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Dirichlet divisor problem


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Mean Square of the Remainder Term in the Dirichlet Divisor Problem

par Yuk-Kam Lau and Kai-Man Tsang

1. Introduction and Main Results

Let \( d(n) \) denote the divisor function. In this paper we shall consider a remainder term associated with the mean square of the error term \( \Delta(x) \) in the Dirichlet divisor problem, which is defined as

\[
\Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1).
\]

Here \( \gamma \) is Euler's constant. The upper bound \( \Delta(x) \ll x^{1/2} \) was first obtained by Dirichlet in 1838. This was gradually sharpened by many authors in the ensuing one and a half century. Iwaniec and Mozzochi [5] proved in 1988 that \( \Delta(x) \ll x^{7/22+\varepsilon} \) for any \( \varepsilon > 0 \), by employing intricated techniques for the estimation of certain exponential sums. Such methods, however, do not seem capable of proving the conjectured best bound: \( \Delta(x) \ll x^{1/4+\varepsilon} \).

Besides this problem, there are plenty of papers written on other interesting properties of \( \Delta(x) \). For instance, Tong [9] showed that \( \Delta(x) \) changes sign at least once in every interval of the form \([X, X + c_0\sqrt{X}]\) where \( c_0 \) is a certain positive constant. Recently Heath-Brown and Tsang [2] showed that this is essentially best possible: the length of the intervals cannot be reduced to \( o(\sqrt{X} \log^{-5} X) \). In contrast to this erratic behaviour, \( \Delta(x) \), when considered in the mean, has very nice asymptotic formula. A classical result of Tong [10] says that

\[
\int_2^X \Delta(x)^2 \, dx = \left( (6\pi^2)^{-1} \sum_{m=1}^{\infty} d(m)^2 m^{-3/2} \right) X^{3/2} + F(X)
\]

(1.1)

with \( F(X) \ll X \log^5 X \). The order of the remainder term \( F(X) \) has significant connection with that of \( \Delta(x) \). Indeed, Ivic’s argument in Theorem

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3.8 of [4] shows that $\triangle(x) \ll (U \log x)^{1/3}$ for any upper bound $U$ of $F(X)$. Thus from the result $\triangle(x) = \Omega(x^{1/4})$ we infer that

\[
F(X) = \Omega(X^{3/4} / \log X).
\]

Ivić conjectured that $F(X) \ll X^{3/4+\varepsilon}$ is true for any $\varepsilon > 0$. This is a very strong bound since it implies $\triangle(x) \ll x^{1/4+\varepsilon}$. There are not many results on $F(X)$ in the literature. Tong’s bound was slightly improved to $F(X) \ll X \log^4 X$ by Preissmann [7] in 1988. However, the gap between this and the $\Omega$-result (1.2) is still very wide.

In this paper we shall prove the following.

**THEOREM 1.** We have

\[
F(X) = \Omega_-(X \log^2 X).
\]

**THEOREM 2.** For $X \geq 2$ we have

\[
\int_2^X F(x) dx = -(8\pi^2)^{-1} X^2 \log^2 X + c_1 X^2 \log X + \mathcal{O}(X^2)
\]

for a certain constant $c_1$.

Theorem 1, which is a direct consequence of Theorem 2, disproves the above conjecture of Ivić. Unfortunately we are still unable to obtain a comparable $\Omega_+$-result for $F(x)$. In fact we believe that there is an asymptotic formula for $F(x)$ of the form

\[
F(x) = -(4\pi^2)^{-1} x \log^2 x + c_2 x \log x + \mathcal{O}(x)
\]

with a certain constant $c_2$. In a forthcoming paper, the second author [11] proves that

\[
\int_X^{2X} (F(x + \sqrt{X}) - F(x))^2 dx = X^3.
\]

Using Preissmann’s bound we see easily that

\[
\int_X^{2X} (F(x + \sqrt{X}) - F(x)) dx = \int_{2X}^{2X+\sqrt{X}} F(x) dx - \int_X^{X+\sqrt{X}} F(x) dx
\]

\[
\ll X^{3/2} \log^4 X.
\]
These two results together shows that $F(x + \sqrt{X}) - F(x)$ changes signs in $[X, 2X]$ and

$$F(x + \sqrt{X}) - F(x) = \Omega(X).$$

Consequently, if (1.3) is true the $O$-term on the right hand side is oscillatory and cannot be reduced.

