ and in which $v$ splits completely, we claim that $4[[L : L]$. Then by Corollary 2, $r_F(S) + |S| − m − 2 \geq 0$, and the result will follow from Theorem 1.

To establish the claim, we first show that $v$ is not split completely in $L_S/F$. When $v$ is real, this is clear since the definition of $L_S$ implies that $\sqrt{−1} \in L_S$ and thus $L_S$ is totally imaginary. When $v$ is finite, we proceed by contradiction. Suppose $v$ splits completely in $L_S/F$. Then $v$ is unramified in $L_S$, so clearly $L_S = L_S^0$. From Corollary 2, we then get $2^p(F(S)+|S|) = [L_S : F] = [L_S^0 : F] = 2^p(F(S^0)+|S^0|) = 2^p(F(S^0)+|S|−1)$, so that $r_F(S) = r_F(S^0) − 1$. Now $2^p(F(S^0) = [Cl_F(S^0)/Cl_F(S^0)]$, so the class $[v]$
of \( v \) must be non-trivial in this group. By class field theory, \( v \) is then not split completely in the maximal unramified multiquadratic extension of \( F \) in which every finite prime of \( S^0 \) splits completely. However, this extension is contained in \( \mathcal{L}_S \), by Lemma 1, and we have assumed that \( v \) splits completely in \( \mathcal{L}_S \), a contradiction.

Let \( \mathcal{L}^\alpha_\mathcal{S} \) denote the splitting field of \( v \) in \( \mathcal{L}_S \). By the claim we have just established, \( 2|[\mathcal{L}_S : \mathcal{L}^\alpha_\mathcal{S}] \). Since \( v \) splits completely in \( L \subset \mathcal{L}_S \), we also have \( \mathcal{L}^\alpha_\mathcal{S} \supset L \) and \( 2|[[\mathcal{L}^\alpha_\mathcal{S} : L] \) unless \( L = \mathcal{L}^\alpha_\mathcal{S} \). Thus \( 4|[\mathcal{L}_S : L] \) unless \( L = \mathcal{L}^\alpha_\mathcal{S} \). From the definitions and Lemma 1 again, it follows that \( L \subset \mathcal{M} \subset \mathcal{L}^\alpha_\mathcal{S} \). Thus in the exceptional case of \( L = \mathcal{L}^\alpha_\mathcal{S} \), we have \( L = \mathcal{M} \), the maximal multiquadratic extension of \( F \) which is unramified outside of \( S \) and in which \( v \) splits completely.

9. The biquadratic case: proof of Theorem 4

We now assume that \( m = 2 \) and turn to the proof of Theorem 4. Since \( |S| \geq 3 \), Theorem 1 reduces us to the case where \( |S| = 3 \) and \( r_F(S) = 0 \). By Remark 2, we may assume that some prime \( p_2 \) over \( 2 \) ramifies in \( L/F \), so that \( S = \{\infty_1, \infty_2, p_2\} \), and we must have \( v = \infty_1 \) splitting in \( L/F \). (This is the only time we will make use of [3].) Then by Proposition 2, we have that \( M_i \) is odd for \( \eta_i \neq 1 \). Thus

\[
\varepsilon = \sqrt{\eta_1^{M_1(w_L/w_1)}} \sqrt{\eta_2^{M_2(w_L/w_2)}} , \quad \text{with both } M_i \text{ odd.}
\]

By Theorem 2, \( \varepsilon \in L \) satisfies \( St'(L/F) \) and indeed the proof shows that we may take \( \sqrt{\eta_i^{(w_L/w_i)}} \in L \) for \( i = 1, 2 \).

Temporarily fix \( i = 1 \) or \( 2 \). Notice that \( \infty_2 \) ramifies in \( K_i \), and only one other prime (namely \( p_2 \)) is allowed to ramify over \( F \). But some other prime must ramify, for otherwise \( K_i \) is contained in the ray class field for \( F \) modulo \( \infty_2 \). But the ray class group modulo \( \infty_2 \) is the same as the ray class group modulo 1, due to the presence of the unit \(-1 \). This would imply that \( K_i \) is unramified at \( \infty_2 \), a contradiction. Thus \( \eta_i \neq 1 \).

It remains to check that \( L(\varepsilon^{1/w_L}) \) is abelian over \( F \). For this we use a standard lemma (see [5, p. 83, Prop. 1.2]).

