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A Reciprocity Congruence for an Analogue of the Dedekind Sum and quadratic reciprocity

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1. INTRODUCTION

In Berndt [1] and Goldberg [2] the sum $S_4(d, c)$ is defined for $c > 0$ by

$$S_4(d, c) = \sum_{j=1}^{c-1} (-1)^{[dj/c]}.$$ 

The sum is one of several involved in the multiplier systems for transformations of the logarithms of the classical theta-functions. We define two of them here. Let $q = e^{\pi iz}$ and for $\text{Im}(z) > 0$ define

$$\theta_3(z) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$ 

We prove a new reciprocity theorem for the sum $S_4(d, c)$. As an application of the theorem, we deduce the law of quadratic reciprocity.

Let $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 1$ and $Vz = (az + b)/(cz + d)$. We use the principal branch of the logarithm at all times. Berndt [1] proved that...
if \( b \) is even, then

\[
(1) \quad \log \theta_4(Vz) = \log \theta_4(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(d,c).
\]

Goldberg [2] gives the following formula. If \( a \) is even and \( b, c \) and \( d \) are odd, then

\[
(2) \quad \log \theta_3(Vz) = \log \theta_4(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(d,c).
\]

With the reciprocity theorem for the Dedekind sum \( s(d,c) \) and connections between \( s(d,c) \) and the Legendre-Jacobi symbol \( \left( \frac{d}{c} \right)_c \), one can deduce the quadratic reciprocity law (See Rademacher and Grosswald [3], pp. 34-35). Berndt [1] proved elegant reciprocity theorems for several sums that arise in the transformation formulas of the logarithms of the classical theta-functions. None of these reciprocity theorems allow for both of the arguments in the sum to be odd. For the application of the sums arising in the theta-function transformations to quadratic reciprocity, however, we need a new reciprocity theorem where \( c \) and \( d \) are both odd. And, unlike the Dedekind sum, the sum \( S_4(d,c) \) does not possess a reciprocity theorem. However, a reciprocity relation modulo 8 for the sum \( S_4(d,c) \) does exist and this is sufficient to deduce the quadratic reciprocity theorem. For additional properties of \( S_4(d,c) \) see [7] and [6].

To establish the connection between \( S_4(d,c) \) and the Legendre-Jacobi symbol, we turn to Rademacher’s book [8] (pp. 180-182) for the needed results. We use the same double subscript notation as Rademacher to state the theorem. For \( \text{Im}(z) > 0 \) and \( \mu, \nu \in \{0,1\} \), let

\[
\theta_{\mu,\nu}(z) = \sum_{n=-\infty}^{\infty} (-1)^\nu n e^{(n+\mu/2)^2 \pi iz}.
\]

Thus

\[
(3) \quad \theta_{0,0}(z) = \theta_3(z) \quad \text{and} \quad \theta_{0,1}(z) = \theta_4(z).
\]

Note that we may allow integers other than 1 and 0 as subscripts since it can be shown that

\[
(4) \quad \theta_{\mu+2,\nu}(z) = (-1)^\nu \theta_{\mu,\nu}(z) \quad \text{and} \quad \theta_{\mu,\nu+2}(z) = \theta_{\mu,\nu}(z).
\]

We state one of Rademacher’s transformation formulas.

**Theorem 1.** If \( V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( c \) is odd and positive, then

\[
\theta_{1+a,1-b}(Vz) = -i^b e^{\pi tbd/4} \epsilon_1 \sqrt{\frac{cz + d}{t}} \theta_4(z),
\]
where
\[ \epsilon_1 = \epsilon_1(a, b, c, d) = \left( \frac{d}{c} \right)^{(c-3)/2} e^{(\pi i/4)c(a+d)}. \]

2. Reciprocity Theorem for \( S_4(d, c) \)

The next result is the foundation on which the new reciprocity theorem is based.