One of the key ingredients in our argument is an asymptotic formula for the sum

$$\sum_{m \leq x} d(m)d(m + h).$$

Such a sum has been investigated by several authors in connection with other problems in analytic number theory. In our proof we use a result of Heath-Brown [1] which is quite sufficient for our purpose. (see (2.12)-(2.15) below)

2. Notations and some Preparation

Throughout the paper, $\varepsilon$ denotes an arbitrary small positive number which need not be the same at each occurrence. The symbols $c_0, c_1, c_2, \ldots$ etc. denote certain constants. We shall also use the well-known inequality $d(n) \ll n^\varepsilon$ from time to time without explicit reference. The constants implicit in the symbols $O$ and $\ll$ depend at most on $\varepsilon$.

A useful formula for studying problems concerning $\Delta(x)$ was obtained by Voronoi [12] at the beginning of this century. The formula expresses $\Delta(x)$ as an infinite series involving the Bessel functions. In practice, the following truncated form of the formula

$$\Delta(x) = (\pi \sqrt{2})^{-1} x^{1/4} \sum_{n \leq N} d(n) n^{-3/4} \cos(4\pi \sqrt{nx} - \pi/4)$$

$$+ O(x^\varepsilon + x^{1/2+\varepsilon} N^{-1/2})$$

for $1 \leq N \ll x$ is quite sufficient. However, for our present problem, the above $O$-term is far too large and we shall use instead the following approximation to $\Delta(x)$ given by Meurman [6, Lemma 3].

**Lemma 1.** For $x \geq 1$ and $M \gg x$, let

$$\delta_M(x) = (\pi \sqrt{2})^{-1} x^{1/4} \sum_{n \leq M} d(n) n^{-3/4} \cos(4\pi \sqrt{nx} - \pi/4).$$
Then \( \Delta(x) = \delta_M(x) + R(x) \) where \( R(x) \ll x^{-1/4} \) if \( \|x\| \gg x^{5/2} M^{-1/2} \) and \( R(x) \ll x^\varepsilon \) otherwise.

Using this we obtain

**LEMMA 2.** Let \( x \geq 2 \) and \( x^7 \ll M \ll x^{100} \). Then

\[
\int_2^x \Delta(u)^2 du = \int_2^x \delta_M(u)^2 du + O(x).
\]

**Proof.** Firstly,

\[
\int_2^x \Delta(u)^2 du = \int_2^x \delta_M(u)^2 du + 2 \int_2^x \delta_M(u)R(u)du + \int_2^x R(u)^2 du.
\]

Next, by Lemma 1, we have

\[
(2.1) \quad \int_2^x R(u)^2 du \ll \sum_{n=2}^{[x] + 1} n^\varepsilon n^{5/2} M^{-1/2} + \int_2^x (u^{-1/4})^2 du \ll \sqrt{x}.
\]

Moreover, following the argument of [3, Theorem 13.5] we show that

\[
\int_2^x \delta_M(u)^2 du \asymp x^{3/2}
\]

for \( M \ll x^{100} \). Thus, by Cauchy-Schwarz’s inequality and (2.1) we have

\[
\int_2^x \delta_M(u)R(u)du \ll x
\]

and hence our lemma.

Square out \( \delta_M(u) \) and then integrate term by term, we get

\[
\int_2^x \delta_M(u)^2 du
\]

\[
= (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u} \cos (4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}) du
\]

\[
+ (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u} \sin (4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}) du.
\]
In the first double sum the diagonal terms yield a total contribution of

\[
(4\pi^2)^{-1} \sum_{m \leq M} d(m)^2 m^{-3/2} \frac{2}{3} (x^{3/2} - 2^{3/2})
\]

\[
= (6\pi^2)^{-1} \sum_{m=1}^{\infty} d(m)^2 m^{-3/2} x^{3/2} + \mathcal{O}(x^{3/2} M^{\epsilon-1/2} + 1).
\]

Here the main term is the same as that in (1.1). Hence by Lemma 2, we can write

\[
F(x) = S_1(x) + S_2(x) + \mathcal{O}(x),
\]

where for any \( y \geq 2, \)

\[
S_1(y) = (2\pi^2)^{-1} \sum_{m \leq n \leq M} d(m)d(n)(mn)^{-3/4} \times \int_2^y \sqrt{u} \cos (4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}) du,
\]

and

\[
S_2(y) = (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \times \int_2^y \sqrt{u} \sin (4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}) du.
\]