**Lemma 2.** Suppose \( L/F \) is a finite abelian extension of number fields with Galois group \( G \). Let \( A \) be the annihilator ideal of the group of roots of unity \( \mu_L \) considered as a module over the group ring \( \mathbb{Z}[G] \). Let \( T \) be a set of \( \mathbb{Z}[G] \)-generators for \( A \). Then an element \( u \) in the multiplicative group \( L^* \) has the property that \( L(u^{1/w_L})/F \) is abelian, if and only if there exists a collection of \( a_\alpha \in L^* \), indexed by \( \alpha \in T \) such that both of the following conditions hold:

\begin{itemize}
  \item[a.] \( a_\alpha^{w_L} = u^\alpha \quad \forall \alpha \in T \)
  \item[b.] \( a_\alpha = a_\beta \quad \forall \alpha, \beta \in T \).
\end{itemize}
Recall that $\tau \in G$ is the complex conjugation in $L$ over $\infty_2$. Also let $\tau_1$ be the element of order 2 in $G$ which fixes $K_1$. Thus $\tau$ and $\tau_1$ generate $G$.

First consider the number of roots of unity $w_L$ in $L$. Suppose $w_L > 2$. Then $L$ has no real embeddings, and the split prime $\infty_1$ of $F$ must be complex, while $\infty_2$ is real. Hence $F$ is a non-Galois cubic extension of $\mathbb{Q}$ and $[L : \mathbb{Q}] = 12$.

If $L$ contains a $p$th root of unity $\zeta_p$ for some odd prime $p$, then $L/F$ must ramify at some prime over $p$, because $F(\zeta_p)/F$ does. But the only finite prime which can ramify in $L/F$ is $p_2 \in S$. If $L$ contains a 16th root of unity, then $[L : \mathbb{Q}] = 12$ must be divisible by $\phi(16) = 8$, a contradiction. Thus the number of roots of unity in $L$ is $w_L = 2, 4$, or 8.

**Case 1:** $w_L = 2$. When $w_L = 2$, the above arguments show that we may take $K(\sqrt{\eta_1}) = L = K(\sqrt{\eta_2})$. Since $\tau$ is a complex conjugation and $\eta_i^{1+\tau} = 1$ for $i = 1, 2$; we conclude that $\sqrt{\eta_1^{1+\tau}} = 1$ for $i = 1, 2$. Using this, one can verify that the conditions of Lemma 2 hold for $e = \sqrt{\eta_1^{M_1}} \sqrt{\eta_2^{M_2}} \in L$ upon setting $a_{w_L} = e$, $a_{1+\tau} = 1$, and $a_{1+\tau_1} = \sqrt{\eta_1^{M_1}}$. Thus $St(L/F, S)$ holds in this case.

Now if $4|w_L$, then $L$ contains $F(\sqrt{-1})$, which must be either $K_1$ or $K_2$. By renumbering, we may assume that it is $K_2$. Thus $w_1 = 2$.

**Case 2:** $w_L = 4$. Now $w_1 = 2$, $w_2 = 4$, and $e = \eta_1^{M_1} \eta_2^{M_2} \in L$. We have noticed above that $L$ contains the square roots of all the roots of unity in the $K_i$. Thus $L$ contains an 8th root of unity $\zeta_8$. Put $a_{w_L} = e$, $a_{\tau+1} = 1$, and $a_{\tau_1-3} = \zeta_8^{\beta_1^{M_1}} \zeta_8^{\beta_2^{M_2}}$. The argument above shows that $\sqrt{\eta_1^{M_1}}$ and $\zeta_8$ lie in $L$, but not $L$, although their squares lie in $L$. Thus $\sqrt{\eta_1^{M_1}} \in L$, since $M_1$ is odd. Also $\sqrt{\eta_2^{M_2}} \in L$, so we have confirmed that $a_{\tau_1-3} \in L$. Again the conditions of Lemma 2 hold and $St(L/F, S)$ is proved in this case.

**Case 3:** $w_L = 8$. Now $w_1 = 2$, $w_2 = 4$, and $e = \eta_1^{2M_1} \eta_2^{2M_2} \in L$. Put $a_{w_L} = e$, $a_{\tau+1} = 1$, and $a_{\tau_1-3} = (\sqrt{\eta_1^{M_1}} (\sqrt{\eta_2^{M_2}})^{-M_2}$. We know that $\sqrt{\eta_1^{M_1}}$ and $\sqrt{\eta_2^{M_2}}$ lie in $L$, but not in $L$, although their squares lie in $L$. Thus $a_{\tau_1-3} \in L$, since $M_1$ and $M_2$ are odd. Again the conditions of Lemma 2 hold and $St(L/F, S)$ is proved in this case.

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