**Theorem 2.** Let \( c \) and \( d \) be positive, coprime, odd integers. Then
\[ S_4(c, d) + S_4(d, c) = -1 + S_4(c^2, cd + 1). \]

**Proof.** Choose \( a \) and \( b \) with \( b > d \), \( b \) even and \( ad - bc = 1 \) and set
\[ V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} -d^2 & cd - 1 \\ cd + 1 & -c^2 \end{bmatrix}. \]

Then
\[ VW = \begin{bmatrix} b - d & c - a \\ d & -c \end{bmatrix}. \]

Note that in each matrix the upper right entry is even and the determinant is 1. We use (1) with \( V \) replaced with \( VW \) to find that
\[ \log \theta_4((VW)z) = \log \theta_4(z) + \frac{1}{2} \log(dz - c) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(-c, d). \]

Then we apply (1) with \( z \) replaced by \( Wz \) to see that
\[ \log \theta_4(V(Wz)) = \log \theta_4(Wz) + \frac{1}{2} \log(c(Wz) + d) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(d, c), \]
and finally we use (1) with \( V \) replaced by \( W \) to deduce that
\[ \log \theta_4(Wz) = \log \theta_4(z) + \frac{1}{2} \log((cd + 1)z - c^2) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(-c^2, cd + 1). \]

We replace \( \log \theta_4(Wz) \) in (6) with (7) and then combine the result with (5) to conclude that
\[ -\frac{1}{4} \pi i S_4(-c, d) = -\frac{1}{4} \pi i S_4(-c^2, cd + 1) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(d, c). \]

We have used the following lemma [5] to conclude that there are no branch changes with the logarithms, so the complete cancellation is justified.
Lemma 1. Let $A, B, C,$ and $D$ be real with $A$ and $B$ not both zero and $C > 0$. Then for $\text{Im}(z) > 0$,
\[ \arg((Az + B)/(Cz + D)) = \arg(Az + B) - \arg(Cz + D) + 2\pi k, \]
where $k$ is independent of $z$ and
\[
k = \begin{cases} 1, & \text{if } A \leq 0 \text{ and } AD - BC > 0, \\ 0, & \text{otherwise.} \end{cases}
\]

Next we multiply (8) by $4/(\pi i)$, rearrange, and use the fact that $S_4(-c, d) = -S_4(c, d)$ to conclude that
\[ S_4(c, d) + S_4(d, c) = -1 + S_4(c^2, cd + 1). \]

The symmetry between $c$ and $d$ on the left-hand side of the equation in Theorem 3 leads immediately to the next result.

Corollary 1. If $c$ and $d$ are coprime, odd, positive integers, then
\[ S_4(c^2, cd + 1) = S_4(d^2, cd + 1). \]

The final step towards the desired reciprocity relation is the following congruence relation.

Lemma 2. Let $d$ be an odd prime and $c > d$ be an odd, positive integer coprime to $d$. Then
\[ S_4(d^2, cd + 1) \equiv cd \pmod{8}. \]

Proof. Using the definition of $S_4(d, c)$ and the fact that
\[ [x] - 2\left[\frac{x}{2}\right] = \begin{cases} 0, & \text{if } [x] \text{ is even,} \\ 1, & \text{if } [x] \text{ is odd,} \end{cases} \]
we see that
\[
(9) \quad S_4(d^2, cd + 1) = \sum_{j=1}^{cd}(-1)^{\left\lfloor \frac{jd^2}{cd+1} \right\rfloor} 
= \#\{\lfloor jd^2/(cd + 1) \rfloor \text{ even}\} - \#\{\lfloor jd^2/(cd + 1) \rfloor \text{ odd}\} 
= cd - 2 \sum_{j=1}^{cd} \left\lfloor \frac{jd^2}{cd+1} \right\rfloor + 4 \sum_{j=1}^{cd} \left\lfloor \frac{jd^2}{2(cd + 1)} \right\rfloor.
\]