From now on, we let \( X \) to be a sufficiently large number, \( M = X^7 \) and \( L = \log X \). For any \( \nu \geq 0 \), let

\[
g(\nu) = \nu^{-3/2} J_{3/2}(\nu) - 4\nu^{-5/2} J_{5/2}(\nu),
\]

where \( J_k \) denotes the Bessel function of order \( k \). It is well-known that [13, §§3.3, 3.4]

\[
J_k(z) \ll \min(|z|^k, |z|^{-1/2})
\]

for any real \( z \). Hence,

\[
g(\nu) \ll \min(1, \nu^{-2})
\]

for any \( \nu \geq 0 \).
LEMMA 3. We have

\begin{equation}
\int_0^X F(x)dx = \sqrt{2\pi}^{-3/2}X^{5/2} \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}g(\theta_{m,n}) + O(X^2)
\end{equation}

where \(\theta_{m,n} = 4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}\).

Proof. By [8, Lemma 4.2], for any real \(\alpha\) and \(y, y \geq 2\) we have

\begin{equation}
\int_2^y \sqrt{u}e^{i\alpha\sqrt{u}}du \ll y|\alpha|^{-1}.
\end{equation}

We first obtain some preliminary bounds for \(S_1(y)\) and \(S_2(y)\). According to (2.3) and on applying (2.8), we have

\begin{equation}
S_1(y) \ll y \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}(\sqrt{n} - \sqrt{m})^{-1}
\end{equation}

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S_1(y) \ll y \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}(\sqrt{n} - \sqrt{m})^{-1}
\end{equation}

\begin{equation}
\ll yM^\epsilon \left\{ \sum_{m<n\leq 2m\leq M} (mn)^{-3/4}(\sqrt{n} + \sqrt{m})(n-m)^{-1} + \sum_{2m<n\leq M} m^{-3/4n^{-5/4}} \right\}
\ll yM^\epsilon \left\{ \sum_{m\leq M/2} m^{-1} \sum_{m<n\leq M} (n-m)^{-1} + \log M \right\} \ll yM^\epsilon.
\end{equation}

Similarly,

\begin{equation}
S_2(y) \ll y \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4}(\sqrt{n} + \sqrt{m})^{-1} \ll yM^\epsilon.
\end{equation}

Next, for \(x \in [\sqrt{X}, X]\) we have \(x^7 \ll M \ll x^{14}\) so that, by (2.2)

\begin{equation}
\int_{\sqrt{X}}^X F(x)dx = \int_{\sqrt{X}}^X S_1(x)dx + \int_{\sqrt{X}}^X S_2(x)dx + O(X^2)
\end{equation}

\begin{equation}
= \int_2^X S_1(x)dx + \int_2^X S_2(x)dx + O(XM^\epsilon + X^2),
\end{equation}

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since, by (2.9) and (2.10), $\int_2^{\sqrt{X}} S_1(x)\,dx \ll X M^\varepsilon$. The main term on the right hand side of (2.7) arises from $\int_2^X S_1(x)\,dx$. Indeed, by (2.3),

$$\int_2^X S_1(x)\,dx = (2\pi^2)^{-1} \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4} \times \int_2^X \int_2^x \sqrt{u} \cos \left(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}\right)\,dudx .$$

Write $\theta = 4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}$ for short. Then the above double integral is equal to

$$\int_2^X (X - u) \sqrt{u} \cos \left(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}\right)\,du$$

$$= 2X^{5/2} \int_{\sqrt{2/X}}^1 (1 - v^2)v^2 \cos(\theta v)\,dv$$

$$= 2X^{5/2} \left\{ \int_0^1 (1 - v^2) \cos(\theta v)\,dv - \int_0^1 (1 - v^2)^2 \cos(\theta v)\,dv \right\} .$$

By the well-known integral representation

$$J_{k+\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi}} \left( \frac{z}{2} \right)^{k+\frac{1}{2}} \frac{1}{k!} \int_0^1 (1 - v^2)^k \cos(zv)\,dv , \quad k = 0, 1, 2, \ldots$$

for the Bessel functions [13, §3.3], the first two integrals on the right hand side is equal to

$$\sqrt{2\pi} \left( \theta^{-3/2} J_{3/2}(\theta) - 4\theta^{-5/2} J_{5/2}(\theta) \right) = \sqrt{2\pi} g(\theta) ,$$

by (2.5). Moreover, using integration by parts we find that

$$\int_0^{\sqrt{2/X}} (1 - v^2)v^2 \cos(\theta v)\,dv \ll X^{-1} \theta^{-1} .$$

Hence

$$\int_2^X (X - u) \sqrt{u} \cos \left(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}\right)\,du = 2\sqrt{2\pi} X^{5/2} g(\theta) + O(X^{3/2} \theta^{-1}) ,$$
and then
\[ \int_2^X S_1(x)dx = \sqrt{2\pi}^{-3/2} X^{5/2} \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}g(\theta) + O\left( X \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}(\sqrt{n} - \sqrt{m})^{-1} \right). \]

The sum inside the $O$-term can be treated by the argument in (2.9), and we then find that the $O$-term is bounded by $XM^\epsilon$, which is smaller than that on the right hand side of (2.7).

In view of (2.11) and (2.7), it remains to bound the two integrals
\[ \int_0^{\sqrt{X}} F(x)dx \quad \text{and} \quad \int_2^X S_2(x)dx \]
by $X^2$. By Preissmann’s bound, we have
\[ \int_0^{\sqrt{X}} F(x)dx \ll X \log^4 X \]
which is acceptable. Next, by (2.4),
\[ \int_2^X S_2(x)dx = (4\pi^2)^{-1} \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4} \times \]
\[ \int_2^X (X-u)\sqrt{u} \sin \left( 4\pi(\sqrt{n} + \sqrt{m})\sqrt{u} \right) du \]
\[ = (2\pi^2)^{-1} X^{5/2} \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4} \times \]
\[ \int_{\sqrt{2}/X}^1 (1-u)v^2 \sin \left( 4\pi(\sqrt{n} + \sqrt{m})\sqrt{X}v \right) dv. \]
The inner integral, on applying integration by parts twice, is found to be
\[ \ll X^{-3/2}(\sqrt{n} + \sqrt{m})^{-1} + X^{-1}(\sqrt{n} + \sqrt{m})^{-2}. \]
Thus,
\[ \int_2^X S_2(x)dx \ll X \sum_{m\leq n\leq M} d(m)d(n)(mn)^{-3/4}n^{-1/2} + X^{3/2} \sum_{m\leq n\leq M} d(m)d(n)(mn)^{-3/4}n^{-1} \ll XM^\epsilon + X^{3/2} \ll X^{3/2}. \]
This completes the proof of Lemma 3.

For any $y > 0$, let
\begin{equation}
\psi_h(y) = \sum_{m \leq y} d(m)d(m + h).
\end{equation}

In his work on the fourth power moment of the Riemann zeta-function on the critical line, Heath-Brown [1] proved that
\begin{equation}
\psi_h(y) = I_h(y) + E_h(y),
\end{equation}
where the main term $I_h(y)$ is of the form
\begin{equation}
I_h(y) = y \sum_{i=0}^{2} \log^i y \sum_{d|h} d^{-1} (\alpha_{i0} + \alpha_{i1} \log d + \alpha_{i2} \log^2 d)
\end{equation}
for certain constants $\alpha_{ij}$, and the remainder $E_h(y)$ satisfies
\begin{equation}
E_h(y) \ll y^{5/6+\varepsilon}
\end{equation}
uniformly for $1 \leq h \leq y^{5/6}$. In particular $\alpha_{20} = 6\pi^{-2}$, $\alpha_{21} = \alpha_{22} = 0$. We note that $I_h(y)$ is roughly of order $y \log y$. In our proof of Theorem 2 in §3 we shall need $I'_h(y)$, the derivative of $I_h(y)$. By (2.14)
\begin{equation}
I'_h(y) = a_2(h) \log^2 y + a_1(h) \log y + a_0(h)
\end{equation}
where
\begin{align*}
a_2(h) &= 6\pi^{-2} \sum_{d|h} d^{-1}, \\
a_1(h) &= \sum_{d|h} d^{-1} (12\pi^{-2} + \alpha_{10} + \alpha_{11} \log d + \alpha_{12} \log^2 d), \\
a_0(h) &= \sum_{d|h} d^{-1} \sum_{j=0}^{2} (\alpha_{0j} + \alpha_{1j}) \log^j d.
\end{align*}

For any $y > 0$, $Q > 3$ let
\begin{equation}
\xi(y, Q) = \sum_{h \leq y} h^{-1} (4a_2(h) \log^2 Qh + 2a_1(h) \log Qh + a_0(h)).
\end{equation}
LEMMA 4. We have

\[
\xi(y, Q) = \frac{4}{3} \log^3 Q y + c_3 \log^2 Q y - \frac{4}{3} \log^3 Q + c_4 \log^2 Q + c_5 \log Q \\
+ c_6 \log y + c_7 + \mathcal{O}(y^{-1} \log^3 y \log^2 Q y).
\]

Proof. In the argument below we use the symbol \( c \) to denote a certain constant which may not be the same at each occurrence.