If $m$ and $n$ are positive and coprime integers, recall that (see [4] p. 186, for example)
\[
(10) \quad \sum_{j=1}^{n-1} \left[ \frac{mj}{n} \right] = \frac{(n-1)(m-1)}{2}.
\]
We now apply (10) to the first of the final two sums in (9) and separate the terms in the second sum on the right according to the parity of \( j \). Thus (9) becomes

\[
S_4(d^2, cd + 1) = cd - (d^2 - 1)cd + 4 \sum_{j=1}^{(cd-1)/2} \left[ \frac{j d^2}{cd + 1} \right] \\
+ 4 \sum_{j=1}^{(cd+1)/2} \left[ \frac{j d^2}{cd + 1} - \frac{d^2}{2(cd + 1)} \right].
\]

Let

\[
N = \# \left\{ j = 1, 2, \ldots, \frac{cd + 1}{2} : \left\{ \frac{j d^2}{cd + 1} \right\} < \frac{d^2}{2(cd + 1)} \right\},
\]

where \( \{x\} = x - [x] \) denotes the fractional part of \( x \). The final sum in (11) can be simplified since \( d < c \) and \( [d^2/2] = (d^2 - 1)/2 \) for \( d \) odd. Note that

\[
\sum_{j=1}^{(cd+1)/2} \left[ \frac{j d^2}{cd + 1} - \frac{d^2}{2(cd + 1)} \right] = \sum_{j=1}^{(cd-1)/2} \left[ \frac{j d^2}{cd + 1} \right] + \frac{d^2 - 1}{2} - N.
\]

We conclude from (11) and (13) that

\[
S_4(d^2, cd + 1) = cd - (d^2 - 1)cd + 8 \sum_{j=1}^{(cd-1)/2} \left[ \frac{j d^2}{cd + 1} \right] + 2(d^2 - 1) - 4N.
\]

Since \( d \) is odd and thus \( d^2 \equiv 1 \pmod{8} \), we deduce the congruence

\[
S_4(d^2, cd + 1) \equiv cd + 4N \pmod{8}.
\]

We claim that \( N \) is even. Let

\[
n = \# \left\{ k = 1, 2, \ldots, \frac{d^2 - 1}{2} : LPR_{d^2}(k(cd + 1)) > \frac{d^2}{2} \right\},
\]

where \( LPR_l(m) \) denotes the least positive residue of \( m \) modulo the positive integer \( l \). Observe that

\[
\left\{ \frac{j d^2}{cd + 1} \right\} < \frac{d^2}{2(cd + 1)}
\]

if and only if there exists a positive integer \( k \) such that

\[
2k(cd + 1) < 2jd^2 < 2k(cd + 1) + d^2.
\]

We rewrite (16) in the form

\[
\frac{k(cd + 1)}{d^2} < j < \frac{k(cd + 1)}{d^2} + \frac{1}{2}.
\]
Since the interval \( (k(cd + 1)/d^2, k(cd + 1)/d^2 + 1/2) \) has length 1/2, it contains an integer \( j \) if and only if \( k \) satisfies
\[
\left\{ \frac{k(cd + 1)}{d^2} \right\} > \frac{1}{2},
\]
or, in other words, if and only if \( LPR_{d^2}(k(cd + 1)) > d^2/2 \). Thus \( N = n \).

The final claim in the proof of the congruence is the following: The number \( n \) defined above is even.

The proof of this claim is a careful application of the method used in the standard proof of Gauss's Lemma (see [4], p. 133, for example).

Let \( r_1, r_2, \ldots, r_n \) be the \( n \) residues of \( k(cd + 1) \), \( 1 \leq k \leq (d^2 - 1)/2 \), falling in the upper half of the least positive residue system \((\mod d^2)\) and let \( s_1, s_2, \ldots, s_l \) be those in the lower half. Then \( n + l = (d^2 - 1)/2 \). Next we consider \( r_i' = d^2 - r_i \) for \( i = 1, 2, \ldots, n \). Each of the \( r_i' \) are distinct and we can say further that no \( r_i' = s_j \) for any \( i \) and \( j \).

Note that there are \( [(d^2 - 1)/(2d)] = (d - 1)/2 \) positive multiples of \( d \) that are less than \( (d^2 - 1)/2 \). Now for \( k = md, 1 \leq m \leq (d - 1)/2 \),
\[
(18) \quad k(cd + 1) = md(cd + 1) \equiv md \pmod{d^2}.
\]

From (18), we conclude that
\[
LPR_{d^2}(md(cd + 1)) = md < \frac{d^2}{2}.
\]

Thus all of the residues produced when \( k \) is a multiple of \( d \) are among the \( s_j \) and are in fact the positive multiples of \( d \) that are less than \( d^2/2 \). We remove them from the list \( \{s_1, \ldots, s_l\} \), reindex this set and put
\[
l' = l - \frac{d - 1}{2}.
\]

So now we consider the \( n \) \( r_i \)'s, and the \( l' \) \( s_j \)'s, with \( n + l' = d(d - 1)/2 \) residues altogether. By the distinctness of the \( r_i' \), we conclude that the \( r_i' \) and the \( s_j \) are some rearrangement of the numbers \( 1, 2, \ldots, (d^2 - 1)/2 \) with the multiples of \( d \) removed. Thus
\[
r_1'r_2' \cdots r_n's_1s_2 \cdots s_l = \prod_{1 \leq \alpha \leq (d^2-1)/2 \atop (d,\alpha)=1} \alpha;
\]
or, by definition of the \( r_i' \),
\[
(d^2 - r_1)(d^2 - r_2) \cdots (d^2 - r_n)s_1s_2 \cdots s_l = \prod_{1 \leq \alpha \leq (d^2-1)/2 \atop (d,\alpha)=1} \alpha,
\]
yielding the congruence
\[(19) \quad (-1)^n r_1 r_2 \cdots r_n s_1 s_2 \cdots s_n \equiv \prod_{1 \leq \alpha \leq (d^2 - 1)/2, \ (d, \alpha) = 1} \alpha \pmod{d^2}.\]

Now we rewrite (19) with the $r_i$ and the $s_j$ in their original form, with some possible rearrangement, to get
\[(20) \quad (-1)^n (cd + 1) \cdot 2(cd + 1) \cdots \frac{d^2 - 1}{2} (cd + 1) \equiv \prod_{1 \leq \alpha \leq (d^2 - 1)/2, \ (d, \alpha) = 1} \alpha \pmod{d^2}.

Recall the exclusion of the multiples of $d$ on each side. Because $d$ is prime, we may cancel the common factors from each side of the congruence (20) and conclude that
\[(21) \quad (cd + 1)^{d(d-1)/2} \equiv (-1)^n \pmod{d^2}.

But by the binomial theorem,
\[(22) \quad (cd + 1)^{d(d-1)/2} \equiv 1 + \frac{d(d-1)}{2} cd \equiv 1 \pmod{d^2}.

From (21) and (22), we deduce that $n$ is even and the proof of the claim is complete. And thus, from (14), we have
\[S_4(d^2, cd + 1) \equiv cd \pmod{8}.

We now assemble Theorem 3, Corollary 1, and Lemma 2 to reach our reciprocity result.

**Theorem 3.** Let $d$ be an odd prime and $c > d$ be an odd, positive integer coprime to $d$. Then
\[S_4(c, d) + S_4(d, c) \equiv -1 + cd \pmod{8}.

3. THE LAW OF QUADRATIC RECIPROCity

As an application of Theorem 4, we offer a new proof of the law of quadratic reciprocity.