Firstly, for \( j = 0, 1, 2 \) there are constants \( \beta_0, \beta_1, \beta_2 \) such that

\[
(2.19) \quad \sum_{h \leq y} a_j(h) = \beta_j y + B_j(y)
\]

with \( B_j(y) \ll \log^3 y \). (Note \( B_j(1^-) = -\beta_j \)). Indeed, by (2.17),

\[
\sum_{h \leq y} a_0(h) = \sum_{d \leq y} d^{-1} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) (\log^j d) (yd^{-1} + \mathcal{O}(1))
\]

\[
= y \sum_{d \leq y} d^{-2} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d + \mathcal{O}\left( \sum_{d \leq y} d^{-1} \log^2 d \right)
\]

\[
= y \sum_{d=1}^{\infty} d^{-2} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d + \mathcal{O}\left( y \sum_{d > y} d^{-2} \log^2 d \right) +
\]

\[
+ \mathcal{O}(\log^3 y)
\]

\[
= \beta_0 y + \mathcal{O}(\log^3 y)
\]

with

\[
\beta_0 = \sum_{d=1}^{\infty} d^{-2} \sum_{j=0}^2 (\alpha_{0j} + \alpha_{1j}) \log^j d .
\]

Similar argument establishes (2.19) for \( j = 1 \) and 2. Further we find that \( \beta_2 = 1 \).
Next by Riemann Stieltjes integration and (2.19), we have

\[
\sum_{h \leq y} a_2(h)h^{-1}\log^2 Qh
= \int_1^y t^{-1}\log^2 Qtdt + \left[ t^{-1}\log^2 Qt B_2(t) \right]_1^y
- \int_1^y B_2(t)t^{-2}(2\log Qt - \log^2 Qt)dt
= \frac{1}{3}(\log^3 Qy - \log^3 Q) + \log^2 Q + O(y^{-1}\log^3 y\log^2 Qy)
- \int_1^y B_2(t)t^{-2}(-\log^2 Q + 2(1 - \log t)\log Q + 2\log t - \log^2 t)dt
= \frac{1}{3}(\log^3 Qy - \log^3 Q) + \log^2 Q + c\log^2 Q + c\log Q + c
+ O\left(\int_1^\infty (\log t)t^{-2}(\log^2 Q + \log^2 t)dt\right) + O(y^{-1}\log^3 y\log^2 Qy)
= \frac{1}{3}(\log^3 Qy - \log^3 Q) + c\log^2 Q + c\log Q + c + O(y^{-1}\log^3 y\log^2 Qy) .
\]

In the same way, we find that

\[
\sum_{h \leq y} a_1(h)h^{-1}\log Qh = \frac{1}{2}\beta_1(\log^2 Qy - \log^2 Q) + c\log Q + c + O(y^{-1}\log^3 y\log Qy)
\]

and

\[
\sum_{h \leq y} a_0(h)h^{-1} = \beta_0 \log y + c + O(y^{-1}\log^3 y) .
\]

Collecting all these in (2.18) our lemma follows.

Lastly we evaluate some integrals involving the function \( g(\nu) \).

**Lemma 5.** We have

\[
\int_0^\infty g(\nu)d\nu = 0 ,
\]

\[
\int_0^\infty g(\nu) \log \nu d\nu = -\sqrt{\pi}2^{-7/2} .
\]
Proof. It is known that [13, §13.24]
\[ \int_0^\infty J_k(\nu) \nu^{s-k-1} d\nu = \frac{\Gamma\left(\frac{s}{2}\right)2^{s-k-1}}{\Gamma(k - \frac{s}{2} + 1)} \]
for \(0 < \Re s < \Re k + 1/2\). Hence
\[
\int_0^\infty (\nu^{-k}J_k(\nu) - (2k + 1)\nu^{-k-1}J_{k+1}(\nu)) \nu^s d\nu = -s2^{s-k-1}\Gamma\left(\frac{s+1}{2}\right)/\Gamma(k - \frac{s}{2} + \frac{3}{2})
\]
for \(-1 < \Re s < \Re k - 1/2\). Setting \(k = 3/2\) and in view of (2.5) we have

\[
(2.20) \quad \int_0^\infty g(\nu) \nu^s d\nu = -s2^{s-5/2}\Gamma\left(\frac{s+1}{2}\right)/\Gamma(3 - \frac{s}{2})
\]
for \(-1 < \Re s < 1\). On putting \(s = 0\) we get \(\int_0^\infty g(\nu)d\nu = 0\). The remaining integral is equal to
\[
\frac{d}{ds}\left(\int_0^\infty g(\nu) \nu^s d\nu\right)|_{s=0}
\]
which can be evaluated by differentiating the right hand side of (2.20).

3. Proof of Theorem 2

We shall now complete the proof of Theorem 2 by evaluating the double sum

\[
T = \sum_{m<n \leq M} u_{m,n}
\]
in Lemma 3, where
\[
u_{m,n} = d(m)d(n)(mn)^{-3/4}g(4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}^\theta).
\]
In view of Lemma 3, we can allow errors of order up to \(X^{-1/2}\) in the course of our analysis.
First of all, we consider those terms $u_{m,n}$ for which $m < n/2$. In this case $\sqrt{n} - \sqrt{m} \asymp \sqrt{n}$ so that, by (2.6)

$$g(4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}) \ll (nX)^{-1}.$$ 

The contribution to $T$ from these $u_{m,n}$ is therefore

$$\ll X^{-1} \sum_{m < n \leq M} d(m)d(n)(mn)^{-3/4}n^{-1} \ll X^{-1},$$

which is acceptable.

For the remaining terms $u_{m,n}$ in $T$, we write $n = m + h$ with $1 \leq h \leq m$. Then

$$T = \sum_{h \leq M/2} \sum_{h \leq m \leq M-h} u_{m,m+h} + O(X^{-1}).$$

For $h \leq m$, we have $4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X} \asymp 2\pi h\sqrt{X/m}$ so that, by (2.6) again

$$g(4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X}) \ll mh^{-2}X^{-1}$$

and each term $u_{m,m+h}$ satisfies

$$u_{m,m+h} \ll M^\varepsilon m^{-3/2}mh^{-2}X^{-1}.$$ 

Thus, the contribution to $T$ from those $u_{m,m+h}$ with $h > \sqrt{M}$ is $\ll X^{-1}M^\varepsilon$ and the error caused by extending the upper limit for the summation on $m$ to $M$ is $O(X^{-1}M^{-1/2+\varepsilon})$. Hence we have

$$T = \sum_{h \leq \sqrt{M}} \sum_{h \leq m \leq M} u_{m,m+h} + O(X^{-1}M^\varepsilon).$$

For simplicity let

$$D_h = h^2XL^{-8}.$$ 

Then we can further write

$$(3.3) \quad T = \sum_{h \leq \sqrt{M}} \sum_{h \leq m \leq \min(D_h,M)} + \sum_{h \leq X^3L^4} \sum_{D_h < m \leq M} + O(X^{-1}M^\varepsilon)$$

$$= \sum_1 + \sum_2 + O(X^{-1}M^\varepsilon),$$
say. Using the same bound (3.2), each term $u_{m,m+h}$ in $\sum_1$ is
\[
\ll (d(m)^2 + d(m + h)^2)(m(m + h))^{-3/4} mh^{-2} X^{-1}
\ll d(m)^2 m^{-1/2} h^{-2} X^{-1} + d(m + h)^2 (m + h)^{-1/2} h^{-2} X^{-1},
\]
since $m + h \asymp m$ for $1 \leq h \leq m$. An application of the well-known estimate
\[
\sum_{m \leq y} d(m)^2 m^{-1/2} \ll \sqrt{y} \log^3 y \quad \text{for } y > 1,
\]
then yields
\[
\sum_1 \ll X^{-1} \sum_{h \leq \sqrt{M}} h^{-2} \sqrt{\min(D_h, M)} \log^3 M \ll X^{-1/2}.
\]
Putting this into (3.3), we have
\[
T = \sum_{h \leq X^{3L^4}} \sum_{D_h < m \leq M} d(m)d(m + h)(m(m + h))^{-3/4} g(\theta_{m,m+h})
\]
\[
+ O(X^{-1/2})
\]
with $\theta_{m,m+h} = 4\pi(\sqrt{m + h} - \sqrt{m})\sqrt{X}$.