**Theorem 4.** Let $c$ and $d$ be distinct odd primes. Then
\[
\left(\frac{c}{d}\right) \left(\frac{d}{c}\right) = (-1)^{\frac{c-1}{2} \frac{d-1}{2}}.
\]
Proof. Given \(c\) and \(d\), there are integers \(a\) and \(b\) with \(b\) even such that \(ad-bc = 1\). Let \(V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) and \(W = \begin{bmatrix} -b & -a \\ d & c \end{bmatrix}\). Note that \(a\) is necessarily odd. From (3), (4) and recalling that \(a, c,\) and \(d\) are odd and \(b\) is even, we see that

\[
\theta_{1+d,1-b}(V z) = \theta_{1+d,1}(V z) = (-1)^{(d+1)/2} \theta_{0,1}(V z) = (-1)^{(d+1)/2} \theta_{4}(V z),
\]

and

\[
\theta_{1+c,1+a}(W z) = \theta_{1+c,0}(W z) = \theta_{0,0}(W z) = \theta_{3}(W z).
\]

From Theorem 1, (23) and (24), we have

\[
\theta_{4}(V z) = (-1)^{(-d-1)/2}(-1)^b e^{\pi i b d/4} \times \left( \frac{d}{c} \right)^{(c-3)/2} e^{\pi i (ac+dc)/4} \sqrt{cz + d} \theta_{4}(z)
\]

and

\[
\theta_{3}(W z) = -i^{-a} e^{-\pi i ac/4} \left( \frac{c}{d} \right)^{(d-3)/2} e^{\pi i (dc-bd)/4} \sqrt{dz + c} \theta_{4}(z).
\]

We also have, from (1) and (2),

\[
\log \theta_{4}(V z) = \log \theta_{4}(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_{4}(d, c)
\]

and

\[
\log \theta_{3}(W z) = \log \theta_{4}(z) + \frac{1}{2} \log(dz + c) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_{4}(c, d).
\]

Next, we multiply (25) and (26) and partially simplify the result to deduce that

\[
\theta_{4}(V z) \theta_{3}(W z) = (-1)^{(-d-1)/2} \sqrt{b} \frac{b}{2} (c+d-6)/2 e^{\pi i d c/2} i^{-1} \times \left( \frac{c}{d} \right) \left( \frac{d}{c} \right) \sqrt{cz + d \sqrt{dz + c}} \theta_{4}(z)^2.
\]

We exponentiate and then multiply (27) and (28) to see that

\[
\theta_{4}(V z) \theta_{3}(W z) = \theta_{4}(z)^2 \sqrt{cz + d \sqrt{dz + c}} e^{-\pi i/2} \times e^{-(\pi i/4)}(S_{4}(c, d) + S_{4}(d, c)).
\]
From (29), (30) and Lemma 1, we deduce that
\begin{equation}
(-1)^{(d-1)/2}e^{(2b-2a+c+d-6)/2}e^{\pi idc/2}e^{-1} \left( \begin{array}{c} c \\ d \end{array} \right) \left( \begin{array}{c} d \\ c \end{array} \right) = e^{-\pi i/2}e^{-(\pi i/4)(S_4(c, d) + S_4(d, c))}.
\end{equation}

Next, we simplify (31) to find that
\begin{equation}
e^{(\pi i/4)(-2a+2b+c-d+2cd)} \left( \begin{array}{c} c \\ d \end{array} \right) \left( \begin{array}{c} d \\ c \end{array} \right) = e^{-(\pi i/4)(S_4(c, d) + S_4(d, c))}.
\end{equation}

Note that since $d$ is odd and $ad-bc = 1$, we have that $a \equiv d + dbc \pmod{8}$. From this fact and the application of Theorem 3 to (32), we conclude that
\begin{equation}
\left( \begin{array}{c} c \\ d \end{array} \right) \left( \begin{array}{c} d \\ c \end{array} \right) = e^{(\pi i/4)(3d-c-2b+2bcd-3cd+1)}.
\end{equation}

A straight-forward calculation shows that
\begin{equation}
3d-c-2b+2bcd-3cd+1 \equiv (c-1)(d-1) \pmod{8}.
\end{equation}

Using (34) in (33), we deduce that
\begin{equation}
\left( \begin{array}{c} c \\ d \end{array} \right) \left( \begin{array}{c} d \\ c \end{array} \right) = e^{(\pi i/4)(c-1)(d-1)} = (-1)^{\frac{c-1}{2}\frac{d-1}{2}}
\end{equation}
as desired.

\[\square\]

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