Next, we transform the above inner sum over $m$ into an integral. By (2.12), (2.13) and Riemann Stieltjes integration we have
\[
\sum_{D_h < m \leq M} = \int_{D_h}^{M} (y(y + h))^{-3/4} g(\theta_{y,y+h}) d\psi_h(y)
\]
\[
= \int_{D_h}^{M} (y(y + h))^{-3/4} g(\theta_{y,y+h}) I_h'(y) dy
\]
\[
+ \left[ (y(y + h))^{-3/4} g(\theta_{y,y+h}) E_h(y) \right]_{D_h}^{M}
\]
\[
- \int_{D_h}^{M} E_h(y) \frac{d}{dy} \left\{ (y(y + h))^{-3/4} g(\theta_{y,y+h}) \right\} dy
\]
\[
= W_1(h) + W_2(h) + W_3(h),
\]
say. We bound $W_2(h)$ by using (2.15) and the trivial estimate $g(\nu) \ll 1$. Whence
\[
W_2(h) \ll M^{-3/2} M^{5/6+\varepsilon} + D_h^{-3/2} D_h^{5/6+\varepsilon} \ll D_h^{-2/3+\varepsilon} \ll h^{-4/3} X^{-2/3+\varepsilon}.
\]
For $W_3(h)$, by [13, §3.2] we have

$$g'(\nu) = -\nu^{-3/2} J_{5/2}(\nu) + 4\nu^{-5/2} J_{7/2}(\nu) \ll \nu \quad \text{for} \quad \nu \geq 0,$$

since $J_k(\nu) \ll \nu^k$. Hence, by $g(\nu) \ll 1$ and (2.15) we have

$$W_3(h) \ll \int_{D_h}^{M} y^{5/6+\varepsilon} \left\{ y^{-5/2} + y^{-3/2} |\theta_{y,y+h}| \right\} dy \ll \int_{D_h}^{M} \left\{ y^{-5/3+\varepsilon} + y^{-2/3+\varepsilon} h y^{-1/2} X^{1/2} y^{-3/2} X^{1/2} \right\} dy \ll h^{-4/3} X^{-2/3+\varepsilon}.$$

In view of (3.4) and (3.5), the contribution to $T$ from $W_2(h)$ and $W_3(h)$ is therefore

$$\ll \sum_{h \leq X^L} h^{-4/3} X^{-2/3+\varepsilon} \ll X^{-2/3+\varepsilon},$$

which is again acceptable. Thus,

$$(3.6) \quad T = \sum_{h \leq X^L} \int_{D_h}^{M} (y(y+h))^{-3/4} g(\theta_{y,y+h}) I'_h(y) dy + O(X^{-1/2}).$$

To evaluate the inner integral, we begin by making the change of variable

$$\omega = \theta_{y,y+h} = 4\pi \left( \sqrt{y + h} - \sqrt{y} \right) \sqrt{X}.$$ 

Then

$$y = 4\pi^2 X \omega^{-2} h^2 - \frac{1}{2} h + (64\pi^2 X)^{-1} \omega^2 = 4\pi^2 X \omega^{-2} h^2 (1 + O(\omega^2 X^{-1} h^{-1})),$$

so that

$$(y(y+h))^{-3/4} = (4\pi^2 X \omega^{-2} h^2 - (64\pi^2 X)^{-1} \omega^2)^{-3/2} = (2\pi h)^{-3} X^{-3/2} \omega^3 (1 + O(\omega^4 X^{-2} h^{-2}))$$

and

$$\frac{dy}{d\omega} = -8\pi^2 X \omega^{-3} h^2 (1 + O(\omega^4 X^{-2} h^{-2})).$$
Moreover, by (2.16)
\[ I'_h(y) = 4a_2(h) \log^2 (2\pi \sqrt{X} \omega^{-1}h) + 2a_1(h) \log(2\pi \sqrt{X} \omega^{-1}h) + a_0(h) + \mathcal{O}(\omega^2 X^{-1}h^{-1}(|a_2(h)|L + |a_1(h)|)) . \]

Set
\[ (3.7) \quad u_1 = 4\pi (\sqrt{M + h} - \sqrt{M}) \sqrt{X} = 2\pi hX^{-3} + \mathcal{O}(h^2 X^{-10}) \]
and
\[ (3.8) \quad u_2 = 4\pi (\sqrt{D_h + h} - \sqrt{D_h}) \sqrt{X} = 2\pi L^4 + \mathcal{O}(h^{-1} X^{-1}L^{12}) . \]

Then with the help of all these estimates we find that
\[
\int_{D_h}^M (y(y + h))^{-3/4} g(\theta_{y,y+h}) I'_h(y) dy \\
= \frac{1}{\pi \sqrt{X}} \int_{u_1}^{u_2} g(\omega) h^{-1} \{4a_2(h) \log^2 (2\pi \sqrt{X} \omega^{-1}h) + 2a_1(h) \log (2\pi \sqrt{X} \omega^{-1}h) + a_0(h)\} d\omega + \mathcal{O}(h^{-2} X^{-3/2+\epsilon}) .
\]

In obtaining the above \(\mathcal{O}\)-term, we have used \(g(\omega) \ll 1\), (3.8) and the observation that \(a_j(h) \ll \log^3 h \ll L^3\). The integration limits \(u_1\) and \(u_2\) can be replaced by \(2\pi hX^{-3}\) and \(2\pi L^4\) respectively, since the error thus caused is
\[
\ll X^{-1/2}h^{-1}L^5(h^2X^{-10} + h^{-1}X^{-1}L^{12}) \ll hX^{-21/2}L^5 + h^{-2}X^{-3/2}L^{17} \ll h^{-2}X^{-3/2+\epsilon} ,
\]
by (3.7) and (3.8). Collecting these into (3.6), we get
\[
T = \frac{1}{\pi \sqrt{X}} \sum_{h \leq X L^3} \int_{2\pi hX^{-3}}^{2\pi L^4} g(\omega) h^{-1} \{4a_2(h) \log^2 (2\pi \sqrt{X} \omega^{-1}h) + 2a_1(h) \log (2\pi \sqrt{X} \omega^{-1}h) + a_0(h)\} d\omega + \mathcal{O}(X^{-1/2}) .
\]

Next we interchange the summation and integration. In view of (2.18) we have
\[ (3.9) \quad T = \frac{1}{\pi \sqrt{X}} \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\xi((\pi)^{-1}\omega X^3, 2\pi \sqrt{X} \omega^{-1}) d\omega + \mathcal{O}(X^{-1/2}) .\]
By Lemma 4, and after some simplifications, we have

\[ \xi((2\pi)^{-1} \omega X^3, 2\pi \sqrt{X} \omega^{-1}) = \log \omega \log^2 X + (c_8 \log \omega + c_9 \log^2 \omega) \log X + \]
\[ + c_{10} \log \omega + c_{11} \log^2 \omega + c_{12} \log^3 \omega + \Phi(X) + \mathcal{O}(\omega^{-1} X^{-3} L^5), \]

where \( \Phi(X) = c_{13} \log^3 X + c_{14} \log^2 X + c_{15} \log X + c_{16} \) and \( c_8, c_9, \ldots, c_{16} \) are certain constants. Finally inserting this into (3.9) we get

\[
T = \frac{1}{\pi \sqrt{X}} \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega) \left\{ \log \omega \log^2 X + (c_8 \log \omega + c_9 \log^2 \omega) \log X + \right. \]
\[ + c_{10} \log \omega + c_{11} \log^2 \omega + c_{12} \log^3 \omega + \Phi(X) \left\} d\omega + \mathcal{O}(X^{-1/2}). \tag{3.10} \]

It remains to evaluate the integrals

\[ K_j = \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega) \log^j \omega d\omega \]

for \( j = 0, 1, 2, 3 \). Writing

\[ K_j = \int_0^\infty g(\omega) \log^j \omega d\omega - \int_0^{2\pi X^{-3}} g(\omega) \log^j \omega d\omega - \int_{2\pi L^4}^\infty g(\omega) \log^j \omega d\omega, \]

we see, by (2.6), that the last two integrals are bounded by \( X^{-3} L^j \) and \( L^{-4+j} \) respectively. Hence, by Lemma 5 we have

\[ K_0 \ll L^{-4}, \quad K_1 = -\sqrt{2}^{-7/2} + \mathcal{O}(L^{-3}) \]

and by (2.6),

\[ K_2, K_3 = \text{constant} + \mathcal{O}(L^{-1}). \]

When these are inserted into (3.10) we obtain

\[ T = -2^{-3} (2\pi X)^{-1/2} \log^2 X + c_{17} X^{-1/2} \log X + \mathcal{O}(X^{-1/2}), \]

and Theorem 2 now follows from (3.1) and Lemma 3.
REFERENCES


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